entropy

Article

When Is an Area Law Not an Area Law?

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Abstract: Entanglement entropy is typically proportional to area, but sometimes it acquires an additional logarithmic pre-factor. We offer some intuitive explanations for these facts.

Keywords: entanglement entropy; area law; Fermi surface; quantum field theory; free fields

1. Introduction

Born of the attempt to understand black hole thermodynamics, the concept of entanglement entropy has since gone out into the wider world and made itself at home in milieus ranging from conformal field theory to lattice-fermions to causal sets. In the course of this odyssey, the so called *area law* that entropy is proportional to horizon area has largely maintained its validity, but not always.

Given a spatial region B and its spatial complement B', the entanglement entropy S between B and B' is said to satisfy an area-law if it is proportional to the area A of the surface ∂B that separates B from B'. Here, of course, one is using the word "area" loosely to denote a quantity which carries dimensions of $[length]^{d-1}$ when the spatial dimension is d (the spacetime dimension is $d+1$). One is also assuming that the quantum field (or spin-system, etc.) under consideration is in, or sufficiently near to, its ground state or "vacuum".

For $d = 1$, when B is a line-segment or half-line, its boundary is reduced to just one or two isolated points, and a strictly valid area law in this case would require S to be independent of the size of B. However, this is not always true. Rather the putative $[length]^0$ scaling of S for $d = 1$ is sometimes replaced by a logarithm, as happens for example if we are dealing with a massless field. Although strictly speaking, a logarithmic scaling is not an area law, it is completely consistent with the rule of thumb that a quantity which scales like x^n for generic n, often scales like $\log x$ when $n = 0$. In that sense, the logarithm for $d = 1$ fits nicely into a general pattern.

What does not fit so nicely, though, is the fact that even in dimensions greater than $1 + 1$, the entanglement entropy can acquire an additional pre-factor of the form $\log L$ where L is some characteristic size of the region B. This happens in particular when the system in question is a collection of free fermions in their ground state at nonzero density (i.e., when they are at zero temperature but nonzero chemical potential). Of course a logarithm represents a relatively mild deviation from a constant, but the discrepancy is big enough that it asks to be explained. Let us, then, try to develop an intuition for when an area law ought to hold, and why.

Note that although some of the arguments we will give herein may be new to a certain extent, this paper is intended primarily as a didactic introduction to certain heuristic viewpoints which have arisen over the years in relation to entanglement entropy. In discussing entanglement entropy with various people, we have found that depending on which community of physicists they belonged to, they have typically been familiar with some of these viewpoints, but unfamiliar with others.

By bringing them together in one place, we hope to make a more unified view available to the community as a whole.

For recent reviews of entanglement entropy and the area law see [1,2]. On the area law for a free scalar field see [3–5]. Note, however, that contrary to the false claim in [5] ("Note added"), the analysis in [4] was not limited to $m^2 > 0$. On the contrary, all values of m^2 were treated together in [4], and the emphasis, if any, was precisely on the massless case. For a cubic region within a cubic lattice of harmonic oscillators see [6]. Logarithmic scaling of entanglement entropy at a quantum critical point in $d = 1$ (spatial dimension) is discussed in [7,8]. For the logarithmic modification of the area law in the presence of a Fermi surface see [9–12]. For a logarithmic prefactor coming from a "Bose surface" rather than "Fermi surface" see [13]. For an extension to a nonrelativistic situation of the "Rindlerian" analysis described below, see [14]. An initial computation of entanglement entropy in a causal set is described in [15].

2. Reasons for the Area Law and Departures from It

Think for example of a spatial region B in the shape of a ball, a disk or a line-segment, depending on the dimension, and consider a massless (gapless) scalar field which we take to be in its "vacuum" or "ground state" and whose entanglement with B' we wish to evaluate (see Figure 1). One might imagine that if the field can be decomposed into quasi-localized "modes" (for example, wavelets [16]), then the entanglement entropy S will come primarily from the modes that "*straddle*" the boundary ∂B, with each such mode contributing on the order of one bit of entropy. The total entropy would then be proportional to the number of straddling modes. How many are there?

