Quantum-Wave Equation and Heisenberg Inequalities of Covariant Quantum Gravity

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Abstract: Key aspects of the manifestly-covariant theory of quantum gravity (Cremaschini and Tessarotto 2015–2017) are investigated. These refer, first, to the establishment of the four-scalar, manifestly-covariant evolution quantum wave equation, denoted as covariant quantum gravity (CQG) wave equation, which advances the quantum state ψ associated with a prescribed background space-time. In this paper, the CQG-wave equation is proved to follow at once by means of a Hamilton–Jacobi quantization of the classical variational tensor field $g \equiv \{g_{\mu\nu}\}$ and its conjugate momentum, referred to as (canonical) g-quantization. The same equation is also shown to be variational and to follow from a synchronous variational principle identified here with the quantum Hamilton variational principle. The corresponding quantum hydrodynamic equations are then obtained upon introducing the Madelung representation for ψ, which provides an equivalent statistical interpretation of the CQG-wave equation. Finally, the quantum state ψ is proven to fulfill generalized Heisenberg inequalities, relating the statistical measurement errors of quantum observables. These are shown to be represented in terms of the standard deviations of the metric tensor $g \equiv \{g_{\mu\nu}\}$ and its quantum conjugate momentum operator.

Keywords: covariant quantum gravity; Hamilton–Jacobi quantization; quantum-wave equation; quantum hydrodynamic equations; Heisenberg inequalities

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1. Introduction

The principles of general covariance and of manifest covariance with respect to the group of local point transformations (LPT-group; [1,2]), i.e., coordinate diffeomorphisms mutually mapping in each other different general relativistic (GR) frames:

$$r \Leftrightarrow r' = f(r), \tag{1}$$

with $r \equiv \{r^\mu\}$ and $r' \equiv \{r'^\mu\}$ denoting four-positions in the two frames, lie at the foundation of all relativistic theories and of the related physical laws.

In fact, although the choice of special coordinate systems is always legitimate for all physical systems, either classical or quantum, the intrinsic objective nature of the physical laws that characterize them, including possible symmetry transformations, makes them manifestly frame independent. For the same reason, since LPTs preserve by construction the differential-manifold structure of

space-time, these principles represent also a cornerstone of the so-called standard formulation of general relativity (SF-GR), namely the Einstein field equations and the corresponding classical treatment of the gravitational field [3–5].

The same principles, however, should apply to relativistic statistical mechanics and the very foundations of quantum field theory. The significance of relativistic statistical mechanics is also of great importance in the framework of general relativity and cosmology, including both classical and quantum theories. This paper deals in particular with the problem of the formulation of the theory of quantum gravity (QG). Despite major theoretical developments achieved in the past, a theory of this type, i.e., fulfilling the same principles, has remained until very recently [6–9] largely unsolved. The fundamental reason is that, as displayed in [8], a corresponding manifestly-covariant, and possibly constraint-free, classical Hamiltonian theory of SF-GR is actually required for the completion of such a task. On the other hand, in the previous mainstream literature, typically only non-manifestly covariant Hamiltonian theories of GR were developed. These are based on suitable decompositions or foliations of space-time, i.e., the adoption of particular subsets of GR-frames or coordinate systems and non-tensor Lagrangian/Hamiltonian variables, which typically involve the singling out of the coordinate time to prescribe the dynamical evolution of metric tensor hypersurfaces (exhaustive developments of the issue can be found in [10–15]).

Nevertheless, the mathematical framework to be adopted for the construction of manifestly-covariant classical Lagrangian and Hamiltonian theories is well-established both for particle dynamics [16–20], as well in continuum field theory, where it is known as the DeDonder–Weyl formalism [21–29]. This type of formulation has been developed and applied consistently to the case of the gravitational field described by SF-GR only recently in [8,9], providing theories of covariant classical gravity (CCQ) and covariant quantum gravity (CQG). In both cases, the classical and quantum Hamiltonian structures are realized in the framework of a so-called background space-time picture, namely requiring that a prescribed space-time $\left(\mathbf{Q}^4, \widehat{g}\right)$ exists whose metric tensor $\widehat{g} \equiv \left\{\widehat{g}_{\mu\nu}(r)\right\}$ is considered a prescribed classical field, which determines the geometric properties of the background space-time and, either classical or quantum, variational tensor fields. More precisely, for this purpose, $\left(\mathbf{Q}^4, \widehat{g}\right)$ is taken as a differentiable Lorentzian manifold with signature $(+, -, -, -)$ or analogous permutations, with $\mathbf{Q}^4$ denoting the four-dimensional Riemann space-time and $\widehat{g}_{\mu\nu}(r)$ the metric tensor.

Nevertheless, historically basic conceptual problems that lie at the very foundation of QG as a quantum theory of the gravitational field as described classically by the Einstein equations have been called into question. As shown in [8], a quantum theory of this type has conceptual implications also for cosmology in connection with the possible existence of massive gravitons associated with a non-vanishing cosmological constant Λ. Therefore, CQG-theory is expected to represent, as well, a candidate quantum theory of the universe, i.e., a basis for quantum cosmology. In view of these considerations, the issues addressed here are the following ones:

- Issue #1: the canonical g-quantization. The issue concerns the quantization, here referred to as g-quantization, of the classical canonical state $x = \{g, \pi\}$, with $\pi \equiv \left\{\pi_{\mu\nu}\right\}$ being the classical reduced-dimensional canonical momentum conjugate to the continuum field tensor $g \equiv \left\{g_{\mu\nu}(r)\right\}$. This is realized by means of a correspondence principle between the classical state $x = \{g, \pi\}$ and the corresponding quantum variables $x^{(q)} = \left\{g^{(q)} \equiv g, \pi^{(q)}\right\}$, with $\pi^{(q)} \equiv \left\{\pi_{\mu\nu}^{(q)}\right\}$ being the corresponding quantum operator. In such a context, the question arises of the unique prescription of the quantum-wave function and the corresponding quantum wave-equation associated with g-quantization. These should be understood respectively as quantum wave-function and quantum wave-equation of the universe and, therefore, to hold for arbitrary realizations of the background space-time. According to [8], the four-scalar (i.e., obtained by saturation of four-tensors) quantum state ψ should dynamically evolve with respect to an invariant proper-time parameter s defining the canonical Hamiltonian flow. Hence, besides $g \equiv \left\{g_{\mu\nu}(r)\right\}$, ψ is parametrized in terms of the prescribed metric tensor $\widehat{g}(r)$ of the background space-time, as well as the four-position r^μ and the proper time s, whose physical meaning in the context of QG remains nevertheless to be specified.

- Issue #2: the quantum Hamilton variational principle. This is about the search for a variational principle for the same quantum wave equation, which may provide an additional "a posteriori" justification of its physical validity. The form of the same equation, in fact, should be consistent with the existence of real symmetric functionals, which are bilinear with respect to the quantum wave function ψ, while satisfying the principle of manifest-covariance for the Hamiltonian functional. In agreement with the variational setting developed for the classical derivation of the Einstein equations, also in the quantum regime, we seek the implementation of a synchronous variational principle characterized by having integral differential four-volume and/or line elements, which are left invariant during synchronous variations (see the details in [6]).

- Issue #3: the g-quantization Heisenberg inequalities. This concerns the problem of quantum measurement and, more precisely, the possible validity of a suitable Heisenberg principle, which may provide inequalities appropriate for the treatment of g-quantization and relate the standard deviations of the quantum observables $x^{(q)}$.

The goal of this paper is to set these problems in the axiomatic framework of CQG-theory earlier developed in [8]. Such a theory, realizing a manifestly-covariant canonical quantization of the classical Hamiltonian state $x = \{g, \pi\}$, is based on the assumption that the corresponding quantum state represented by a single four-scalar wave-function ψ advances in proper-time by means of a non-stationary quantum-wave equation (the CQG-wave equation) or equivalently of a corresponding set of non-stationary quantum hydrodynamic equations.

More precisely, for this purpose, first, in reference to Issue #1, the problem is set in the context of the Hamilton–Jacobi (HJ) g-quantization formulated for the classical canonical state $x = \{g, \pi\}$. As we intend to show, in such a context, the CQG-wave equation emerges uniquely from the classical Hamilton–Jacobi equation by invoking quantization rules relating Poisson brackets to quantum commutators and the definition of conjugate quantum operators of field observables. The resulting quantum wave equation does not require any independent postulation, but rather is found to be naturally associated with the classical Hamiltonian and Hamilton–Jacobi theories of GR. Second, in reference to Issue #2, a suitable quantum Hamilton variational principle is established, which recovers the characteristic variational property of quantum field theory. More precisely, the same quantum wave-equation, as well as the related quantum hydrodynamic equations are shown to be variational both with respect to the quantum wave-function and the corresponding quantum fluid fields. Finally, in reference to Issue #3, a crucial property of quantum measurements performed in the context of g-quantization is displayed, lying in the validity of suitable Heisenberg inequalities. In particular, we intend to show that, for arbitrary quantum wave-functions, which are solutions of the same CQG-wave equation, the standard deviations associated with quantum measurements of $g_{\mu\nu}$ and $\pi_{\mu\nu}^{(q)}$ satisfy two different inequalities, referred to respectively as first and second Heisenberg inequalities.

