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Generation of Wave Groups by Shear Layer Instability

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Abstract: The linear stability theory of wind-wave generation is revisited with an emphasis on the generation of wave groups. The outcome is the fundamental requirement that the group move with a real-valued group velocity. This implies that both the wave frequency and the wavenumber should be complex-valued, and in turn this then leads to a growth rate in the reference frame moving with the group velocity which is in general different from the temporal growth rate. In the weakly nonlinear regime, the amplitude envelope of the wave group is governed by a forced nonlinear Schrödinger equation. The effect of the wind forcing term is to enhance modulation instability both in terms of the wave growth and in terms of the domain of instability in the modulation wavenumber space. Also, the soliton solution for the wave envelope grows in amplitude at twice the linear growth rate.

Keywords: wind waves; wave groups; modulation instability

1. Introduction

The generation of water waves by wind is a fundamental problem of both scientific and operational concern. However, despite much theoretical research, observations and numerical simulations, the theoretical mechanism remains controversial, see the comprehensive reviews by [1,2], and the recent comments by [3–6]. Two main mechanisms are currently invoked. One is a shear flow instability mechanism initially developed by [7] and subsequently adapted for routine use in wave forecasting models, see the review by [2]. In this theory, turbulence in the wind is used only to determine a logarithmic profile for the mean wind profile $u_0(z)$. Then, a monochromatic sinusoidal wave field is assumed, with a real-valued wavenumber k and a complex-valued phase speed, $c = c_r + ic_i$ so that the waves may have a growth rate kc_i. It is found that there is a significant transfer of energy from the wind to the waves at the critical level z_c where $u_0(z_c) = c_r$. Pertinent to the context of this paper, we note that that this was extended to allow for spatial growth instead of temporal growth by [8]. The other is essentially a steady-state theory, developed originally by [9] for separated flow over large amplitude waves, and importantly later adapted for non-separated flow over low-amplitude waves, see [1,4] for instance. Here the wind turbulence is taken into account through an eddy viscosity term in an inner region near the wave surface, and asymmetry in this inner region then allows for an energy flux to the waves.

Neither theory alone has been found completely satisfactory, and in particular, both fail to take account of wave transience and the tendency of waves to develop into wave groups, see [5,6]. This issue was addressed in our preliminary study [10], and is developed further here in the context of a general theory of wave groups for unstable waves. The methodology is based on linear shear flow instability theory, but incorporates from the outset that the waves will have a wave-group structure with both temporal and spatial dependence. In fluid flows this was initiated by [11,12] in the context of shear flows, see the summary by [13] and the reviews by [14,15]. The essential feature that we exploit is that the wave group moves with a real-valued group velocity $c_g = d\omega/dk$ even when for unstable flows the frequency $\omega = kc$ and the wavenumber k are complex-valued.

In Section 2 we develop the general linearized theory for a stratified shear flow, showing that the group velocity is real-valued and presenting some implications. In Section 3 we make a reduction to an air-water system, with constant density in both the air and the water, and no background shear flow in the water. Two special cases are investigated in detail, Kelvin-Helmholtz instability and and a smooth monotonic wind profile. While both these have been heavily studied, the wave-group analysis presented here provides a new perspective. In Section 4 we extend the analysis to the weakly nonlinear regime, and present a forced nonlinear Schrödinger model, which is used to examine modulation instability under wind forcing. We conclude in Section 5.

2. Formulation

We begin with the linear stability theory for a general stratified shear flow, and then develop the theory for the air-water system as a special case. The basic state is the density profile $\rho_0(z)$ and the horizontal shear flow $u_0(z)$ in the x-direction. Then the linearized equations are

$$\rho_0(Du + wu_{0z}) + p_x = 0\,, \tag{1}$$

$$\rho_0 Dv + p_y = 0\,, \tag{2}$$

$$\rho_0 Dw + p_z + g\rho = 0\,, \tag{3}$$

$$D\rho + \rho_{0z} w = 0\,, \tag{4}$$

$$u_x + v_y + w_z = 0\,, \tag{5}$$

$$\text{where} \quad D = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\,. \tag{6}$$

Here, the terms (u, v, w) are the perturbation velocity components in the (x, y, z) directions, ρ is the perturbation density, and p is the perturbation pressure. Equations (1)–(3) represent conservation of momentum, Equation (4) represents conservation of mass, and Equation (5) is the incompressibility condition. The vertical particle displacement ζ is defined in this linearized formulation by

$$D\zeta = w\,. \tag{7}$$

Then the density field is given by integrating Equation (4) to get

$$\rho = -\rho_{0z}\zeta\,. \tag{8}$$

Substituting Equations (7) and (8) into the remaining equations and eliminating u, v, p yields a single equation for ζ,

$$\{\rho_0 D^2 \zeta_z\}_z + \{\rho_0 D^2 \zeta\}_{xx} + \{\rho_0 D^2 \zeta\}_{yy} - g\rho_{0z}\{\zeta_{xx} + \zeta_{yy}\} = 0\,. \tag{9}$$

This equation, together with the boundary conditions that $\zeta = 0$ at $z = -H$ (the bottom of the ocean) and as $z \to \infty$ (the top of the atmosphere) is the basic equation to examine wave groups and linear stability.