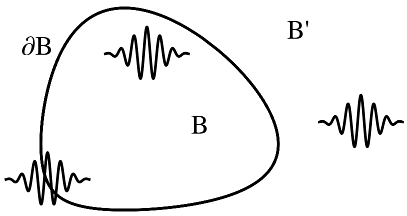

Figure 1. A spatial region B and its boundary ∂B, showing "modes" which do and do not straddle ∂B.

Without some restriction or "cutoff" there are of course infinitely many straddling modes of shorter and shorter wavelength λ. But if we limit ourselves to $\lambda > \ell$ for some $\ell > 0$, the number of modes will be finite, and we can estimate it as follows, assuming for the moment that $d = 3$ and writing L for the diameter of the ball. Let $k = 2\pi/\lambda$ as usual. Then in the range d^3k about k, the number of modes will be approximately $V\, d^3k/(2\pi)^3$, where V is the volume of the ball B. Those which straddle the sphere ∂B will comprise a fraction V'/V of the total, where $V' \approx \lambda A$ with A being the area of ∂B, whence the number of straddling modes will be $\lambda\, A\, d^3k/(2\pi)^3$. Integrating this from $k_{min} = 2\pi/L$ up to $k_{max} = 2\pi/\ell$, and assuming that $\ell \ll L$, yields then the area law,

$$S \propto N \approx \int_{k_{min}}^{k_{max}} \lambda A \frac{d^3k}{(2\pi)^3} = \int_{2\pi/L}^{2\pi/\ell} \frac{2\pi}{k} A \frac{4\pi k^2 dk}{(2\pi)^3} = \frac{A}{2\pi}\left(\left(\frac{2\pi}{\ell}\right)^2 - \left(\frac{2\pi}{L}\right)^2\right) \approx \frac{2\pi A}{\ell^2}$$

In spatial dimension d the integrand will contain $k^{d-2}\, dk$ in place of $k\, dk$, and the integration will yield instead of a coefficient of $(1/\ell^2 - 1/L^2)$ a coefficient of $(1/\ell^{d-1} - 1/L^{d-1})$, except when $d = 1$, in which case the integration will yield a coefficient of $\log(L/\ell)$. Thus, we expect an area law for $d > 1$ and a "log-corrected area law" for $d = 1$.

In dropping the term $1/L^2$, we have of course assumed that $\ell \ll L$, and under this assumption the above derivation shows that almost all of the contributing modes lie very close to the boundary.

When this is true, and provided that the boundary is sufficiently smooth, its detailed shape becomes irrelevant, and our estimate for the sphere should be valid in general.

So far we have been assuming a massless field. With a nonzero mass m, one would expect entanglement to extend at most over a correlation length or "Compton wavelength", $1/mc$, meaning that only modes with wavelengths shorter than $1/mc$ should be counted in estimating the entropy. Consequently, instead of a lower limit $k_{min} \sim 1/L$, our mode-counting integral should now have a lower limit of $\sim mc$, or just m if we set the speed of light c to 1, along with $\hbar$. For $d > 1$ this change to k_{min} makes very little difference, provided that $\ell \ll 1/m$, but for $d = 1$, it has the effect of converting $\log(L/\ell)$ into $\log(1/m\ell)$, a form we will need below.

The analysis we have just gone through explains why an area law arises in general, and also why a logarithmic dependence on L shows up in $1+1$ dimensions when $m = 0$. However, it also leads us to ask why similar reasoning fails in the Fermi-surface case. Before suggesting an answer, let us consider another argument for an area law, which in one way is more precise, but which applies only when the region in question is a halfspace and the quantum field in question is relativistic. In that case, one can view the problem from the perspective of a "Rindler observer", for whom, as is well known, the Minkowski vacuum appears as a thermal state with a local temperature T proportional to the local redshift and given by $\beta \equiv 1/T = 2\pi z$ where z is the distance to the boundary plane (or "horizon"). Now a massless field in a thermal state of *uniform* temperature T carries an entropy density proportional to T^3 when $d = 3$, or to T^d in general. In our case, the temperature is not uniform, but assuming a local analysis is approximately valid, we can simply replace T by $1/2\pi z$ and estimate the total entropy by integrating T^3 over the region of interest.