In this connection, it is important to remark that one of the objectives of CQG-theory is the search for solutions for the quantum-gravity wave-function ψ in contexts of physical and astrophysical interest, for example those characterized by the occurrence of strong-gravity effects and where quantum phenomena may play a relevant role. In particular, this generally involves the determination of both proper-time-dependent and stationary solutions. In the second case, the issue concerns the eigenvalue problems associated with the stationary CQG-wave equation arising both in the case of vacuum, as well as in the presence of external sources. Thus, in particular, the goal concerns the determination of the corresponding energy spectrum of the quantum Hamiltonian operator and the conditions warranting its discrete or continuous structure. Ultimately, a theory of this type should yield information about the phenomenological properties of gravitons and related observational features, like mass, energy and quantum dynamics. An application of this type of CQG-theory was first proposed in [9], where the case of the stationary vacuum CQG-wave equation was studied in a cosmological regime characterized by a non-vanishing cosmological constant Λ. A stationary equation for the CQG-state in terms of the four-scalar invariant-energy eigenvalue was constructed for the corresponding quantum Hamiltonian

operator holding in the harmonic approximation, i.e., assuming to have oscillation of the quantum field $g_{\mu\nu}$ suitably close to the classical background tensor $\widehat{g}_{\mu\nu}$. The conditions for the existence of a discrete invariant-energy spectrum were determined, providing a possible estimate for the graviton mass in the cosmological framework considered, together with the interpretation about the quantum origin of the cosmological constant in terms of the graviton Compton wavelength. The achievements of the present research are intended to provide a theoretical framework that can be applied to further investigate the phenomenology of such physical solutions. In particular, noting that the predicted existence of gravitons by CQG-theory is directly related to the quantum behavior of the field $g_{\mu\nu}$, it follows that the proof of existence of Heisenberg inequalities for $g_{\mu\nu}$ and $\pi_{\mu\nu}^{(q)}$ is expected to constrain in turn also the observational challenges of graviton particles themselves. Accordingly, the dynamics of quantum gravitational fields and quantum gravitons are subject to uncertainty effects, which can be predicted on the basis of the theory proposed below.

The scheme of presentation is as follows. First, in Section 2, the classical Hamiltonian structure of GR is recalled, which is determined in the framework of a manifestly-covariant approach. This leads to the introduction of both Hamilton and Hamilton–Jacobi equations underlying the gravitational field equations of GR. In Section 3, the Hamilton–Jacobi g-quantization scheme is displayed, yielding a relationship between classical and quantum Hamilton–Jacobi representations and yielding in this way a representation of the quantum wave equation in manifestly-covariant form. Section 4 deals with the formulation of a synchronous quantum Hamilton variational principle for the quantum gravity wave equation. In Section 5, the quantum hydrodynamic equations corresponding to the quantum wave equation are derived, upon invoking the Madelung representation for the quantum wave function. The same equations are then shown to be variational, as well, following from the same synchronous variational principle leading to the CQG-wave equation. Then, in Section 6, the validity is proven of suitable Heisenberg inequalities, while concluding remarks are drawn in Section 7.

2. The Classical Hamiltonian Structure of SF-GR

In this section, the Hamiltonian structure of space-time $\{x = (g, \Pi), H\}$ introduced in [7] and the derivation of its corresponding reduced-dimensional representation are recalled. This follows by identifying the variational canonical variables g and Π respectively with the second-order real tensor fields $g \equiv (g_{\mu\nu}(r))$ and the third-order canonical momentum $\Pi \equiv \left(\Pi_{\mu\nu}^{\alpha}(r) \right)$ and H with a suitable variational Hamiltonian density. As a consequence, the resulting continuum Hamilton equations (extended Hamilton equations of SF-GR) take the form:

$$\begin{cases} \nabla_\alpha g^{\mu\nu} = \frac{\partial H}{\partial \Pi_{\mu\nu}^\alpha}, \\ \nabla_\alpha \Pi_{\mu\nu}^\alpha(r) = -\frac{\partial H}{\partial g^{\mu\nu}}, \end{cases} \tag{2}$$

with ∇_α denoting hereon the covariant derivative in which the standard connections (Christoffel symbols) are evaluated with respect to the prescribed metric tensor $\widehat{g}(r) \equiv \left\{ \widehat{g}_{\mu\nu}(r) \right\}$ of the background space-time $\left\{ \mathbf{Q}^4, \widehat{g}(r) \right\}$ (see the related discussion given in [2]). Here, the Hamiltonian density $H \equiv H(x, \widehat{g}(r), r)$ and the related effective kinetic (T) and potential (V) densities are taken respectively of the form:

$$\begin{cases} H \equiv T + V, \\ T \equiv \frac{1}{2\kappa f(h)} \Pi_{\mu\nu}^\alpha \Pi_\alpha^{\mu\nu}, \\ V \equiv \sigma V_o\left(g, \widehat{g}\right) + \sigma V_F\left(g, \widehat{g}, r\right), \end{cases} \tag{3}$$

with κ being the dimensional constant $\kappa = \frac{c^3}{16\pi G}$ and G being the universal gravitational constant. In addition, here, h denotes the "variational" weight-factor first introduced in [6], which is also crucial for the establishment of the same canonical Equation (2) and is actually a suitably-prescribed function of the variational tensor field $g(r) \equiv \left\{ g_{\mu\nu}(r) \right\}$, namely:

$$h = \left(2 - \frac{1}{4} g^{\alpha\beta} g_{\alpha\beta} \right). \tag{4}$$

It is important to recall here that in the framework of the synchronous classical variational principle, variational and prescribed metric tensors are allowed to possess different properties and to satisfy distinctive constraints. In particular, one has that $\widehat{g}^{\alpha\beta} \widehat{g}_{\alpha\beta} = \delta_\alpha^\alpha$, while at the same time, $g^{\alpha\beta} g_{\alpha\beta} \neq \delta_\alpha^\alpha$. As a consequence, in general $h(g) \neq 1$, while it must be $h(\widehat{g}) = 1$ identically. In addition, in Equation (3), $f(h)$ and σ are multiplicative gauge functions, which according to the [8] should be respectively identified with $f(h) = 1$ and $\sigma = -1$. More precisely, according to the notations reported in [2], the two four-scalars $V_o(g, \widehat{g})$ and $V_F(g, \widehat{g}, r)$ represent respectively the vacuum and external-field contributions:

$$\begin{aligned} V_o(g, \widehat{g}) &\equiv h\kappa \left[g^{\mu\nu} \widehat{R}_{\mu\nu} - 2\Lambda \right], \\ V_F(g, \widehat{x}, r) &\equiv h L_F(g, \widehat{x}, r), \end{aligned} \tag{5}$$

with $\widehat{R}_{\mu\nu} \equiv R_{\mu\nu}(g(r))\big|_{g(r) \equiv \widehat{g}(r)}$ and Λ denoting in V_o the Ricci tensor evaluated with respect to the prescribed metric tensor $\widehat{g}(r)$ and the cosmological constant. Accordingly, $L_F(g, \widehat{x}, r)$ denotes a suitable four-scalar function depending on external sources (for classical sources, see the appropriate variational prescriptions reported in [6]).

Finally, for completeness, it is worth recalling that, as shown in [7], the extended Hamilton Equation (2) is variational, i.e., it coincides with the Euler–Lagrange equations of the synchronous classical Hamilton variational principle:

$$\delta S(x, \widehat{x}) = 0, \tag{6}$$

with $S(x, \widehat{x})$ denoting the classical Hamilton variational functional:

$$\begin{aligned} S(x, \widehat{x}) &= \int_{\mathbf{Q}^4} d\Omega L(x, \widehat{g}(r), r), \tag{7} \\ L(x, \widehat{g}(r), r) &= \Pi_{\mu\nu}^\alpha \nabla_\alpha g^{\mu\nu} - H(x, \widehat{g}(r), r). \tag{8} \end{aligned}$$