Next we seek a solution describing a wave group,

$$\zeta = \{A(X, Y, T)\phi(z) + \epsilon\phi^{(2)}(X, Y, T; z) + \cdots\} \exp\left(ikx + ily - i\omega t\right) + \text{c.c.}\,, \qquad \text{where} \quad X = \epsilon x\,,\ Y = \epsilon y\,,\ T = \epsilon t\,. \tag{10}$$

Here c.c. denotes the complex conjugate, and $\epsilon \ll 1$ is a small parameter describing the slow variation of the amplitude $A(X, Y, T)$ relative to the carrier wave. The frequency $\omega = kc$, where c is the phase speed in the x-direction. Both ω and the wavenumbers k, l may be complex-valued, and then the imaginary part of the frequency is the temporal growth rate of an unstable wave. Importantly later we shall set $l = 0$ so the transverse dependence is only in the amplitude envelope. At leading order, we obtain the modal equation, well-known as the Taylor-Goldstein equation,

$$(\rho_0 k^2 W^2 \phi_z)_z - (k^2 + l^2)(g\rho_{0z} + k^2 W^2)\phi = 0\,, \quad W = c - u_0\,. \tag{11}$$

This defines the modal functions and the dispersion relation specifying $\omega = \omega(k, l)$. At the next order in ϵ we obtain the equation determining the wave envelope amplitude $A(X, Y, T)$. However, first we note the integral identity

$$P(\omega, k, l) \equiv \int_{-H}^{\infty} \rho_0 k^2 W^2 (\phi_z^2 + (k^2 + l^2)\phi^2) + (k^2 + l^2) g\rho_{0z}\phi^2 \, dz = 0\,. \tag{12}$$

This can be regarded as an expression of the dispersion relation, $\omega = \omega(k, l)$. In the sequel we will be mainly concerned with the case when $l = 0$, when the modal Equation (11) reduces to

$$(\rho_0 W^2 \phi_z)_z - (g\rho_{0z} + k^2 W^2)\phi = 0\,, \quad W = c - u_0\,, \tag{13}$$

which can be regarded as determining the dispersion relation in the form $c = c(k)$ where $\omega = kc$, and c is the complex-valued phase velocity. Differentiation of $P(\omega(k, l), k, l) = 0$ with respect to k, l, and evaluating at $l = 0$, yields

$$P_\omega c_g + P_k = 0\,, \quad P_l = 0\,, \tag{14}$$

where c_g is the group velocity in the x-direction, and can be expressed in the form,

$$c_g = \frac{d\omega}{dk} = c + k\frac{dc}{dk}\,, \quad k\frac{dc}{dk} = \frac{J}{I}\,, \quad J = -\int_{-H}^{\infty} \rho_0 k^2 W^2 \phi^2 \, dz\,, \quad I = \int_{-H}^{\infty} \rho_0 W(\phi_z^2 + k^2\phi^2) \, dz\,. \tag{15}$$

It is useful to note that the dependence of $P(\omega, k, l)$, and hence of $\omega(k, l)$. on l is through l^2.

At the next order in the asymptotic expansion we obtain a forced Taylor-Goldstein equation for $\phi^{(2)}$ in the independent variable z. A compatibility condition is needed and this yields when $l = 0$,

$$A_T + c_g A_X = 0\,. \tag{16}$$

The details are described in [10] and are omitted here. Instead we note that in this linearized theory, the equation for the envelope amplitude can be obtained more directly from the dispersion relation evaluated at $l = 0$,

$$P(\omega + \epsilon\frac{i\partial}{\partial T}, k - \epsilon\frac{i\partial}{\partial X}, -\epsilon\frac{i\partial}{\partial Y})A(X, Y, T) = 0\,. \tag{17}$$

Expansion in powers of ϵ yields, at the leading order in ϵ,

$$iP_\omega A_T - iP_k A_X - iP_l A_Y = 0\,. \tag{18}$$

The amplitude Equation (16) follows on using Equation (14). It implies that the amplitude envelope propagates with the group velocity, since the solution states that A is constant on the characteristics $dx/dt = c_g$, and so c_g must be real-valued. This is well-known for stable waves, but that it also holds for unstable waves when ω, k may be complex-valued is not so well-known in the fluid dynamics literature. However, see the seminal work on shear flows by [11,12], and the several papers which followed in the reviews by [14,15]. These works mainly solved the linear initial-value problem with Fourier transforms, and then when the long-time asymptotic solution was sought, the method of steepest descent revealed the critical condition that $x/t = c_g$, thus enforcing the group velocity to be real-valued. For unstable waves when the frequency ω is complex-valued this leads to the necessity that the wavenumber k must also be complex-valued, and *vice-versa* . The imaginary parts of $\omega(k)$ and k are linked by the requirement that the group velocity is real-valued.