Since translation parallel to the boundary is a symmetry of the halfspace, the entropy per unit boundary area is the relevant quantity, and it will be given by an integral $\int T^3 dz \sim \int dz/z^3$. (We henceforth abandon any effort to keep track of numerical pre-factors, which in any case depend on the details of the cutoff, unless $d = 1$.) In this situation it seems natural to introduce the UV cutoff by restricting the range of integration to $z > \ell$, i.e., by ignoring points closer than ℓ to the boundary. If we do so then a boundary-portion of area A will contribute an entropy of $A \int_{1/\ell}^{1/L} dz/z^3 \sim A/\ell^2$, as before. For general $d > 1$ the same reasoning yields A/ℓ^{d-1}, while for $d = 1$ it yields $\log(L/\ell)$. Thus we reach exactly the same conclusions as before. In this case, however, it is obvious that the analysis is inapplicable when a Fermi surface is involved, because the nonzero density of electrons (or other fermions) breaks the symmetry of Lorentz-boosts (at least naively).

To progress further it seems necessary to distinguish more carefully between the component of the momentum-vector which is parallel to the boundary surface (and which we will call *tangential*) and the component which is orthogonal to it (and which we will call *longitudinal*). In order to have a definite setup to work with, let us consider a free relativistic scalar field of mass $m \geq 0$ in a halfspace. Also, in order to distinguish between the tangential and longitudinal projections of the wave-vector without introducing cumbersome subscripts like $k_\parallel$ and $k_\perp$, let us use the letter "p" for the tangential wave-vector, while reserving "k" for the longitudinal component. (See Figure 2.)

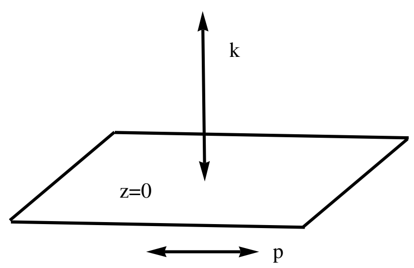

Figure 2. Flat boundary surface and the "tangential" and "longitudinal" wave-vectors p and k.

In view of the translational symmetry along the boundary, it is evident that modes with different values of p will not interact with each other and will contribute independently to the entanglement.

Thus it suffices to estimate the contribution from each p and add them up to obtain the total entropy S per unit boundary area. In this way, the problem is reduced to series of $1+1$ dimensional problems parameterized by the tangential momentum-vector.

A mode with tangential momentum p takes the form $\phi(t,x,z) = e^{ip\cdot x} f(t,z)$ where z is the longitudinal coordinate as before, and x coordinatizes the boundary surface, $z=0$. Substituted in the wave equation, $(\Box - m^2)\phi = 0$, this Ansatz produces the 2D equation,

$$-\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial z^2} - (m^2+p^2) f = 0 \tag{1}$$

In other words, it produces the $d=1$ wave equation with m^2+p^2 playing the role of m^2. The important thing to notice here is that, *with the sole exception of $p=0$, we are always in the massive (gapped) case*, even when the field overall is massless.

Now a massive free field in $d=1$ is one of the examples we've already encountered. Based on our earlier analysis of that case, we expect (and in fact we know) that the entropy will scale as $\log(1/m\ell)$. (Even without any real thought we can be confident on dimensional grounds that S in this case will be some function, $f(m\ell)$. All that the more detailed analysis contributes is the particular form of f. That this form is logarithmic guarantees via (2) that the entropy per unit area is finite, but that would have been true even if $f(x)$ had diverged as a power law $1/x^\gamma$ with $\gamma < d-1$.)

Given this, it is an easy matter to deduce the scaling of the total entropy (and therewith the area law). Returning for simplicity to $m=0$, we obtain from tangential momentum p an effective mass of $\sqrt{m^2+p^2} = \sqrt{p^2} = |p|$ and a corresponding entropy of $S(p) \sim \log(1/\ell|p|)$, valid for $p \ll 1/\ell$. (For $p \gtrsim 1/\ell$, $S(p)$ falls rapidly to zero.) Integrating over p then yields (in $d=3$) an entropy per unit area of

$$\int \frac{d^2p}{(2\pi)^2}\, S(p) \approx \int_0^{1/\ell} p\, dp\, \log(1/\ell p) = \frac{1}{4\ell^2}\,, \tag{2}$$

or corresponding to a portion of the boundary of area A, an entropy $\sim A/\ell^2$. Thus, we confirm our earlier conclusion in a more rigorous manner, and more importantly, in a manner which lends itself to being carried over to the case of a Fermi surface.