2.1. Reduced-Dimensional Hamiltonian Representation

The construction of the reduced dimensional representation for Equation (2) is realized, first, by means of the splitting representation for the canonical momentum $\Pi_{\mu\nu}^\alpha(r)$, i.e., its representation in terms of the direct product of two tensors, respectively, first- and second-order tensors, of the form:

$$\Pi_{\mu\nu}^\alpha(r) = t^\alpha(r) \pi_{\mu\nu}(r). \tag{9}$$

Here, $t^\alpha(r)$ denotes a real four-vector constructed to be a unit four-vector, i.e., such that:

$$\widehat{g}_{\alpha\nu}(r) t^\alpha(r) t^\nu(r) \equiv t^\alpha(r) t_\alpha(r) = 1, \tag{10}$$

and fulfilling identically the divergence-free condition:

$$\nabla_\alpha t^\alpha(r) = 0. \tag{11}$$

The second step consists in the introduction of a proper-time parametrization (*s*-parametrization) for the tensor fields, i.e., of the form:

$$\begin{aligned} \{g(s), \pi(s)\} &= \{g(r(s)), \pi(r(s))\}, \tag{12} \\ \widehat{g}(s) &= \widehat{g}(r(s)), \tag{13} \end{aligned}$$

in terms of suitable space-time curves $\{r(s), t(s), s \in I \equiv \mathbb{R}\}$. The same curves belong to the background space-time $\{\mathbf{Q}^4, \widehat{g}(r)\}$, with s denoting the proper time defined on the same space-time for subluminal trajectories and $t(s)$ satisfying by construction for all $s \in I$ the constraints (10) and (11). A particular realization of such curves is provided by geodetics of the metric $\widehat{g} \equiv \{\widehat{g}_{\mu\nu}(r)\}$, namely curves fulfilling the initial-value problem:

$$\begin{cases} \frac{dr^\mu(s)}{ds} = t^\mu(s), \\ \frac{Dt^\mu(s)}{Ds} = 0, \\ r^\mu(s_o) = r_o^\mu, \\ t^\mu(s_o) = t_o^\mu, \end{cases} \tag{14}$$

where $\left(r_o^\mu, t_o^\mu\right)$ represent respectively an arbitrary four-position of $\{\mathbf{Q}^4, \widehat{g}(r)\}$ and an arbitrary tangent vector fulfilling Equation (10). Since at any point of $\left(\mathbf{Q}^4, \widehat{g}\right)$, infinite geodetic curves exist, this shows that the decomposition (9), which fulfills by construction the constraint Equations (10) and (11), is always possible. Thus, in terms of the parametrization (12) and thanks to Equations (9)–(11), the extended canonical Equation (2) implies the reduced Hamilton equations of SF-GR, namely:

$$\begin{cases} \frac{d}{ds} g^{\mu\nu}(s) = \frac{\partial H_R}{\partial \pi_{\mu\nu}(s)}, \\ \frac{d}{ds} \pi_{\mu\nu}(s) = -\frac{\partial H_R}{\partial g^{\mu\nu}(s)}, \end{cases} \tag{15}$$

subject to the initial conditions:

$$\begin{cases} g^{\mu\nu}(s_o) = g_{(o)}^{\mu\nu}(r(s_o)), \\ \pi_{\mu\nu}(s_o) = \pi_{(o)\mu\nu}(r(s_o)), \end{cases} \tag{16}$$

with $\left\{g_{(o)}^{\mu\nu}(r(s_o)), \pi_{(o)\mu\nu}(r(s_o))\right\}$ denoting suitable initial tensor fields. Here, $\frac{d}{ds}$ denotes for greater generality the covariant s-derivative, i.e., the substantial derivative:

$$\frac{d}{ds} = \left.\frac{\partial}{\partial s}\right|_r + \left. t^\alpha \nabla_\alpha\right|_s, \tag{17}$$

with $\left.\frac{\partial}{\partial s}\right|_r$ denoting the partial s-derivative and $\left. t^\alpha \nabla_\alpha\right|_s$ the convective derivative along a space-time trajectory $\{r(s), t(s), s \in I\}$. In addition, the partial derivatives $\frac{\partial H_R}{\partial \pi_{\mu\nu}(s)}$ and $\frac{\partial H_R}{\partial g^{\mu\nu}(s)}$ are performed with respect to the explicit dependences only. As a consequence H_R takes generally the form:

$$\begin{cases} H = H_R \equiv T_R + V, \\ T_R(x, \widehat{g}) \equiv \frac{1}{2kf(h)} \pi_{\mu\nu} \pi^{\mu\nu}, \\ V(g, \widehat{g}, r, s) \equiv \sigma V_o\left(g, \widehat{g}, r\right) + \sigma V_F\left(g, \widehat{g}, r, s\right), \end{cases} \tag{18}$$

with σV_o and σV_F corresponding respectively to Equation (5) and where, in particular, for greater generality, σV_F is allowed to depend explicitly on s. As a consequence, in the canonical equations (15):

(a) the s-derivatives of the canonical state $x(s) = \{g(s), \pi(s)\}$ must be intended of the type:

$$\frac{d}{ds} x(s) \equiv \left(\left.\frac{\partial}{\partial s}\right|_r + \left. t^\alpha \nabla_\alpha\right|_s\right) x(s), \tag{19}$$

i.e., to include also a possible explicit s-dependence;

(b) the Hamiltonian density H_R by construction is actually taken in the form:

$$H_R = H_R(x, \widehat{g}(r(s)), r(s), s; t), \tag{20}$$

with $t \equiv \{t^\mu\}$ denoting a possible parametric dependence on the tangent four-vector t^μ appearing through $r(s)$. Such a dependence, arising because of Equation (19), is nevertheless implicit and, as such, is not expected to affect the solutions of the same reduced Hamilton equations.

Nevertheless, in case H_R does not depend explicitly on $r \equiv r(s)$, as may occur in the case of vacuum solutions and in the case $V_o = V_o(g, \widehat{g})$, then necessarily, H_R becomes independent of t^μ, as well. Notice, finally, that in order to apply Hamilton–Jacobi quantization (see below), the dimensional-normalization:

$$\begin{cases} g^{\mu\nu} \to \overline{g}^{\mu\nu} = g^{\mu\nu} \\ \pi^{\mu\nu} \to \overline{\pi}^{\mu\nu} = \frac{\alpha L}{k} \pi^{\mu\nu} \\ H_R \to \overline{H}_R \equiv \overline{T}_R + \overline{V} = \frac{\alpha L}{k} H_R \\ \overline{V}(\overline{g}, \widehat{g}, r, s; t) \equiv \sigma \overline{V}_o(\overline{g}, \widehat{g}, r) + \sigma \overline{V}_F(\overline{g}, \widehat{g}, r, s; t) \end{cases} , \tag{21}$$

can be conveniently introduced [2]. Here, $(\overline{g}^{\mu\nu}, \overline{\pi}^{\mu\nu})$ identifies the normalized canonical state, with L and α denoting a four-scalar scale-length and an invariant parameter having the dimension of an action, both to be suitably defined (see related discussion in [2,8]). As a result, the corresponding normalized Hamiltonian density and the effective potentials set in dimensionally-normalized form, denoted by over-bars, are given respectively by:

$$\begin{cases} \overline{H}_R \equiv \overline{T}_R + \overline{V} \\ \overline{T}_R(x, \widehat{g}) \equiv \frac{1}{2\alpha L f(h)} \overline{\pi}_{\mu\nu} \overline{\pi}^{\mu\nu} \\ \overline{V}(g, \widehat{g}, r, s; t) \equiv \sigma \overline{V}_o(g, \widehat{g}, r) + \sigma \overline{V}_F(g, \widehat{g}, r, s; t) \\ \overline{V}_o(\overline{g}, \widehat{g}, r) \equiv h \alpha L f(h) \left[g^{\mu\nu} \widehat{R}_{\mu\nu} - 2\Lambda \right] \\ \overline{V}_F(g, \widehat{g}, r, s; t) \equiv \frac{\alpha L}{k} V_F(g, \widehat{g}, r, s; t) . \end{cases} \tag{22}$$

In the remainder, the constitutive Equation (22) will be adopted throughout the paper, dropping for simplicity the over-bar notation.

2.2. Hamilton–Jacobi Equation

As shown in [8], the reduced Hamilton Equation (15) of SF-GR is equivalent to the continuum Hamilton–Jacobi equation for the Hamilton principal function $\mathcal{S} \equiv \mathcal{S}(g(s), \widehat{g}(s), r(s), s; t)$ constructed in such a way that the canonical momentum takes the form:

$$\pi^{\mu\nu} = \frac{\partial \mathcal{S}(g(s), \widehat{g}(s), r(s), s; t)}{\partial g_{\mu\nu}(s)}. \tag{23}$$

This requires more precisely denoting $H_R \equiv H_R\left(g(s), \frac{\partial \mathcal{S}(g(s), \widehat{g}(s), r(s), s; [t])}{\partial g}, \widehat{g}(s), r(s), s; [t]\right)$, with the consequence that the Hamilton principal function $\mathcal{S}$ must be a solution of the classical Hamilton–Jacobi equation:

$$\frac{\partial \mathcal{S}}{\partial s} + H_R = 0, \tag{24}$$

where $\frac{\partial}{\partial s} \equiv \frac{d}{ds}$ identifies the substantial, i.e., total covariant s-derivative defined by Equation (17), and the same s-derivative is performed keeping constant both $g(s)$ and $\widehat{g}(s)$. In the particular case in which $r(s)$ and t are ignorable for the Hamiltonian density H_R (i.e., as for vacuum solutions), then also the Hamilton principal function becomes of the form $\mathcal{S} = \mathcal{S}(g(s), \widehat{g}(s), s)$.