At the next order in the expansion we get that

$$i(A_T + c_g A_X) + \epsilon(\lambda A_{XX} + \sigma A_{YY}) = 0\,, \quad \lambda = \frac{\omega_{kk}}{2}\,, \quad \sigma = \frac{\omega_{ll}}{2}. \tag{19}$$

Here we note that $\omega_{kk} = c_{gk}$ while

$$\omega_{ll} = -\frac{P_{ll}}{P_\omega} = \frac{\int_{-H}^{\infty} \rho_0 W^2 \phi_z^2 \, dz}{\int_{-H}^{\infty} \rho_0 k W(\phi_z^2 + k^2\phi^2)\, dz}\,, \tag{20}$$

If there is no shear flow $u_0(z)$ this is just $\omega_{ll} = c_g/k$. see Equation (15). Since c_g is real-valued, we can make a transformation to put Equation (19) into the canonical form,

$$\xi = X - c_g T\,, \quad \tau = \epsilon T\,, \tag{21}$$

$$iA_\tau + \lambda A_{\xi\xi} + \sigma A_{YY} = 0\,. \tag{22}$$

This is the well-known linear Schödinger equation for the evolution of weakly dispersive stable wave packets, but it is not widely known that it also holds for unstable wave packets.

Next we put $k = k_r + ik_i$, $\omega = \omega_r + i\omega_i$, and assume that $|k_i| \ll k_r$ and $|\omega_i| \ll \omega_r$, where without loss of generality we assume that $k_r > 0, \omega_r > 0$. For consistency with the modulation scaling in the wave packet expansion Equation (10) we anticipate that the spatial and temporal growth rates $|k_i|, |\omega_i|$ are at least $O(\epsilon)$. Then we extract the imaginary part of the phase in Equation (10), so that

$$\begin{gathered} \exp{(ikx + ily - i\omega t)} = E \exp{(i\theta)}\,, \quad E = \exp{(-k_i(x - c_g t) + \delta t)}\,, \\ \delta = \omega_i - c_g k_i\,, \quad \theta = k_r x - \omega_r t\,. \end{gathered} \tag{23}$$

In the reference frame moving withe group velocity, δ is the growth rate, and taking account that c_g is real-valued, can be written as

$$\delta = \mathrm{Im}[\omega - c_g k] = -\mathrm{Im}[k^2 \frac{dc}{dk}]\,. \tag{24}$$

Importantly this must be evaluated on those complex-valued wavenumbers k such that c_g is real-valued. Since θ is a real-valued phase, we write $B = EA$ so that Equation (10) becomes,

$$\zeta = \{B(X, Y, T)\phi(z) + \cdots\} \exp{(i\theta)} + \text{c.c.}\,, \tag{25}$$

and the wave packet Equation (16) becomes, to leading order.

$$B_T + c_g B_X - \tilde{\delta} B = 0\,, \quad \tilde{\delta} = \frac{\delta}{\epsilon}\,. \tag{26}$$

Note that the growth rate $\tilde{\delta}$ is at least $O(1)$. The same substitution converts the linear Schrödinger Equation (22) into

$$i(B_\tau - \Delta B) + \lambda B_{\xi\xi} + \sigma B_{YY} = 0\,, \quad \Delta = \frac{\tilde{\delta}}{\epsilon} = \frac{\delta}{\epsilon^2}\,. \tag{27}$$

For consistency we must now assume that $|k_i|, |\omega_i|$ are at least $O(\epsilon^2)$, so that the growth rate Δ is at least $O(1)$. The coefficients λ, σ are real-valued to leading order in ϵ,

As already noted the dispersion relation $\omega = \omega(k)$ must be examined in the complex k-plane, and even for relatively simple expressions, this can be a complex task in general. However, if, as here, we assume that the the imaginary parts of $\omega(k)$ and k are small, then simple approximate expressions can be derived. This, putting $k = k_r + ik_i$ and expanding,

$$\omega_i(k) \approx \gamma + c_g(k)k_i + \mathrm{Im}[\sigma(k_r)]k_i^2 + \cdots\,, \quad \gamma = \mathrm{Im}[\omega(k_r)]\,, \tag{28}$$

where γ is the temporal growth rate, and is often written as $k_r\mathrm{Im}[c(k_r)]$. It follows that $\delta \approx \gamma$, and since $\sigma(k_r)$ is real-valued to leading order, the error term is $O(k_i^3)$. This differs from the corresponding growth rate in [10] due to a different interpretation of the temporal growth rate; [10] defined this as $k_r\mathrm{Im}[c(k)]$ which includes a contribution from k_i. The interpretation here agrees with that in [3,16–19], who also studied water wave groups under wind forcing. The derivation here is quite general and applies to all physical systems which support wave groups. The condition that the group velocity be real-valued implies that

$$k_i\mathrm{Re}[c_{gk}(k_r)] \approx -\mathrm{Im}[c_g(k_r)] = -\mathrm{Im}[c(k_r) + \frac{J(k_r)}{I(k_r)}] \tag{29}$$

where we note that $c_g(k) \approx c_g(k_r)$ to the same level of approximation, and the right-hand side has used the expression Equation (15).

