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On Analytic Functions Involving the q-Ruscheweyeh Derivative

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Abstract: In this paper, we use concepts of q-calculus to introduce a certain type of q-difference operator, and using it define some subclasses of analytic functions. Inclusion relations, coefficient result, and some other interesting properties of these classes are studied.

Keywords: univalent; q-starlike; q-difference operator; subordination

MSC: 30C45

1. Introduction

Let A denote the class of functions f which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and are of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad (z \in E). \tag{1}$$

One-to-one analytic functions in this class are usually called univalent. A function $f \in A$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ in E if it satisfies the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in E).$$

We denote this class by $S^*(\alpha)$. In particular, for $\alpha = 0$, we have $S^*(0) = S^*$, the well-known class of starlike functions. The class $C(\alpha)$, $(0 \leq \alpha < 1)$ consisting of convex functions of order α can be defined by the relation

$$f \in C(\alpha), \quad \text{if and only if,} \quad zf' \in S^*(\alpha).$$

Let $f_1, f_2 \in A$. If there exists a Schwartz function $\phi(z)$ analytic in E with $\phi(0) = 0$ such that $|\phi(z)| < 1$ for all $z \in E$ such that $f_1(z) = f_2(\phi(z))$, then we say that $f_1(z)$ is subordinate to $f_2(z)$ and write

$$f_1(z) \prec f_2(z),$$

where $\prec$ denotes subordination.

Let f and g be analytic in E with $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$. Then, the convolution $*$ (or Hadamard product) of f and g is defined as

$$(f * g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m.$$

For $n \in \mathbb{N}_\circ = \{0, 1, 2, 3, ...\}$, let

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$$

so that

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}.$$

The operator D^n is called the Ruscheweyh derivative of order n, see Reference [1]. For the applications of the Ruscheweyh differential operator in geometric function theory, see References [2–4].

In this paper, we generalize the operator D^n by using q-calculus concepts. Recently, q-calculus has attracted the attention of many researchers in the field of geometric function theory. q-Derivatives and q-integrals play an important and significant role in the study of quantum groups and q-deformed super-algebras, the study of fractal and multi-fractal measures, and in chaotic dynamical systems. The name q-calculus also appears in other contexts; see References [5–13]. The most sophisticated tool that derives functions in non-integer order is the well-known fractional calculus; see References [1,12–16]. One can find numerous applications of the q-operator in real-world problems as well as in problems defined on complex plains.

Ismail et al. [15] generalized the class S^* with the concept of q-derivative and called it S_q^* of q-starlike functions. Here, we give some basic definitions and results of q-calculus which we shall use in our results. For more details, see References [12,13,17–22].

If $q \in (0,1)$ is fixed, then a subset B of $\mathbb{C}$ is called q-geometric, if $qz \in B$ whenever $z \in B$ and B contains all geometric sequences $\{zq^m\}_0^\infty$, $zq \in B$. Jackson [9,10] defined q-derivative and q-integral of f on the set B as follows:

$$\partial_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \quad (z \neq 0, z \in (0,1)), \tag{2}$$

and

$$\int_0^z f(t)\partial_q t = z(1-q)\sum_{m=0}^{\infty} q^m f(zq^m),$$

provided that the series converges.

It can easily be seen that for $m \in \mathbb{N} = \{1, 2, 3, ...\}$ and $z \in E$

$$\partial_q\left\{\sum_{m=1}^{\infty} a_m z^m\right\} = \sum_{m=1}^{\infty} [m,q] a_m z^{m-1},$$

where

$$[m,q] = \frac{1-q^m}{1-q} = 1 + \sum_{i=1}^{m-1} q^i, \quad [0,q] = 0. \tag{3}$$

For any non-negative integer m, the q-number shift factorial is defined by

$$[m,q]! = \begin{cases} 1, & m = 0 \\ [1,q][2,q][3,q]...[m,q], & m \in \mathbb{N}. \end{cases}$$

Furthermore, the q-generalized Pochhamer symbol for $x > 0$ is given as

$$[m,q]_m = \begin{cases} 1, & m = 0 \\ [x,q][x+1,q]...[x+m-1,q], & m \in \mathbb{N}. \end{cases}$$

Let the function F be defined as

$$F_{n+1,q}(z) = z + \sum_{m=2}^{\infty} \frac{[n+1,q]_{m-1}}{[m-1,q]!} z^m, \tag{4}$$

where the series is absolutely convergent in E.