3. Hamilton–Jacobi g-Quantization

The key starting point for the quantization of the Hamiltonian system $\{x, H_R\}$ developed by CQG-theory [9] concerns the notion of the quantum state itself. This is to be intended as the quantum state of a particle with spin-two, which in the context of CQG is identified with a massive graviton, i.e., having rest-mass $m_o > 0$. In particular, this refers to the assumption that such a state can be

represented by a single complex four-scalar quantum wave-function ψ. That such a prescription is indeed possible in the context of a first-quantization approach adopted by CQG-theory [9], having a classical background space-time $\widehat{g}_{\mu\nu}$, which is distinguished from the variational/quantum field $g_{\mu\nu}$, follows from the fact that in such a case, ψ can always be identified with the tensor product of the form $\psi(s) = \widehat{g}_{\mu\nu}\psi^{\mu\nu}(s)$. This establishes the relationship between the four-scalar wave function $\psi(s)$ and the corresponding second-order four-tensor $\psi^{\mu\nu}(s)$. In particular, one readily obtains that necessarily $\psi^{\mu\nu}(s) = \frac{1}{4}\psi(s)\widehat{g}^{\mu\nu}$, so that in the present description, the four-tensor $\psi^{\mu\nu}(s)$ has to be regarded as a derived quantity in terms of the wave-function $\psi(s)$, which is solved by the CQG-wave equation. Regarding the functional setting, i.e., the prescription of the functional class of admissible wave-functions $\{\psi\}$, it is assumed that ψ are smoothly differentiable complex functions, which are parametrized in terms of the space-time curves $\{r(s), t(s), s \in I \equiv \mathbb{R}\}$ indicated above and fulfilling suitable boundary conditions for $s \to \pm\infty$ and on the improper hyper-surface of the configuration space U_g (see below). In particular, $\psi(s)$ is assumed to be a smooth function of the form $\psi(s) = \psi(g, \widehat{g}(s), r, s; t)$, with $g = \{g_{\mu\nu}\}$, s and $t = \{t^{\mu}\}$ being independent variables that span respectively the configuration space U_g, the time axis I and the tangent space TU_g, while:

$$\rho(s) \equiv |\psi(s)|^2 \tag{25}$$

identifies the corresponding quantum probability density function (quantum PDF) on the configuration space. Here, we shall assume for definiteness that the variational tensors $g_{\mu\nu}$ spanning U_g are symmetric, so that U_g is identified with the subset of a linear space $\mathbb{R}^{10}$. In addition, one can always require $g_{\mu\nu}$ to be also non-singular. This means, in other words, that for any unit four-vector t^{μ} fulfilling Equation (10), it must be that:

$$\Theta\left(|g_{\mu\nu}t^{\mu}t^{\nu}|^2\right) = 1, \tag{26}$$

with $\Theta(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x \leq 0) \end{cases}$ denoting the strong Heaviside step function. In the context of g-quantization, the configuration space is identified with U_g, so that by construction, its quantum probability is defined as:

$$P(U_g) = \langle\psi|\psi\rangle \equiv \int\limits_{U_g} d(g)\rho(s) = 1, \tag{27}$$

with $d(g) \equiv \prod\limits_{\mu,\nu=1,4} d\widehat{g}_{\mu\nu}$ denoting the canonical measure on U_g. In addition, both r and $\widehat{g}(r)$ are considered functions of s, i.e., evaluated along the classical space-time curves $\{r(s), t(s), s \in I \equiv \mathbb{R}\}$, so that $r \equiv r(s) = \{r^{\mu}(s)\}$ and $\widehat{g}(s) \equiv \widehat{g}(r(s))$. Finally, the wave-functions $\psi(s)$ span by assumption a Hilbert space Γ_{ψ}, which is finite-dimensional in the sense that it is defined on a continuum configuration space U_g having a finite dimension. This is endowed with the scalar product:

$$\langle\psi_a|\psi_b\rangle \equiv \int\limits_{U_g} d(g)\psi_a^*(g, \widehat{g}(r), r(s), s)\psi_b(g, \widehat{g}(r), r(s), s), \tag{28}$$

with $\psi_{a,b}(s) \equiv \psi_{a,b}(g, \widehat{g}(r), r(s), s)$ being arbitrary elements of the Hilbert space Γ_{ψ}, where as usual, ψ_a^* denotes the complex conjugate of ψ_a.

Based on these premises, the formal construction of CQG-theory is then based on the adoption of two distinctive axioms related respectively to the following two prescriptions:

- First, the canonical quantization rule, herein referred to as the g-quantization rule, prescribing the mapping between the classical and quantum Hamiltonian structures:

$$\left(x = \{g, \pi\}, H_R\right) \Rightarrow \left(x^{(q)} = \left\{g^{(q)}, \pi^{(q)}\right\}, H_R^{(q)}\right). \tag{29}$$

This mapping is realized by the CQG-correspondence principle, namely:

$$
\begin{cases}
g_{\mu\nu} \to g^{(q)}_{\mu\nu} \equiv g_{\mu\nu} \\
\pi_{\mu\nu} \to \pi^{(q)}_{\mu\nu} \equiv -i\hbar \frac{\partial}{\partial g^{\mu\nu}} \\
H_R \to H^{(q)}_R = \frac{1}{f(h)} T^{(q)}_R(\pi, \widehat{g}) + V
\end{cases} .
\tag{30}
$$

- Second, the quantum-wave equation advancing in proper-time the same quantum state. This is provided by the CQG-wave equation:

$$
i\hbar \frac{\partial}{\partial s} \psi(s) = \left[H^{(q)}_R, \psi(s) \right] \equiv H^{(q)}_R \psi(s),
\tag{31}
$$

with $\frac{\partial}{\partial s}$ denoting again the total covariant $s-$derivative defined by Equation (17) and $[A, B]$ being the quantum commutator $[A, B] \equiv AB - BA$.

The goal of this section is to set the CQG-theory in the context of a Hamilton–Jacobi quantization scheme (see, for example, [30]), which permits us to determine immediately the precise form of the resulting quantum wave equation displayed in Equation (31). The starting point is the prescription of the relevant classical canonical momenta in the context of the Hamilton–Jacobi approach. By direct inspection of the Hamilton–Jacobi equation recalled above in Equation (24), it follows that g-quantization can be achieved in terms of the two classical canonical momenta $\frac{\partial S(g,\widehat{g},(r),s;t)}{\partial g_{\mu\nu}}$ and $\frac{\partial S(g,\widehat{g},(r),s;t)}{\partial s}$, identifying, respectively, $\pi_{\mu\nu}$ and the canonical momentum conjugate to the proper-time s. In the context of the Hamilton–Jacobi quantization, the appropriate correspondence principle for g-quantization, establishing the relationship between the classical and quantum momenta and Hamiltonian functions, is then provided by the mapping:

$$
\pi_{\mu\nu} \equiv \frac{\partial S\left(g, \widehat{g}, (s), r, s; [t]\right)}{\partial g^{\mu\nu}} \to \pi^{(q)}_{\mu\nu} \equiv -i\hbar \frac{\partial}{\partial g^{\mu\nu}},
\tag{32}
$$

$$
p \equiv -\frac{\partial S\left(g, \widehat{g}(s), r, s; [t]\right)}{\partial s} \to p^{(q)} \equiv -i\hbar \frac{\partial}{\partial s},
\tag{33}
$$

$$
H_R\left(g, \frac{\partial S\left(g, \widehat{g}(s), r, s; [t]\right)}{\partial g}, \widehat{g}(s), r, s; [t] \right) \to H^{(q)}_R.
\tag{34}
$$

with $\pi^{(q)}_{\mu\nu}$, $p^{(q)}$ and $H^{(q)}_R$ identifying the quantum canonical momenta conjugate to $g_{\mu\nu}$ and s, respectively, and the quantum Hamiltonian operator. This will be referred to as *Hamilton–Jacobi g-quantization*. Notice that, in the same mapping, the "coordinates" $g \equiv \{g_{\mu\nu}\}$ and s remain unchanged, i.e., so that they still formally coincide with the classical ones. The mapping realized by Equations (32)–(34) implies the simultaneous validity of the two fundamental commutator relations:

$$
\left[\pi^{(q)\alpha\beta}, g_{\mu\nu} \right] = -i\hbar \delta^\alpha_\mu \delta^\beta_\nu,
\tag{35}
$$

$$
\left[p^{(q)}, s \right] = -i\hbar,
\tag{36}
$$

together with:

$$
\left[g^{\alpha\beta}, g_{\mu\nu} \right] = \left[\pi^{(q)\alpha\beta}, \pi^{(q)}_{\mu\nu} \right] = 0.
\tag{37}
$$

Notice that since both $\pi^{(q)\alpha\beta}$ and $g^{(q)}_{\mu\nu}$ are symmetric, Equation (35) holds for arbitrary permutations of the indexes. As a consequence, based on the classical Hamilton–Jacobi Equation (24), the additional mapping:

$$
\boxed{\frac{\partial S}{\partial s} + H_R = 0} \Rightarrow \boxed{\left\{ p^{(q)} + H^{(q)}_R \right\} \psi(s) = 0}
\tag{38}
$$

follows, which implies the validity of the quantum-wave equation indicated above (see Equation (31)). We stress that the same CQG-wave equation exhibits a number of distinctive properties:

1. It is manifestly covariant, i.e., it retains its form under the action of arbitrary local point transformations (1), which preserve by construction the differential manifold of the space-time $\left(\mathbf{Q}^4, \widehat{g}(r)\right)$.
2. It is an evolution equation, which is parametrized in terms of the proper-time s, i.e., the Riemann distance, which is associated with the background space-time for subluminal trajectories.
3. It advances in proper-time the four-scalar wave-function $\psi(s)$, the associated configuration-space quantum PDF being prescribed by Equation (25).
4. The same wave equation holds in principle for arbitrary initial conditions, as well as for arbitrary external sources, as is appropriate for the treatment of problems of QG and quantum cosmology.