3. Air-Water System

For an air-water system, we follow the formulation of [10] and write,

$$\rho_0(z) = \rho_a\mathcal{H}(z) + \rho_w\mathcal{H}(-z)\,, \quad \rho_{0z} = (\rho_a - \rho_w)\delta(z)\,. \tag{30}$$

Here ρ_a, ρ_w are the constant air and water density respectively, the undisturbed air-water interface is at $z = 0$, $\mathcal{H}(z)$ is the Heaviside function and $\delta(z)$ is the Dirac delta function. The water is bounded below at $z = -H$, and the air is unbounded above. Continuity of ζ at the interface $z = 0$ implies that ϕ is continuous across $z = 0$. Since the modal Equation (11) is homogeneous, without loss of generality we can set $\phi(z = 0) = 1$. Then in the air ($z > 0$) and water ($z < 0$) the modal Equation (11) collapses to the Rayleigh equation

$$(W^2\phi_z)_z - k^2W^2\phi = 0\,, \quad W = c - u_0\,. \tag{31}$$

The dynamical boundary condition at $z = 0$ is found by integrating Equation (31) across $z = 0$, with the outcome

$$[\rho_0W^2\phi_z]^+_- = g(\rho_a - \rho_w)\phi(0)\,. \tag{32}$$

The system Equation (31), Equation (32) is supplemented by the boundary conditions that $\phi = 0$ at $z = -H$ and that $\phi \to 0$ as $z \to \infty$. It remains to specify the shear flow $u_0(z)$. In the water, there is no background current, so that $u_0(z) = 0$, $-H \leq z \leq 0$, so that the solution of Equation (31) which satisfies the boundary condition $\phi(-H) = 0$ is

$$\phi = \frac{\sinh(k(z+H))}{\sinh(kH)}\,, \quad -H < z < 0\,. \tag{33}$$

Here we recall that we have set $\phi(0) = 1$ without loss of generality. The boundary condition Equation (32) then reduces to

$$s(c - u_0(0+))^2\phi_z(0+) = g\{\frac{c^2}{c_0^2} - (1-s)\}\,, \quad c_0^2 = \frac{g}{k}\tanh(kH)\,. \tag{34}$$

If there is no air ($s = 0$), then this reduces to the usual water wave dispersion relation $c^2 = c_0^2$ and in that limit the waves are stable and k is real-valued. Since $s \ll 1$ it follows that for unstable waves $\omega_i = \mathrm{Im}[\omega]$ and $k_i = \mathrm{Im}[k]$ are $O(s)$, and so s is a convenient small parameter, which we will later link to the modulation parameter ϵ. The integral identity Equation (12) at $l = 0$ reduces to

$$\tilde{P} \equiv s\int_0^\infty W^2(\phi_z^2 + k^2\phi^2)\,dz + g\{\frac{c^2}{c_0^2} - (1-s)\} = 0\,. \tag{35}$$

Similarly, the expression Equation (15) for the group velocity reduces to

$$
\begin{aligned}
c_g &= c + \frac{J}{I}\,, \\
J &= -s\int_0^\infty k^2W^2\phi^2\,dz + \frac{gkc^2}{c_0^3}\frac{dc_0}{dk}\,, \\
I &= s\int_0^\infty W(\phi_z^2 + k^2\phi^2)\,dz) + \frac{gc}{c_0^2}\,.
\end{aligned} \tag{36}
$$

Note that in the limit $s \to 0$ this becomes $c_{0g} = c_0 + kdc_0/dk$. the group velocity for unforced water waves. It remains to specify the wind profile $u_0(z)$ in $z > 0$, and we will reconsider two well-known cases.

3.1. Kelvin-Helmholtz Instability

First, assume that $u_0 = U > 0$ where U is a constant. This Helmholtz profile is not usually regarded as a relevant model for water waves, see [7], but with the inclusion of interfacial surface tension it is may become of some practical interest, see [13,15,20,21]. It is useful here as it leads to an explicit expression for the dispersion relation, which can then be analyzed for complex-valued ω and k. For this choice of u_0 the modal Equation (31) in $z > 0$ has the solution

$$\phi = \exp\left(-kz\right), \quad 0 < z < \infty\,, \tag{37}$$

valid for $k_r > 0$. Application of the boundary condition Equation (34) leads to the dispersion relation

$$sk(c-U)^2 + k\coth\left(kH\right)c^2 = g(1-s) + \Sigma k^2\,, \tag{38}$$

Here we have included the effects of interfacial surface tension with a coefficient Σ, see [13,15,20,21]. This is a quadratic equation for $c(k)$ with solution

$$(s+\coth\left(kH\right))c = sU \pm \{(s+\coth\left(kH\right))(\frac{g(1-s)}{k} + \Sigma k) - s\coth\left(kH\right)U^2\}^{1/2}\,. \tag{39}$$