In light of the foregoing analysis, it seems possible to put one's mathematical finger on the reason why a logarithmic divergence (and hence a departure from a simple area law) shows up in $d=1$, but not in higher dimensions. If we take Equation (2) as our starting point, then it's evident that a logarithm is mathematically present in all dimensions, but since it is associated only with vanishing tangential momentum, its contribution to the entanglement entropy is "drowned out" when $d \geq 2$. As a result, one obtains an area law in higher dimensions, whether or not the field is massless. (The integrand $\log(1/\ell p)$ in (2) is not correct all the way down to $p=0$. Rather it goes over to the form $\log(L/\ell)$ for $p \ll 1/L$. However, that portion of the integral yields (approximately) only $\int_0^{1/L} p\, dp\, \log(L/l) = \log(L/l)/2L^2$, which is negligible in comparison to $1/4\ell^2$ when $\ell \ll L$.)

Although the logarithmic scaling in $d=1$ is well established, and although we have tried above to provide qualitative explanations for it, one might still wonder whether there is anything more to be gleaned from it on an intuitive, or even a quantitative level. To that end, we would like to offer some remarks concerning the massless scalarfield in $d=1$. When $m=0$, the entropy can depend only on L/ℓ, or possibly the ratio of either L or ℓ to some further infrared scale that might enter the problem. The only real question, then, is why the dependence turns out to be logarithmic. Such a dependence fits nicely into the ℓ^{1-d} hierarchy, but can one give a less formal reason? More specifically, can we relate the logarithmic scaling of the entropy to the "infrared nonlocality" that accompanies masslessness in $1+1$ dimensions?

A first symptom of this "nonlocality" is the fact that, rather than falling off with separation, the Green function for a massless scalar field grows logarithmically. A second symptom, closely related to the first, is that the Minkowski vacuum fails to exist when $m=0$: neither in full $\mathbb{M}^2$

nor in $\mathbb{S}^1 \times \mathbb{R}$ is there a normalizable state of minimum energy. (Such a state does exist when the spacetime is a halfspace or a strip, with Dirichlet boundary conditions.) For this reason, we believe that most computations claiming to evaluate the vacuum entanglement entropy in $1+1$ dimensions are incomplete and need to be corrected. Indeed, numerical simulations [17] strongly suggest that in these two cases the entanglement entropy will diverge in the absence of some sort of infrared regulator (the divergence being logarithmic in the regulator). In the analysis above, we have implicitly imposed such a regulator, for example Dirichlet conditions at one end of the spatial interval $[0, L]$, or as in [18] the choice of "Sorkin–Johnston-vacuum" in some large "causal diamond" containing $[0, L]$.)

These symptoms of "nonlocality" must be related to the logarithmic scaling of the entropy, but can one be more precise about how? Is there, perhaps, a more detailed breakdown of how the entanglement entropy arises that might also carry over to other cases of interest? For example, could one identify some sort of "falloff of entanglement with distance" which would yield the entropy when integrated over all pairs of points on either side of the boundary? If so, and if the entanglement fell off as $1/r^2$, then logarithmic scaling would follow. (In a rather different vein, reference [19] suggests that the logarithmic scaling in $d = 1$ might be connected to a similar looking corner-contribution in $d = 2$.

The analysis that led us to Equation (2) parallels the original proof of the area law in [3,4]. As in those papers, most of our discussion here so far has been limited to the case of a free scalar field (in its vacuum state or sufficiently near to it). Let us now try to carry our analysis over to a "gas" of electrons, treated as non-interacting, and held at zero temperature but *nonzero*, nonrelativistic density.