In particular, an interesting comparison can be made with the Wheeler–DeWitt wave equation [31,32]. Indeed, it is well-known that the Wheeler–DeWitt wave equation realizes a Schrödinger-like evolution equation advancing in time the dynamics of the so-called "wave function of the universe" [33]. Despite its formal analogy, however, basic differences emerge, the most important one being that the Wheeler–DeWitt wave equation is not manifestly covariant. The reason, as earlier discussed (see [9]), is that its time evolution is parametrized with respect to the coordinate-time t, which is not an invariant four-scalar.

4. The Quantum Hamilton Variational Principle

The goal of this section is to prove that the CQG-wave Equation (31) admits also a variational formulation in terms of a suitably-prescribed synchronous variational principle. In this regard, a basic prerequisite is that the variational principle yielding this equation should be manifestly covariant. In other words, the corresponding variational functional, the appropriate class of variations, as well as the resulting Euler–Lagrange equations should all be covariant with respect to the LPT-group. As we intend to show here, such a requirement is non-trivial and restricts the class of admissible quantum-wave equations for quantum gravity.

A key feature of the CQG-wave equation is that it is a hyperbolic evolution equation, which prescribes the dependence of the quantum state $\psi = \psi(s)$ in terms of the proper-time s, which is associated with the background curved space-time $\left(\mathbf{Q}^4, \widehat{g}_{\mu\nu}(r)\right)$ and parametrizes the space-time curves $\{r(s), t(s), s \in I \equiv \mathbb{R}\}$. Such an equation is a first-order partial differential equation with respect to s, to be supplemented by suitable initial conditions, namely prescribing for all $r(s_o) = r_o \in \left(\mathbf{Q}^4, \widehat{g}(r)\right)$ the condition $\psi(s_o) = \psi_o(g, \widehat{g}(r_o), r_o; t)$, as well as boundary conditions at infinity on the improper boundary of configuration space U_g, i.e., letting $\lim_{g \to \infty} \psi(g, \widehat{g}(s), r(s), s; t) = 0$.

Let us now show that the form of Equation (31) indicated above warrants that it is also variational, i.e., that in analogy with quantum-wave equations known in quantum mechanics, it admits a manifestly-covariant variational formulation. For the construction of the variational principle, one can indeed adopt a method analogous to that followed in [30] in the case of relativistic quantum mechanics. The prerequisites are set by the following prescriptions:

Prescription 1. *A scalar product is defined on a suitable extended configuration space, identified with the direct product* $U_g \times I_{(s_o, s_1)}$*,* U_g *being the configuration space defined above and* $I_{(s_o, s_1)}$ *the subset of the real axis* $I \equiv \mathbb{R}$*, namely the interval* $I_{(s_o, s_1)} = [s_o, s_1]$*.*

Prescription 2. *A real functional* $Q(\psi, \psi^*)$ *of the variational wave function* $\psi \equiv \psi(s)$ *is defined, which is symmetric with respect to the same scalar product on the set* U_g*. The same function* ψ *is required to belong to the functional class* $\{\psi(s)\}$ *of the* $C^{(2)}$ *(continuous twice-differentiable) complex functions prescribed so that the associated quantum PDF* $\rho(s)$ *(see Equation (25)) fulfills on* U_g *the normalization condition (27).*

Prescription 3. *The same real functional and the corresponding Lagrangian density* (L_Q) *are all four-scalars with respect to generic background space-time* $\left(\mathbf{Q}^4, \widehat{g}(r)\right)$*.*

Prescription 4. *The Lagrangian density can be represented in terms of the first-order differential operators associated with the quantum canonical momenta, respectively $\pi_{\mu\nu}^{(q)}$ and $p^{(q)}$ (see Equations (32) and (33)) or their complex conjugates $\pi_{\mu\nu}^{(q)*}$ and $p^{(q)*}$, acting respectively on ψ and its complex conjugate wave-function ψ^*.*

In the present framework, the variational functional corresponding to these prescriptions is provided by a real symmetric four-scalar Hamilton functional of the form:

$$Q\left(\psi,\psi^*\right) = \int_{s_o}^{s_1} ds \int_{U_g} d(g) L_Q\left(\mathbf{w},\mathbf{w}^*\right), \tag{39}$$

where $s_o, s_1 \in I$ are arbitrary proper times, such that $s_o < s_1$ (so that in particular, a possible choice is realized by setting $s_o = -\infty$, $s_1 = +\infty$), while $d(g)$ represents the four-scalar volume element of configuration space U_g. Furthermore, denoting $\psi^* \equiv \psi^*(s)$ the complex conjugate of $\psi \equiv \psi(s)$ and $\mathbf{w} \equiv \left(\psi, \frac{\partial \psi}{\partial g_{\mu\nu}}, \frac{\partial \psi}{\partial s}\right)$, then $\mathbf{w}^*$ is its complex conjugate. In particular, for consistency with Prescriptions 2–4, $L_Q\left(\mathbf{w},\mathbf{w}^*\right)$ must be identified with the four-scalar real Lagrangian density:

$$L_Q\left(\mathbf{w},\mathbf{w}^*\right) = i\hbar \frac{1}{2} \left[\psi \frac{\partial}{\partial s}\psi^* - \psi^* \frac{\partial}{\partial s}\psi\right] + \frac{1}{2\alpha L}\left(i\hbar \frac{\partial}{\partial g_{\mu\nu}}\right)\psi^*\left(-i\hbar \frac{\partial}{\partial g^{\mu\nu}}\right)\psi + V\psi^*\psi. \tag{40}$$

In addition, consistent with Prescription 1, the integral occurring in the Hamilton functional $Q\left(\psi,\psi^*\right)$ is performed respectively: (a) with respect to s, with integration carried out both with respect to the explicit and implicit dependences contained in $\psi(g, \widehat{g}(s), r(s), s; t)$; and (b) with respect to the explicit dependence in terms of $g \equiv \{g_{\mu\nu}\}$. This means that $Q\left(\psi,\psi^*\right)$ can be equivalently represented as:

$$Q\left(\psi,\psi^*\right) = \int_{s_o}^{s_1} ds \left\langle \psi\middle| K_Q \psi\right\rangle, \tag{41}$$

with $\left\langle \psi\middle| K_Q \psi\right\rangle$ representing the quantum expectation value (see [8]) and K_Q being here the quantum operator:

$$K_Q = K_Q^{(1)} + \frac{1}{2\alpha L}\left(i\hbar \frac{\overleftarrow{\partial}}{\partial g^{\mu\nu}}\right)\left(-i\hbar \frac{\overrightarrow{\partial}}{\partial g^{\mu\nu}}\right) + V, \tag{42}$$

$$K_Q^{(1)} = i\hbar \frac{1}{2}\left[\frac{\overleftarrow{\partial}}{\partial s} - \frac{\overrightarrow{\partial}}{\partial s}\right], \tag{43}$$

and where $\left(\frac{\overleftarrow{\partial}}{\partial s}, \frac{\overleftarrow{\partial}}{\partial g_{\mu\nu}}\right)$ and $\left(\frac{\overrightarrow{\partial}}{\partial s}, \frac{\overrightarrow{\partial}}{\partial g^{\mu\nu}}\right)$ identify the "bra" and "ket" operators acting on $\langle \psi| \equiv \psi^*$ and $\psi \equiv |\psi\rangle$, respectively. Hence, $Q\left(\psi,\psi^*\right)$ can be equivalently represented in terms of the quantum Hamiltonian operator $H_R^{(q)}$ and the scalar product defined for the waves functions belonging to $\{\psi\}$; namely, it takes the form:

$$Q\left(\psi,\psi^*\right) = \int_{s_o}^{s_1} ds \left\langle \psi\middle| \left[-i\hbar \frac{\partial}{\partial s} + H_R^{(q)}\right]\psi\right\rangle, \tag{44}$$

with these operators $-i\hbar \frac{\partial}{\partial s}$ and $H_R^{(q)}$ to be intended as ket operators acting on $\psi \equiv |\psi\rangle$.

As a basic consequence, the same operators, and hence also K_Q, are all symmetric, while all of the involved quantum operators have a four-tensor nature. This means that the Hamilton functional (39) (or equivalently, Equation (44)) is such that: (1) $Q\left(\psi,\psi^*\right)$ is a real four-scalar functional, with all of the quantum operators and variables being represented by four-tensor quantities; (2) the quantum operators $i\hbar \frac{\partial}{\partial s}$ and $H_R^{(q)}$, and hence, also K_Q, are all symmetric with respect to the same scalar product.