There is now temporal instability when the argument of the term in the brackets $\{\cdots\}$, evaluated at $k = k_r$, is negative, this being the well-known Kelvin-Helmholtz instability. The subsequent analysis is simplified if we take the deep-water limit $H \to \infty$, that is $\coth\left(kH\right) \approx 1$ since $k_r > 0$. Then Equation (39) reduces to

$$(1+s)c = sU \pm \{(1+s)(\frac{g(1-s)}{k} + \Sigma k) - sU^2\}^{1/2}\,. \tag{40}$$

It is useful to define a dimensionless wavenumber

$$K = k\frac{sU^2}{g(1-s^2)}\,, \tag{41}$$

so that Equation (40) becomes

$$(1+s)c = sU \pm is^{1/2}U\{1 - \frac{1}{K} - BK\}^{1/2}\,, \quad B = \frac{\Sigma g(1-s^2)(1+s)}{s^2U^4}\,. \tag{42}$$

Here B is a dimensionless Bond number. Temporal instability occurs when the term in brackets $\{\cdots\} > 0$, which requires that $B < 1/4$ and then defines a wavenumber band,

$$K_- < K_r < K_+\,, \quad BK_\pm = \frac{1}{2} \pm \{\frac{1}{4} - B\}^{1/2}\,. \tag{43}$$

Within this band the temporal growth rate is

$$\gamma = k_r c_i\,, \quad (1+s)c_i = \pm s^{1/2} U\{1 - \frac{1}{K_r} - BK_r\}^{1/2}\,. \tag{44}$$

The expression Equation (15) for the group velocity becomes

$$(1+s)c_g = sU \pm i\frac{s^{1/2}U}{2}\frac{2 - K^{-1} - 3BK}{\{1 - K^{-1} - BK\}^{1/2}}\,. \tag{45}$$

The requirement that c_g be real-valued implies that

$$\mathrm{Re}[\frac{2 - K^{-1} - 3BK}{\{1 - K^{-1} - BK\}^{1/2}}] = 0\,, \tag{46}$$

which determines a relationship between K_r and K_i. The growth rate Equation (24) becomes

$$\delta = -\mathrm{Im}[k^2\frac{dc}{dk}] = \mp\frac{g(1-s)}{s^{1/2}U}\mathrm{Re}[\frac{(1 - BK^2)}{\{1 - K^{-1} - BK\}^{1/2}}]\,. \tag{47}$$

Importantly, we note that since for these unstable waves $0 < B < 1/4$, and from Equation (43) BK is $O(1)$, we infer that the growth rate δ scales with $g(1-s)/s^{1/2}U$ for large surface tension (B is $O(1/4)$), but for small surface tension as $B \to 0$ scales with $s^{3/2}U^3/(1+s)^2\Sigma$.

Although Equation (46) is a relatively simple expression, an analytical solution appears still to be beyond reach. Hence instead we follow the example of [15] and examine the dispersion relation in the vicinity of the onset of instability, that is, we set $B = 1/4 - \Delta, K = 2 + \kappa$ and $0 < \Delta \leq 1, \kappa << 1$. The expression Equation (42) for the phase speed becomes

$$(1+s)c \approx sU \pm is^{1/2}U\frac{\{32\Delta - \kappa^2\}^{1/2}}{4}\,, \tag{48}$$

and there is instability when $\Delta > 0$ in the wavenumber band $\kappa_r^2 < 32\Delta$. The expression Equation (64) for the group velocity reduces to

$$(1+s)c_g \approx sU \mp is^{1/2}U\frac{\kappa}{\{32\Delta - \kappa^2\}^{1/2}}\,. \tag{49}$$

For this to be real-valued, we require that either $\kappa = \kappa_r$ is real-valued and $\kappa_r^2 > 32\Delta$, or that $\kappa = i\kappa_i$ is pure imaginary. The former leads to stable waves and is excluded here, and so the latter is adopted when Equation (49) becomes

$$(1+s)c_g \approx sU + s^{1/2}U\frac{\kappa_i}{\{32\Delta + \kappa_i^2\}^{1/2}}\,. \tag{50}$$

The growth rate Equation (47) becomes

$$\delta \approx \mp\frac{4g(1-s)}{s^{1/2}U}[\frac{(4\Delta + \kappa_i^2)}{\{32\Delta + \kappa_i^2\}^{1/2}}]\,. \tag{51}$$

The corresponding temporal instability growth rate is Equation (44), and in this approximation becomes

$$\gamma \approx \pm\frac{g(1-s)}{s^{1/2}U}\{2\Delta\}^{1/2}\,, \tag{52}$$

These differ in magnitude even when $\kappa_i = 0$, and curiously the branches ($\pm$) which are unstable/stable for the temporal growth rate γ interchange for the growth rate δ in the group velocity reference frame.