The q-Ruscheweyh differential operator $D_q^n : A \to A$ of order $n \in \mathbb{N}_\circ$, $q \in (0,1)$ and for f given by (1) is defined as

$$\begin{aligned} D_q^n f(z) &= F_{n+1,q}(z) * f(z) \\ &= z + \sum_{m=2}^{\infty} \frac{[n+1,q]_{m-1}}{[m-1,q]!} a_m z^m, \quad \text{see Reference [2].} \end{aligned} \tag{5}$$

In addition,

$$D_q^0 f(z) = f(z) \quad \text{and} \quad D_q^1 f(z) = z\partial_q f(z).$$

Equation (5) can be written as

$$D_q^n f(z) = \frac{z\partial_q^n (z^{n-1} f(z))}{[n,q]!}, \quad n \in \mathbb{N}.$$

Since $\lim_{q\to 1^-} F_{n+1,q}(z) = \frac{z}{(1-z)^{n+1}}$, it follows that

$$\lim_{q\to 1} D_q^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = D^n f(z).$$

Throughout this paper, it is assumed that $q \in (0,1)$ and $z \in E$, unless otherwise stated.

2. Main Results

In this section, some new classes of analytic functions involving the q-Ruscheweyh derivative are introduced and some new results are derived.

Definition 1. *Let $f \in A$. Then, f is said to belong to the class ST_q, if*

$$\left| \frac{z}{f(z)} (\partial_q f)(z) - \frac{q}{1-q^2} \right| \le \frac{q}{1-q^2}, \quad \forall z \in E, \tag{6}$$

where $\partial_q f(z)$ is defined by (2) on the set B, $q \in (0,1)$.

Remark 1. *We note that as $q \to 1^-$, the disc $|w - \frac{q}{1-q^2}| \le \frac{q}{1-q^2}$ becomes the right half plane $\Re\{w\} > \alpha$, $\alpha \in (\frac{1}{2}, 1)$ and the class ST_q reduces to $S^*(\frac{1}{2})$.*

Following the similar method used in Reference [17], we note from (6) that $f \in ST_q$, if and only if

$$\frac{z\partial_q f(z)}{f(z)} \prec \frac{1}{1-qz}. \tag{7}$$

It can be seen from (7) that the transformation $\frac{1}{1-qz}$ maps $|z| = r$ onto the circle with center $C(r) = \frac{qr^2}{1-q^2r^2}$ and the radius $\sigma(r) = \frac{qr}{1-q^2r^2}$, which can be written as

$$\frac{1-qr+qr^2}{(1-qr)(1+qr)} \leq \left\{\Re\frac{z\partial_q f(z)}{f(z)}\right\} \leq \frac{1+qr+qr^2}{(1-qr)(1+qr)}. \tag{8}$$

Now, with $\partial_q(\log f(z)) = \frac{\partial_q f(z)}{f(z)}$, $\Re\frac{\partial_q f(z)}{f(z)} = r\frac{\partial_q \log|f(z)|}{dr}$ and some computation, (8) yields

$$\frac{1}{r} + \frac{q}{1+qr} \leq \frac{\partial_q \log|f(z)|}{dr} \leq \frac{1}{r} + \frac{q}{1-qr}. \tag{9}$$

Taking the q-integral on both sides of (9) together with some simplifications, we obtain the following result for the class ST_q.