Thus, introducing the functional class of variations $\{\psi_1 \equiv \psi + \delta\psi\}$, it is assumed that all functions ψ_1 belong to the class of admissible wave-functions $\{\psi\}$ indicated above, while the variation $\delta\psi \equiv \delta\psi(g, \widehat{g}(s), r(s), s; t)$ is considered as a four-scalar complex function. In other words, the same quantum wave Equation (31) must be uniquely determined by means of the quantum Hamilton variational principle:

$$\delta Q\left(\psi, \psi^*\right) = 0, \tag{45}$$

to hold in a suitable class of variations in which the variations $\delta\psi$ (and similarly $\delta\psi^*$) are considered as independent four-scalar functions, with $\delta Q\left(\psi, \psi^*\right)$ denoting the Frechet derivative evaluated with respect to the same variations, namely letting:

$$\delta Q\left(\psi, \psi^*\right) = \lim_{\alpha \to 0} \frac{Q\left(\psi + \alpha\delta\psi, \psi^* + \alpha\delta\psi^*\right) - Q\left(\psi, \psi^*\right)}{\alpha}. \tag{46}$$

The same equation is required to realize a so-called synchronous variational principle, i.e., performed again in terms of a synchronous variation operator δ [6], which in this case leaves unchanged the line element ds. This implies that the equation:

$$\delta\left(ds\right) = 0 \tag{47}$$

must hold identically. Such a differential constraint can be fulfilled by suitable prescription of the variations $\delta\psi$ and $\delta\psi^*$ and in particular requiring that the parameter α in the Frechet derivative (46) is independent of s. Moreover, we shall assume that the same variations ($\delta\psi$ and $\delta\psi^*$) are considered independent and vanish on the boundary of Q^4 and at infinity for $s \to \pm\infty$. Then, by elementary algebra, it follows that the wave function ψ, which is extremal for the functional $Q\left(\psi, \psi^*\right)$, must fulfill the Euler–Lagrange equations:

$$\frac{\delta Q\left(\psi, \psi^*\right)}{\delta\psi^*} = 0, \tag{48}$$

$$\frac{\delta Q\left(\psi, \psi^*\right)}{\delta\psi} = 0. \tag{49}$$

This proves at the same time the variational and manifest-covariant properties of the quantum-wave equation (31). In fact, thanks to the symmetry property of the Hamilton functional $Q\left(\psi, \psi^*\right)$, Equations (48) and (49) coincide respectively with Equation (31) and its complex conjugate, so that the variational character of Equation (31) is established. Furthermore, thanks to the four-tensor property of the same functional $Q\left(\psi, \psi^*\right)$ and of the quantum variables and operators, the same Euler–Lagrange Equations (48) and (49) are all manifestly covariant.

These properties are distinctive of CQG-theory and depart from previous QG literature. For example, let us consider again the comparison with the Wheeler–DeWitt wave equation [31,32]. The latter equation is indeed variational. Nevertheless, despite that fact that the same variational principle is in some sense analogous to (45), it is also not manifestly-covariant. This arises, again, because the Wheeler–DeWitt wave equation itself is based on the so-called $3 + 1$ foliation of space-time, so that the variational principle occurring for the same equation is not manifestly covariant. This problem however does not arise in the framework of the CQG-theory considered here.

5. The Variational Quantum Hydrodynamic Equations

A pre-requisite for establishing Heisenberg inequalities in the framework of CQG-theory is the introduction of the concept of a quantum probability density function and the derivation of the corresponding set of quantum hydrodynamic equations (CQG-QHE) implied by the CQG-wave equation, to be prescribed in conservative form in order to warrant the conservation of quantum probability.

In the present setting, the quantum probability density function (PDF) associated with the CQG-state is identified with the real function $\rho(s) \equiv \rho(g, \widehat{g}(r), r(s), s)$ and is prescribed as:

$$\rho(s) \equiv |\psi(s)|^2, \tag{50}$$

in formal analogy with the customary definition of quantum PDF holding in non-relativistic quantum mechanics. Here, $\rho(s)$ is a four-scalar and represents the probability density of the Lagrangian field variable $g \equiv \{g_{\mu\nu}\}$ in the volume element $d(g)$ belonging to the configuration space U_g. By assumption, the probability $P(A)$ of an arbitrary subset $A \subseteq U_g$ is normalized, in the sense that for arbitrary $(\widehat{g}(r), r(s), s)$, it must be:

$$P(A) \equiv \langle \psi | \delta_A(g) \psi \rangle \equiv \int_{U_g} d(g) \rho(s) \delta_A(g), \tag{51}$$

with $\delta_A(g)$ being the characteristic function of the set A. In a similar way, given the validity of Equation (50), one can define the real function $S^{(q)}(s) \equiv S^{(q)}(g, \widehat{g}(r), r(s), s)$ as:

$$S^{(q)}(s) \equiv \hbar \operatorname{arcsin h} \left\{ \frac{\psi(s) - \psi^*(s)}{2\sqrt{\rho(s)}} \right\}, \tag{52}$$

which represents, on the configuration space U_g, the quantum phase-function associated with the same CQG-state $\psi(s)$. Hence, in terms of the real four-scalar field functions $\rho(s)$ and $S^{(q)}(s)$ prescribed respectively by Equations (50) and (52), the CQG-state defined by the complex function $\psi(s)$ can be cast in the equivalent form of the Madelung exponential representation:

$$\psi(s) = \sqrt{\rho(s)} \exp\left\{ \frac{i}{\hbar} S^{(q)}(s) \right\}, \tag{53}$$

i.e., the quantum fluid fields $\left\{ \rho(s), S^{(q)}(s) \right\}$ identifying respectively the quantum PDF and the quantum phase-function. Once the Madelung representation is invoked, it is possible to replace the single CQG-wave Equation (31) for the complex wave-function $\psi(s)$ with the equivalent set of CQG-QHE realized respectively by a continuity equation for the real quantum PDF $\rho(s)$ and a quantum Hamilton–Jacobi equation for the real quantum phase-function $S^{(q)}(s)$.

In particular, invoking the Madelung representation for $\psi(s)$, given the definition of the Hamiltonian operator $H_R^{(q)}$ according to Equation (30) and imposing the constraint condition $f(h) = 1$, the following set of real PDEs is obtained from the CQG-wave equation:

$$\frac{\partial \rho(s)}{\partial s} + \frac{\partial}{\partial g_{\mu\nu}} \left(\rho(s) V_{\mu\nu}(s) \right) = 0, \tag{54}$$

$$\frac{\partial S^{(q)}(s)}{\partial s} + H_c^{(q)} = 0. \tag{55}$$

These are referred to as the CQG-quantum continuity equation and the CQG-quantum Hamilton–Jacobi equation advancing in proper-time, respectively $\rho(s)$ and $S^{(q)}(s)$. Here, the quantum hydrodynamics fields $\rho(s) \equiv \rho(g, \widehat{g}, s)$ and $S^{(q)}(s) \equiv S^{(q)}(g, \widehat{g}, s)$ are assumed to depend smoothly on the tensor field $g \equiv \{g_{\mu\nu}\}$ spanning the configuration space U_g and in addition to admitting a Lagrangian path-parametrization in terms of the geodetics $r(s) \equiv \{r^\mu(s)\}$ associated locally with the prescribed field $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$. The four-tensor $V_{\mu\nu}(s)$ is prescribed instead as:

$$V_{\mu\nu}(s) = \frac{1}{\alpha L} \frac{\partial S^{(q)}}{\partial g^{\mu\nu}}. \tag{56}$$

Finally, $H_c^{(q)}$ identifies the effective quantum Hamiltonian density:

$$H_c^{(q)} = \frac{1}{2\alpha L} \frac{\partial S^{(q)}}{\partial g^{\mu\nu}} \frac{\partial S^{(q)}}{\partial g_{\mu\nu}} + V_{QM} + V, \tag{57}$$

with $V \equiv V(g, \widehat{g}(r), r, s)$ being the effective potential density and V_{QM} a potential density denoted as Bohm-like effective quantum potential, which is prescribed as:

$$V_{QM}(g, \widehat{g}(r), r, s) = \frac{\hbar^2}{8\alpha L} \frac{\partial \ln \rho}{\partial g^{\mu\nu}} \frac{\partial \ln \rho}{\partial g_{\mu\nu}} - \frac{\hbar^2}{4\alpha L} \frac{1}{\rho} \frac{\partial^2 \rho}{\partial g_{\mu\nu} \partial g^{\mu\nu}}. \tag{58}$$

The effective potential V_{QM} is analogous to the well-known Bohm potential met in non-relativistic quantum mechanics (see, for example, [34,35]), its physical origin arising due to the non-uniformity of the quantum PDF ρ, namely such that generally it must be $\frac{\partial}{\partial g_{\mu\nu}} \rho(s) \neq 0$. Finally, it must be stressed that, as discussed in [9], the constraint condition $f(h) = 1$ is required in order to warrant the quantum unitarity principle, namely the conservation of quantum probability, thus resolving at the quantum level the indeterminacy on the prescription of the function $f(h)$ arising in the classical Hamilton–Jacobi theory of GR (see also [8]).