3.2. Monotonic Wind Profile

The usual theories such as those in [2,7,22] assume that the wind profile $u_0(z)$ is continuous, monotonically increasing with height z and vanishes at $z = 0, u_0(0) = 0$. However, there are then no simple explicit analytic expressions available for the modal function $\phi(z)$ and hence for the dispersion relation. Instead it is customary to take the limit $s \to 0$ when $c_i \to 0$. Then various approximations have been used, most of which require evaluation of the modal function near a critical level z_c where $u_0(z_c) = c_r \approx c_0(k_r)$ and there is a singularity. Here we attempt to avoid this limit, and use an approximation similar to those used by [2,10].

We make a further assumption that $u_0(z) = U_0 > 0$, a constant, for $z > z_0$, where $U_0 > c_0(k_r)$. In the zone $0 < z < z_0$ the term $W^2k^2\phi$ in Equation (31) is neglected, and then an approximate solution is,

$$\phi \approx 1 + D\int_0^z \frac{dy}{W^2(y)}\,, \quad 0 < z < z_0\,, \tag{53}$$

where the constant D is determined by matching at $z = z_0$. Formally, this is valid when $k_r z \ll 1$, and in particular, $k_r z_c \ll 1$. In the limit $c_i \to 0$ the second term in Equation (53) is singular at $z = z_c$ and is evaluated by assuming that $c_i > 0$ (evaluated at $k = k_r$), and then taking the limit $c_i \to 0+$. This yields the Frobenius expansion

$$\phi \sim D_c\{\frac{1}{z - z_c} + \frac{u_{0zz}(z_c)}{u_{0z}(z_c)}\log{(z - z_c)} + \cdots\}\,, \quad z \to z_c\,, \quad D = -D_c u_{0z}^2(z_c) \tag{54}$$

Here the branch of the logarithm when $z - z_c < 0$ must be chosen corresponding to the requirement that the growth rate $\gamma = k_r c_i > 0$, that is

$$\log{(z - z_c)} = \log|z - z_c| - i\pi \mathrm{sign}[u_{0z}(z_c)]\,, \quad \text{when} \quad z < z_c\,. \tag{55}$$

Then for $z > z_c$,

$$\phi = 1 + \{\mathcal{P}\int_0^z \frac{dy}{(c_r - u_0(y))^2} - \frac{i\pi u_{0zz}(z_c)}{|u_{0z}(z_c)|^3}\}\,, \quad z_c < z < z_0\,. \tag{56}$$

where $\mathcal{P}\int$ denotes the principal value integral, and we recall that $u_0(z_c) = c_r$. In $z > z_0$,

$$\phi = D_1 \exp{(-k(z - z_0))}\,, \quad z > z_0\,. \tag{57}$$

Across $z = z_0$ both ϕ and ϕ_z are continuous and so

$$D_1 = 1 + D\int_0^{z_0} \frac{dz}{W^2(z)}\,, \quad -kD_1 = \frac{D}{(c - U_0)^2}\,. \tag{58}$$

Elimination of D_1 yields the expression for D,

$$D\{\frac{1}{k(c - U_0)^2} + \int_0^{z_0} \frac{dz}{W^2(z)}\} = -1\,. \tag{59}$$

Finally, since $c^2\phi_z(0+) = D$. substitution into Equation (34) yields the dispersion relation

$$\mathcal{D}(c, k) \equiv g\{\frac{c^2}{c_0^2} - (1 - s)\}\{\frac{1}{k(c - U_0)^2} + \int_0^{z_0} \frac{dz}{W^2(z)}\} + s = 0\,. \tag{60}$$

This is equivalent to Equation (35) under the approximations used here. It simplifies considerably in the limits $z_0 \to \infty, U_0 \to \infty$ where we assume that in this joint limit the integral term converges. Then Equation (60) becomes

$$\mathcal{D}(c,k) \equiv g\{\frac{c^2}{c_0^2} - (1-s)\}\{\int_0^\infty \frac{dz}{W^2(z)}\} + s = 0\,. \tag{61}$$

Following the analysis in [10] it is useful to put

$$\frac{c^2}{c_0^2} = 1 - s + s\mu\,, \quad g\mu\{\int_0^\infty \frac{dz}{W^2(z)}\} = -1\,, \tag{62}$$

where we note that μ is independent of s. In the limit $c_i \to 0$ with $k = k_r$ real-valued, after using Equation (56) and that $s \ll 1$, we get the temporal growth rate

$$\gamma = k_r c_i\,, \quad \frac{c_i}{c_0} = -\frac{s\pi u_{0zz}(z_c)}{2g|u_{0z}(z_c)|^3}\{\mathcal{P}\int_0^\infty \frac{dz}{W^2(z)}\}^{-2}\,. \tag{63}$$

This quite simple expression for γ agrees with that in [10] after appropriate simplifications in [10] are made. In general, it offers a potentially quite useful explicit expression for the temporal growth rate, although it cannot be used for the commonly invoked logarithmic profile as then the integral term does not converge.