Theorem 1. *Let $f \in ST_q$. Then,*

$$\frac{1}{(1+qr)^{qq_1}} \leq \left|\frac{f(z)}{z}\right| \leq \frac{1}{(1-qr)^{qq_1}}, \quad q_1 = \frac{1-q}{\log q^{-1}}. \tag{10}$$

Since $\lim_{q\to 1^-}\{\frac{1-q}{\log q^{-1}}\} = 1$, we obtain the well known distortion result for $f \in S^*(\frac{1}{2})$ as

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}.$$

Definition 2. *Let $f \in A$. Then, f is said to belong to the class $S_q^*(n,\alpha)$, if and only if*

$$\Re\left\{\frac{z\partial_q(D_q^n f(z))}{D_q^n f(z)}\right\} > \alpha, \quad 0 \leq \alpha < 1, \quad z \in E \tag{11}$$

and $D_q^n f$ is defined by (5).

As a special case, we have $f \in S_q^*(n,\alpha)$, $\alpha = \frac{1}{1+q}$, if and only if $D_q^n \in ST_q$, $z \in E$.

The following identity can easily be obtained from (5).

$$z\partial_q(D_q^n f(z)) = \left(1 + \frac{[n,q]}{q^n}\right) D_q^{n+1} f(z) - \frac{[n,q]}{q^n} D_q^n f(z). \tag{12}$$

If $q \to 1^-$, then

$$z(D^n f(z)) = (n+1)D^{n+1}f(z) - nD^n f(z),$$

which is the well known identity of the Ruscheweyeh derivative operator D^n.

Remark 2. *$\cap_{n=0}^{\infty} S_q^*(n,\alpha) = \{id\}$, where id is the identity function. Then, it follows trivially that $z \in S_q^*(n,\alpha)$ for $n \in \mathbb{N}_\circ$. On the contrary, assume $f \in \cap_{n=0}^{\infty} S_q^*(n,\alpha)$ with $f(z)$ given by* (1)*. Then, from* (5)*, we deduce that $f(z) = z$.*

With a similar argument used in Reference [23], it can easily be shown that

$$\cap_{0<q<1} S_q^*(n,\alpha) = S^*(n,\alpha).$$

Theorem 2. *$S_q^*(n+1,\alpha) \subset S_q^*(n,\alpha)$, $\alpha = \frac{1}{1+q}$, $n \in \mathbb{N}_\circ$.*

Proof. Let $f \in S_q^*(n+1, \alpha)$, $\alpha = \frac{1}{1+q}$.
Set

$$\frac{z\partial_q(D_q^n f(z))}{D_q^n f(z)} = p(z), \tag{13}$$

where $p(z)$ is analytic in E with $p(0) = 1$.
We can show that

$$p(z) \prec \frac{1}{1-qz}, \quad z \in E.$$

Differentiating (13) q-logarithmically and using identity (12), we have

$$\frac{z\partial_q(D_q^{n+1} f(z))}{D_q^{n+1} f(z)} = p(z) + \frac{z\partial_q p(z)}{p(z) + N_q}, \quad N_q = \frac{[n,q]}{q^n}. \tag{14}$$

Let

$$p(z) = \frac{1}{1 - q\phi(z)}. \tag{15}$$

Clearly, $\phi(z)$ is analytic in E and $\phi(0) = 0$. We can show S$|\phi(z)| < 1, \forall z \in E$.

Suppose on the contrary that there exists a $z_0 \in E$ such that $|\phi(z_0)| = 1$. Since $f \in S_q^*(n+1, \frac{1}{1+q})$, $\Re\{p(z) + \frac{z\partial_q p(z)}{p(z)+N_q}\} > \frac{1}{1+q}$, for $z \in E$ and $N_q = \frac{[n,q]}{q^n}$.

Now, from (15), it follows that

$$p(z) + \frac{z\partial_q p(z)}{p(z) + N_q} = \frac{1}{1 - q\phi(z)} + \frac{qz\partial_q \phi(z)}{(1 - q\phi(z))[(N_q + 1) - N_q q\phi(z)]}.$$

At $z = z_0$, we have

$$\Re\left\{p(z) + \frac{z\partial_q p(z)}{p(z) + N_q}\right\}_{z=z_0} = \Re\left\{\frac{1}{1 - q\phi(z_0)} + \frac{qz_0\partial_q \phi(z_0)}{(1 - q\phi(z_0))[(1 + N_q) - qN_q\phi(z_0)]}\right.. \tag{16}$$