A theoretical implication of the validity of the variational principle (45) can be obtained if in the functional (44), the wave functions ψ and ψ^* are represented in terms of the Madelung representation (53). In fact, in terms of the same fields, the Hamilton functional (44) then becomes:

$$Q(\psi, \psi^*) = \int_{s_o}^{s_1} ds \int_{U_g} d(g) \left\{ \frac{i\hbar}{2} \left[\frac{\partial \rho(s)}{\partial s} + \frac{\partial}{\partial g_{\mu\nu}} \left(\rho(s) V_{\mu\nu}(s) \right) \right] + \rho(s) \left[\frac{\partial S^{(q)}(s)}{\partial s} + H_c^{(q)} \right] \right\}, \tag{59}$$

with $V_{\mu\nu}(s)$ and $H_c^{(q)}$ being respectively the second-order four-tensor and the effective quantum Hamiltonian defined by Equations (56) and (57). For definiteness, let us require, consistent with Prescription 2, that the normalization (27) applies. As a consequence, the functional class $\{\psi(s)\}$ must be prescribed so that the boundary condition $\rho(s) \equiv 0$ on the improper hypersurface of U_g is fulfilled. As a consequence, in Equation (59), it follows identically that:

$$\int_{s_o}^{s_1} ds \int_{U_g} d(g) \frac{\partial \rho(s)}{\partial s} \equiv 0, \tag{60}$$

and similarly:

$$\int_{s_o}^{s_1} ds \int_{U_g} d(g) \frac{\partial}{\partial g_{\mu\nu}} \left(\rho(s) V_{\mu\nu}(s) \right) \equiv 0, \tag{61}$$

the two equations, thus yielding that:

$$Q(\psi, \psi^*) = \int_{-\infty}^{+\infty} ds \int_{U_g} d(g) \rho(s) \left[\frac{\partial S^{(q)}(s)}{\partial s} + H_c^{(q)} \right]. \tag{62}$$

Then, one can show that the Euler–Lagrange equations implied by the quantum Hamilton variational principle (45), written for the two real fields $\rho(s)$ and $S^{(q)}(s)$, namely:

$$\frac{\delta Q(\psi, \psi^*)}{\delta S^{(q)}} = 0, \tag{63}$$

$$\frac{\delta Q(\psi, \psi^*)}{\delta \rho} = 0, \tag{64}$$

necessarily recover the quantum hydrodynamic equations, i.e., respectively Equations (54) and (55). In fact, the first Equation (63) manifestly requires:

$$-\frac{\partial}{\partial s}\rho(s) - \frac{\partial}{\partial g_{\mu\nu}}\left(\rho(s)\frac{1}{\alpha L}\frac{\partial S^{(q)}}{\partial g^{\mu\nu}}\right) = 0, \tag{65}$$

which coincides with the quantum continuity Equation (54). Second, noting that the variation of the Bohm-like potential vanishes identically, the second Equation (64) recovers exactly Equation (55). Therefore, also in terms of the fluid fields $\left\{\rho(s), S^{(q)}(s)\right\}$ the functional $Q\left(\psi, \psi^*\right)$ is variational.

Concerning the CQG-quantum Hamilton–Jacobi equation determined here, one notices that the same equation generalizes the classical GR-Hamilton–Jacobi equation determined in [8] in the framework of the manifestly-covariant Hamiltonian theory of GR. As a consequence, Equation (55) must imply the validity of corresponding Hamilton equations to be expressed in terms of the effective quantum Hamiltonian density $H_c^{(q)}$. Nevertheless, due to the presence of the Bohm-like effective quantum potential $V_{QM}(g, \widehat{g}(r), r, s)$, the latter now generally must depend explicitly on the proper time s. This contribution is expected to give rise to proper-time-dependent solutions of the non-stationary CQG-wave equation, while the Bohm-like quantum potential vanishes in the semiclassical limit prescribed letting $\hbar \rightarrow 0$. This theoretical feature establishes a logical consistency of the CQG-theory with the classical Hamilton–Jacobi equation recalled above (see Equation (24)) and earlier determined in [9], while at the same time, the connection between CQG-QHE and the corresponding CQG-wave equation marks a strong analogy of the present quantum theory with the Schrödinger equation and the generalized Klein–Gordon equation reported in [30] holding for relativistic quantum mechanics. Finally, an important issue must be mentioned, related to the fluid description underlying the CQG-wave equation. This concerns a generalization of the background space-time picture adopted in this work, toward realization of second-quantization theory, i.e., in which quantum sources are taken into account. Specifically, this includes quantization and quantum modifications of the background space-time, as well as the inclusion of specific possible quantum particle sources (such as the Hawking radiation, graviton sources, etc.). The fluid description obtained here can be adopted for investigating such a route, providing the basis for a statistical description of quantum gravity theory characterized by second-quantization effects and the independence of the geometrical background.

6. Generalized Heisenberg Inequalities for *g*-Quantization

In this section, we present proofs of Heisenberg inequalities from first principles holding for the CQG-wave equation in the framework of *g*-quantization. More precisely, the problem is addressed whether, as a consequence of the strict positivity and smoothness of the quantum PDF $\rho(s)$ and in analogy with standard quantum mechanics, the quantum state $\psi(s)$ might/should satisfy suitable Heisenberg inequalities, which are related to the fluctuations (and corresponding standard deviations) of the Lagrangian variable $g_{\mu\nu}$ and of the conjugate quantum canonical momentum $\pi_{\mu\nu}^{(q)}$ and if the same inequalities might place a constraint on the proper-time evolution of the quantum state.

For this purpose, we determine preliminarily the expectation values and corresponding fluctuations (i.e., the squared of the standard deviations), which are associated with the generalized Lagrangian coordinates $g_{\mu\nu}$. These are respectively prescribed in terms of the expectation values:

$$\widetilde{g}_{\mu\nu} \equiv \left\langle g_{\mu\nu} \right\rangle \equiv \left\langle \psi | g_{\mu\nu} \psi \right\rangle, \tag{66}$$

and:

$$\left\langle \left(\Delta g_{\mu\nu}\right)^2 \right\rangle \equiv \left\langle \psi \left| \left(g_{\mu\nu} - \widetilde{g}_{\mu\nu}\right)\left(g_{(\mu)(\nu)} - \widetilde{g}_{(\mu)(\nu)}\right) \psi \right\rangle. \tag{67}$$

The corresponding weighted configuration-space integrals are then given by the following expressions:

$$\widetilde{g}_{\mu\nu} = \int_{U_g} d(g)\rho g_{\mu\nu}, \tag{68}$$

$$\left\langle \left(\Delta g_{\mu\nu}\right)^2 \right\rangle = \int_{U_g} d(g)\rho \left(g_{\mu\nu} - \widetilde{g}_{\mu\nu}\right) \left(g_{(\mu)(\nu)} - \widetilde{g}_{(\mu)(\nu)}\right). \tag{69}$$

Similar calculations can be performed for the conjugate quantum momenta, namely for the expectation values:

$$\widetilde{\pi}_{\mu\nu} \equiv \left\langle \pi_{\mu\nu}^{(q)} \right\rangle \equiv \left\langle \psi | \pi_{\mu\nu}^{(q)} \psi \right\rangle, \tag{70}$$

and:

$$\left\langle \left(\Delta \pi_{\mu\nu}^{(q)}\right)^2 \right\rangle \equiv \left\langle \psi \left| \left(\pi_{\mu\nu}^{(q)} - \widetilde{\pi}_{\mu\nu}\right) \left(\pi_{(\mu)(\nu)}^{(q)} - \widetilde{\pi}_{(\mu)(\nu)}\right) \psi \right\rangle. \tag{71}$$

Straightforward calculations yield in this case:

$$\begin{aligned}
\widetilde{\pi}_{\mu\nu} &= \int_{U_g} d(g)\psi^* \left(-i\hbar \frac{\partial}{\partial g^{\mu\nu}}\right)\psi \\
&= \int_{U_g} d(g)\rho \left(\frac{\partial S^{(q)}}{\partial g^{\mu\nu}} - \frac{i\hbar}{2}\frac{\partial \ln\rho}{\partial g^{\mu\nu}}\right) = \int_{U_g} d(g)\rho \frac{\partial S^{(q)}}{\partial g^{\mu\nu}},
\end{aligned} \tag{72}$$

and respectively:

$$\begin{aligned}
\left\langle \left(\Delta \pi_{\mu\nu}^{(q)}\right)^2 \right\rangle &= \int_{U_g} d(g)\psi^* \left(-i\hbar \frac{\partial}{\partial g^{\mu\nu}} - \widetilde{\pi}_{\mu\nu}\right)\left(-i\hbar \frac{\partial}{\partial g^{(\mu)(\nu)}} - \widetilde{\pi}_{(\mu)(\nu)}\right)\psi \\
&= \int_{U_g} d(g)\rho \left(-i\hbar \frac{\partial \ln\rho}{\partial g^{\mu\nu}} + \frac{\partial S^{(q)}}{\partial g^{\mu\nu}} - \widetilde{\pi}_{\mu\nu}\right)\left(-i\hbar \frac{\partial \ln\rho}{\partial g^{(\mu)(\nu)}} + \frac{\partial S^{(q)}}{\partial g^{(\mu)(\nu)}} - \widetilde{\pi}_{(\mu)(\nu)}\right) \\
&\quad + \int_{U_g} d(g)\rho \left(-\frac{\hbar^2}{2}\frac{\partial^2 \ln\rho}{\partial g^{(\mu)(\nu)}\partial g^{\mu\nu}} - i\hbar \frac{\partial^2 S^{(q)}}{\partial g^{(\mu)(\nu)}\partial g^{\mu\nu}}\right).
\end{aligned} \tag{73}$$