Our interest here is in the group velocity $c_g = c + kdc/dk$ and then using Equation (62)

$$c_g \approx c_{0g} + \frac{sc_0}{2}(\mu - 1)\,, \tag{64}$$

where we note that here $d\mu/dk = 0$ Equation (62), and $c_{g0} = c_0 + kdc_0/dk$ is the water wave-group velocity, but evaluated here for a complex-valued k. We now require that this be real-valued, and putting $k = k_r + ik_i$ and expanding for $|k_i| \ll k_r$ we get that

$$k_i \frac{dc_{0g}}{dk_r}(k_r) + c_i = 0\,, \tag{65}$$

where $k_r c_i = \gamma$ is the temporal growth rate defined by Equation (63). Please note that $k_i > 0$ here. In this same limit $|k_i| \ll k_r$, the growth rate $\delta \approx \gamma$, see Equation (28) and the following discussion.

4. Nonlinear Schrödinger Equation

When this linearized analysis is extended to the weakly nonlinear regime, we expect that the linear Schrödinger equation (Equation (27)) will be replaced by the wind-forced nonlinear Schrödinger equation, see [3,10,16–19] for related studies in the one-dimensional context,

$$i(B_\tau - \Delta B) + \lambda B_{\xi\xi} + \sigma B_{YY} + \nu|B|^2 B = 0 \tag{66}$$

Here the nonlinear coefficient ν is the Stokes amplitude-dependent frequency correction, which in the present context to leading order will just be that for water waves. In the deep-water limit as $H \to \infty$, $\lambda = -c_0/8k_r$, $\sigma = c_0/4k_r$ and $\nu = -c_0 k_r^3/2$ where $c_0^2 = g/k_r$. Formally, the derivation of Equation (66) requires a re-scaling in which $\tau = \epsilon^2 t, \xi = \epsilon(x - c_g t)$ as in Equation (21), and the amplitude B is scaled with ϵ. This suggests that for the monotonic wind profiles of Section 3.2 we put $s = O(\epsilon^2)$. Since $s \approx 1.275 \times 10^{-3}$ this implies a restriction to waves with amplitudes of non-dimensional order 0.036. However, the analysis of the Kelvin-Helmholtz profiles in Section 3.1 does not require this link between s and ϵ.

The nonlinear and dispersive terms in Equation (66) are not sufficient to control the exponential growth of a localized wave packet, since

$$\frac{d}{dT}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |B|^2\, d\xi dY = 2\Delta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |B|^2\, d\xi dY\,. \tag{67}$$

Further the modulation instability, present when $\nu\lambda > 0$ (as for deep-water waves) in the absence of wind, is enhanced in the presence of wind, see [16] for the one-dimensional case. To see this, first transform Equation (66) into

$$B = \tilde{B}\exp{(\Delta\tau)}\,, \quad s = \frac{\exp{(2\Delta\tau)}-1}{2\Delta}\,,$$
$$i\tilde{B}_s + \lambda F\tilde{B}_{\xi\xi} + \sigma F\tilde{B}_{YY} + \nu|\tilde{B}|^2\tilde{B} = 0\,, \quad \text{where} \quad F = \frac{1}{1+2\Delta s}\,. \tag{68}$$

In this transformed system, the energy expression Equation (67) becomes a conservation law

$$\frac{d}{dT}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\tilde{B}|^2\,d\xi dY = 0,. \tag{69}$$

This has the "plane wave" solution $\tilde{B} = B_0\exp{(-i\nu|B_0|^2 s)}$. Modulation instability is then found by putting $\tilde{B} = B_0(1+b)\exp{(-i\nu|B_0|^2 s)}$ into Equation (68) and linearizing in b, so that

$$ib_s + \lambda F b_{\xi\xi} + \sigma F b_{YY} + \nu|B_0|^2(b+b^*) = 0\,. \tag{70}$$

Then we seek solutions of the form $b = (p(\tau)+iq(\tau))\cos{(K\xi + LY)}$ where p, q are real-valued, and find that

$$\{\frac{p_s}{F}\}_s + M(MF - 2\nu|B_0|^2)p = 0\,, \quad q = \frac{p_s}{MF}\,, \quad M = \lambda K^2 + \sigma L^2\,. \tag{71}$$