For the one-electron states, we can do exactly the same decomposition into tangential modes as before and thereby reduce the calculation again to a succession of $d = 1$ problems. (Equivalently we are decomposing the field operator ψ that creates the 1-electron states.) Making for the 1-electron Schrödinger equation the same substitution as we made earlier for the Klein-Gordon equation, then teaches us that the effect of the tangential momentum p in the present, nonrelativistic case is just to shift the zero of energy upward by $p^2/2m$, as if we had added a constant potential term, $V = p^2/2m$, to the Hamiltonian.

Consequently, for a given value of p, the $d = 1$ free-particle energy levels will only be filled up to a maximum energy of $E_f(p) = E_f - p^2/2m$, i.e., they will be filled up to a longitudinal momentum k such that $k^2/2m = E_f(p)$. The Fermi level E_f is effectively shifted downward, but otherwise nothing changes. See Figure 3. (Relativistically, the dispersion relation itself would also become a function of p, but it would remain convex.) Thus each 1d problem has its own Fermi "surface", its location being set by p^2. If we can solve for the entropy $S(p)$ in this situation, we have only to integrate the result, exactly as before.

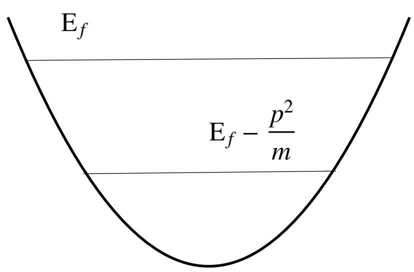

Figure 3. The effective Fermi level in k is shifted downward by $p^2/2m$.

Now for each given p (except very near $E_f(p) = 0$) we can approximate the dispersion relation in the neighborhood of $E = E_f(p)$ by a straight line. Near $k = \sqrt{2mE_f(p)}$ our 1d problem thus looks like that of a relativistic massless field, for which we already know the answer. If we accept this picture [this is perhaps the weakest link in the argument!], we can immediately conclude that $S(p)$ scales like $\log(L/\ell)$. But unlike in Equation (2), this scaling law is independent of the tangential momentum p, and that being the case, the integral $\int dp\, p\, S(p)$ will scale the same way. The logarithm is thus not drowned out and remains as an overall factor when we sum up the contributions, while the summation itself produces a factor of $1/\ell^2$ as before. The entanglement entropy will therefore scale like $\log(L/\ell)\, A/\ell^2$ rather than just A/ℓ^2.

It remains to determine, at least in order of magnitude, the value of the cutoff ℓ. In a continuum quantum field theory, any cutoff must come from outside the theory. Either it is put in by hand, or it emerges from some deeper reality. In the earlier case of a scalar field, our cutoffs were ad hoc and introduced by hand, but in the present situation the deeper reality is known, and it implies that the continuum approximation will fail when one reaches a distance-scale comparable to the mean separation between electrons. This therefore is the appropriate value of ℓ, and since it coincides with the scale set by the "Fermi momentum" p_F, we can also think of ℓ as representing $1/p_F$.

With the determination of ℓ our discussion of the fermion gas has reached its destination. But what might raise doubts about our explanation of the logarithmic prefactor is that in concentrating on the approximately linear dispersion relation near the effective Fermi surface, it seemingly attributes the entanglement entropy entirely to the "shallow electrons" in the Dirac sea, whereas it seems clear that "deeper electrons" must be contributing most of the entropy since they are the majority (at least in a 1d context). What perhaps rescues the explanation however, is that the deeper electrons do in fact make their presence felt via their influence on the cutoff, because it's the total number-density n that determines ℓ through the relation $\ell \sim n^{-1/3}$. In this way the proposed explanation attains a certain logical coherence.

Before concluding we would like to comment on an aspect of our treatment that in a certain sense is unsatisfactory, namely its appeal to space as opposed to spacetime, with the consequent use of a cutoff that fails to be Lorentz invariant. For the purposes of condensed matter, this is not a problem, of course, but in the context of relativistic quantum field theory, or of black hole entropy, a more four-dimensional treatment would be desirable. Such a method was introduced in [20] and illustrated in [18] by a $1+1$ dimensional calculation of entanglement entropy. Taking this as a point of reference, it would be interesting to ask whether the intuitive explanations offered above could also be cast into spatio-temporal form.

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