From the last expression, it follows in particular that the fluctuation $\left\langle \left(\Delta \pi_{\mu\nu}^{(q)}\right)^2 \right\rangle$ can be represented as:

$$\left\langle \left(\Delta \pi_{\mu\nu}^{(q)}\right)^2 \right\rangle = \left\langle \left(\Delta \pi_{\mu\nu}\right)^2 \right\rangle_1 + \left\langle \left(\Delta \pi_{\mu\nu}\right)^2 \right\rangle_2, \tag{74}$$

with $\left\langle \left(\Delta \pi_{\mu\nu}\right)^2 \right\rangle_1$ and $\left\langle \left(\Delta \pi_{\mu\nu}\right)^2 \right\rangle_2$ denoting respectively the two weighted integrals:

$$\left\langle \left(\Delta \pi_{\mu\nu}\right)^2 \right\rangle_1 = \frac{\hbar^2}{4}\int_{U_g} d(g)\rho \frac{\partial \ln\rho}{\partial g^{\mu\nu}}\frac{\partial \ln\rho}{\partial g^{(\mu)(\nu)}}, \tag{75}$$

$$\left\langle \left(\Delta \pi_{\mu\nu}\right)^2 \right\rangle_2 = \int_{U_g} d(g)\rho \left(\frac{\partial S^{(q)}}{\partial g^{\mu\nu}} - \widetilde{\pi}_{\mu\nu}\right)\left(\frac{\partial S^{(q)}}{\partial g^{(\mu)(\nu)}} - \widetilde{\pi}_{(\mu)(\nu)}\right). \tag{76}$$

We intend to prove that the following fundamental inequality holds:

$$\left\langle \left(\Delta g_{(\mu)(\nu)}\right)^2 \right\rangle \left\langle \left(\Delta \pi_{\mu\nu}\right)^2 \right\rangle_1 \geq \frac{\hbar^2}{4}, \tag{77}$$

which implies in turn also that:

$$\left\langle \left(\Delta g_{(\mu)(\nu)}\right)^2 \right\rangle \left\langle \left(\Delta \pi_{\mu\nu}^{(q)}\right)^2 \right\rangle \geq \frac{\hbar^2}{4}. \tag{78}$$

The two inequalities (77) and (78) will be referred to respectively as first and second Heisenberg inequalities for g-quantization. The last inequality can also be represented in terms of the corresponding standard deviations $\sigma_{g_{\mu\nu}}$ and $\sigma_{\pi_{\mu\nu}}$, namely:

$$\begin{cases} \sigma_{g_{\mu\nu}} \equiv \sqrt{\left\langle \left(\Delta g_{(\mu)(\nu)} \right)^2 \right\rangle} \\ \sigma_{\pi_{\mu\nu}} \equiv \sqrt{\left\langle \left(\Delta \pi_{\mu\nu} \right)^2 \right\rangle} \end{cases}, \tag{79}$$

thus yielding the equivalent Heisenberg inequality for the standard deviations in the customary formal representation:

$$\sigma_{g_{(\mu)(\nu)}} \sigma_{\pi_{\mu\nu}} \geq \frac{\hbar}{2}. \tag{80}$$

The proof of the two inequalities (77) and (78) is analogous to that given in [36]. For this purpose, one notices, first, that by construction, the conservation of $\rho(s)$ is warranted by the validity of the quantum continuity Equation (54). Hence, thanks to the normalization of $\rho(s)$ according to Equation (51) which holds for all $s \in I \equiv \mathbb{R}$, for arbitrary proper-time s, integration by parts in the previous configuration-space integral delivers equivalently also the identity:

$$- \int_{U_g} d(g) \rho \left(g_{\mu\nu} - \widetilde{g}_{\mu\nu} \right) \frac{\partial \ln \rho}{\partial g_{(\mu)(\nu)}} = 1. \tag{81}$$

Hence, the inequality:

$$\int_{U_g} d(g) \rho \sqrt{\left(g_{\mu\nu} - \widetilde{g}_{\mu\nu} \right) \left(g_{(\mu)(\nu)} - \widetilde{g}_{(\mu)(\nu)} \right)} \sqrt{\frac{\partial \ln \rho}{\partial g_{(\mu)(\nu)}} \frac{\partial \ln \rho}{\partial g_{(\mu)(\nu)}}} \geq 1 \tag{82}$$

manifestly must hold identically for arbitrary s, as well. Next, Schwartz's inequality delivers:

$$\int_{U_g} d(g) \rho \left(g_{\mu\nu} - \widetilde{g}_{\mu\nu} \right) \left(g_{(\mu)(\nu)} - \widetilde{g}_{(\mu)(\nu)} \right) \int_{U_g} d(g) \rho \frac{\partial \ln \rho}{\partial g_{\mu\nu}} \frac{\partial \ln \rho}{\partial g_{(\mu)(\nu)}} \geq 1. \tag{83}$$

Upon multiplying term by term the LHS of the previous equation with respect to $\frac{\hbar^2}{4}$, an inequality follows, which exactly coincides for arbitrary s with Equation (77). Since by construction $\left\langle \left(\Delta \pi_{\mu\nu} \right)^2 \right\rangle_2 \geq 0$, the second Heisenberg inequality (78) necessarily holds, as well. Such inequalities hold for arbitrary $s \in I \equiv \mathbb{R}$. Hence, no possible constraint can arise on the dynamical proper-time evolution wave-function $\psi(s)$ as a consequence of the validity of the same inequalities.

The physical interpretation, and consequent implications, implied by the validity of the Heisenberg inequalities (77) and (78) (or equivalently (80)) is of crucial importance for the prescription of quantum measurements in the context of CQG g-quantization. This occurs if, in analogy with the original Heisenberg interpretation (see [36]), the standard deviations $\sigma_{g_{\mu\nu}}$ and $\sigma_{\pi_{\mu\nu}}$ defined above are interpreted as quantum measurement errors. Then, the inequality (80) states the impossibility of realizing simultaneous quantum measurements for the mutually-conjugated Lagrangian coordinate and corresponding momentum, so that both the canonical variables cannot be measured exactly simultaneously, i.e., with vanishing standard deviations. In other words, the product of the corresponding fluctuations/standard deviations is necessarily non-zero since they must satisfy respectively the fundamental inequalities (78) and (80).

7. Conclusions

In this paper, a number of issues have been posed, which are intimately related to quantum gravity and the statistical interpretation of the corresponding relativistic quantum wave equation.

The goal has been to set them in the framework of the recently-developed axiomatic approach to the quantization of the metric field tensor (referred to here as g-quantization) achieved in the context of covariant quantum gravity (CQG-theory). Such a theory relies on the adoption of the principle of manifest covariance for the formulation of relativistic statistical mechanics and in particular for the prescription of a quantum field theory, which is appropriate in the case of the gravitational field.

For this purpose, first, we have shown that the CQG-quantum wave equation laying at the basis of CQG-theory can be obtained in the framework of a Hamilton–Jacobi quantization scheme, i.e., based on the construction of the classical Hamilton–Jacobi equation and the corresponding manifestly-covariant Hamiltonian structure associated with the Einstein field equations. This feature is crucial since it permits the identification of a specific quantum wave equation, i.e., the evolution equation, which advances the quantum state. In the framework of the background space-time (first-quantization) manifestly-covariant approach adopted here, the latter has been identified with a single four-scalar wave function ψ.

The second step has concerned a crucial aspect of CQG-theory and quantum field theory alike, namely the establishment of a variational formulation for the related quantum-wave equation. In this paper, we have shown that such an approach can be achieved by means of a suitable variational principle denoted as quantum Hamilton variational principle. Such a principle, as the analog of the classical Hamilton equations associated with the same Einstein equations, is a synchronous one, i.e., it is prescribed so that the variation operator leaves invariant the relevant volume element in the Hamilton variational functional. In addition, by construction, it exhibits also the property of manifest covariance. As such, it is frame-independent, i.e., it holds for arbitrary coordinate systems. Finally, the same variational principle is of general validity in the sense that it is valid for arbitrary choices of the background space-time, namely arbitrary classical solutions of the Einstein field equations. As a result, by means of suitable variations of the Hamilton variational functional, the quantum wave equation, as well as the corresponding set of quantum hydrodynamic equations, obtained after introducing the Madelung representation for the wave function, are shown to coincide with the corresponding Euler–Lagrange equations.

Finally, in reference to the problem of quantum measurements for CQG-theory, the validity of two Heisenberg inequalities, referred to here respectively as first and second Heisenberg inequalities, has been proven to hold for arbitrary quantum wave-functions, which are solutions of the same CQG-wave equation. More precisely, in the context of the g-quantization considered here, the same inequalities refer to quantum measurements of the tensor field $g \equiv \{g_{\mu\nu}\}$ and its conjugate quantum operator $\pi_{\mu\nu}^{(q)}$, the statistical measurement error estimates being achieved by means of the corresponding standard deviations.

These conclusions are believed to provide theoretical insight to the axiomatic foundations of CQG-theory. It must be stressed that, in the current formulation, CQG-theory is built upon a first-quantization approach assuming the existence of a generic background metric that characterizes the geometrical properties of space-time. This is shown to realize a g-quantization of the canonical state (g, π), which fulfills by construction the quantum unitarity principle and, consequently, the conservation of quantum probability associated with the wave function ψ. As a consequence, no trans-Planckian effects, nor possible information losses arising at event horizons in black-hole space-times are yet included either. Nevertheless, as pointed out above, the inclusion of second-quantization effects, with particular reference to quantum modifications of the background space-time at the Planck length and quantum particle sources, such as Hawking radiation, is in principle possible. In view of these considerations, the features highlighted here suggest that CQG theory may provide fertile theoretical grounds for a variety of applications in quantum gravity and quantum cosmology to be envisaged in future works.

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