When $\Delta = 0, F = 1$, and this yields the usual criterion for modulation instability, namely that $M(M - \nu|B_0|^2) < 0$. That is, since here $\nu < 0, \lambda < 0, \sigma > 0$, there is instability for $M < 0$ and then $|M| < |\nu||B_0|^2$, which defines the well-known instability band in the $K-L$ plane. When $\Delta > 0$, F varies from 1 to 0 as s increases from 0 to ∞. Since as $\tau \to \infty$, $s \to \infty$, there is modulation instability provided only that $M\nu > 0$, that is $M < 0$, and so independent of $|B_0|$. Further the band in the $K-L$ plane opens to the half-space $|\lambda|K^2 > \sigma L^2$. Using the deep-water values for λ, σ this is the region $K^2 > 2L^2$. Although the general solution of Equation (71) can be expressed in terms of modified Bessel functions of imaginary order, see [16], we shall not pursue this here as the main outcome is already clear. However, we note that as $F \to 0$, $p \sim F^{1/2}\exp{(\Sigma F^{-1/2})}$, $\Sigma = |2M\nu|^{1/2}/\delta$. Even taking account of the cancellation of the factor $\tau^{-1/2}$ with the pre-factor $\exp{(\Delta T)}$ in Equation (68), we see that the modulation growth rate is now super-exponential.

This linearized analysis of modulation instability does not indicate the outcome of the wave growth. However, in the absence of wind forcing, and in the context of the one-dimensional (that is, the Y-variation is omitted) nonlinear Schrödinger equation, it is known that modulation instability leads to formation of envelope solitary waves or even rogue waves, modelled by Peregrine breathers, see [23,24] for instance. This has been confirmed in several numerical and laboratory experiments, see [25–27]. We might expect a similar outcome under wind forcing, but a detailed analysis is beyond the scope of this present article. Instead we note that the transformed nonlinear Schrödinger Equation (68) in the one-dimensional context has the slowly varying solitary wave solution

$$\tilde{B} = B_{sol}\mathrm{sech}(\Theta)\exp{(i\Phi)}\,, \quad \Theta = \Gamma(\xi - V\tau)\,, \quad \Phi = K\xi - \Omega T$$
$$V = 2\lambda F\nu K\,, \quad -\Omega + \lambda F K^2 = \lambda F\Gamma^2 = \frac{1}{2}\nu B_{sol}^2\,. \tag{72}$$

The solitary wave parameters are slowly varying functions of s, and are determined by an asymptotic multi-scale analysis, see [28]. The outcome is that

$$B_{sol}^2 \propto \frac{\nu}{\lambda F}\,, \quad \Gamma \propto \frac{\nu}{\lambda F}\,. \tag{73}$$

This can also be established by substituting Equation (72) into the energy expression Equation (69). As $s \to \infty$, $F \to 0$, and so the amplitude $B_{sol} \to \infty$, and the growth rate is exponential as $B_{sol} \sim s^{-1/2} \sim \exp(\Delta\tau)$. Interestingly, this is superposed onto the growth term $\exp(\Delta\tau)$ in the transformation in Equation (68) and so doubles the linear growth rate. This agrees with the super-exponential growth rate of the modulational instability.

5. Discussion

In this paper, we have presented a theory for the description of wave groups for unstable waves, Although this is in the context of a stratified shear flow, the methodology is based on the linear dispersion relation, and so is applicable to many other physical systems. At leading order for plane waves, the system is governed by the Taylor-Goldstein equation determining a dispersion relation for the wave frequency and wavenumber. At the next order in an asymptotic expansion, the main outcome is that, as is well-known for stable waves, the wave envelope propagates with the group velocity, which must then be real-valued. This has the consequence that for unstable waves, both the wave frequency and the wavenumber are complex-valued. The outcome is that the waves are unstable in the reference frame moving with the group velocity, with a growth rate which in general is different from the temporal growth rate where only the wave frequency is complex-valued.

The theory is then explored in the context of an air-water system, with the aim of examining the consequences of this wave-group approach for wind waves. Two specific cases are examined in detail, chosen for their analytical simplicity rather than direct applicability. One is Kelvin-Helmholtz instability where we find that the growth rate for wave groups is quite different from the well-known temporal instability. The other is for a monotonic wind profile, where to achieve analytical tractability, we make some approximations in the calculation of the modal function to lead to an explicit dispersion relation. Even so, here we must exploit the approximation that the ratio of the air density to the water density is a small parameter, and then we find that the growth rate in the reference frame moving with the group velocity and the temporal growth rate are in approximate agreement.

At the next order in the asymptotic expansion, and incorporating weakly nonlinear terms, we obtain a nonlinear Schrödinger equation, the usual equation for stable waves but now incorporating a linear growth term. Although this is not integrable, the plane wave solution is tested for modulation instability. We find that there is an an enhanced growth rate on top of the linear growth rate, and so the overall growth rate is super-exponential. Also, the band width of modulation wavenumbers is considerable widened under the wind forcing. For stable waves governed by the nonlinear Schrödinger equation, wave groups can be described by the soliton solution. In the presence of wind forcing, we find that the soliton amplitude grows at twice the linear growth rate. Overall, although there is still much to be explored in this forced nonlinear Schrödinger equation, we conclude that wind forcing considerably enhances modulation instability and the growth of the wave-group envelope.

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