If $\phi(z_0) = e^{i\theta}$, then

$$\frac{1}{1 - q\phi(z_0)} = \frac{1}{1 - qe^{i\theta}} = \frac{1 - q\cos\theta + iq\sin\theta}{(1 - 2q\cos\theta + q^2)},$$

and

$$\Re\frac{1}{1 - q\phi(z_0)} = \frac{1 - q\cos\theta}{1 - 2q\cos\theta + q^2}. \tag{17}$$

Using q-Jacks's Lemma given in Reference [23], we have

$$z_0\partial_q \phi(z_0) = k\phi(z_0), \quad k \geq 1. \tag{18}$$

Now, with $\theta = \pi$, we have from (16) and (18)

$$\begin{aligned} \Re\left\{\frac{qz_0\partial_q \phi(z_0)}{(1 - q\phi(z_0))((1 + N_q) - N_q q\phi(z_0))}\right\}_{\theta=\pi} &= \frac{-k(q+1)(1 + (1+q)N_q)}{(1+q)^2(1 + (1+q)N_q)^2} \\ &< 0. \end{aligned} \tag{19}$$

Hence, from (17) and (19), it follows that there exists $z_0 \in E$ such that

$$\Re\left\{\frac{z\partial_q(D_q^{n+1}f(z))}{D_q^{n+1}f(z)} - \frac{1}{1+q}\right\} < 0,$$

which is a contradiction to the given hypothesis. Thus, $|\phi(z)| < 1$ for all $z \in E$ and $p(z) \prec \frac{1}{1-qz}$, which proves $f \in S_q^*(n,\alpha)$, $\alpha = \frac{1}{1+q}$. □

Theorem 3. *Let $f \in S_q^*(n,\alpha)$ and let $I_{n,q}f : A \to A$ be defined as*

$$I_{n,q}f(z) = \frac{[n+1,q]}{z^n}\int_0^z t^{n-1}f(t)\mathrm{d}_q t, \quad n \in \mathbb{N}_\circ. \tag{20}$$

Then, $I_n f \in S_q^*(n+1,\alpha)$. For $q \to 1^-$, (20) represents Bernardi operator, see Reference [24].

The proof is straightforward, when we note from (12) and (20) that

$$D_q^n f = D_q^{n+1}(I_{n,q}f).$$

Lemma 1. *Let $f \in ST_q$ and let $f(z)$ be given by (1). Then,*

$$|a_m| \le \frac{C(q)\cdot m^{qq_1}}{[m,q]}, \quad q_1 = \frac{1-q}{\log q^{-1}}, \quad qq_1 > \frac{1}{2},$$

and $C(q)$ is a constant depending only on q.

Proof. By Cauchy theorem, for $z = re^{i\theta}$ and Cauchy–Schwartz inequality, we have

$$\begin{aligned}
[m,q]|a_m| &\le \frac{1}{2\pi r^m}\int_0^{2\pi}|f(z)p(z)|\mathrm{d}\theta, \qquad p(z) \prec \frac{1}{1-qz}, \\
&\le \frac{1}{r^m}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(z)|^2\mathrm{d}\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2\mathrm{d}\theta\right)^{\frac{1}{2}}.
\end{aligned} \tag{21}$$

If $p(z) \prec \frac{1}{1-qz}$ and $p(z) = 1 + \sum_{m=1}^{\infty} c_m z^m$, then $|c_m| \le q$ and

$$\int_0^{2\pi}|p(z)|^2\mathrm{d}\theta \le \frac{1+(4q^2-1)r^2}{1-r^2}. \tag{22}$$

Using Theorem 1 and the subordination principle, we obtain

$$\int_0^{2\pi}|f(z)|^2\mathrm{d}\theta \le \int_0^{2\pi}\frac{r^2\mathrm{d}\theta}{|1-qre^{i\theta}|^{2qq_1}} \le c_1(q)\left(\frac{1}{1-qr}\right)^{2qq_1-1}, \tag{23}$$

where c_1 is a constant.

From (21)–(23), it follows that

$$[m,q]|a_m| \le C(q)\cdot m^{qq_1}, \quad qq_1 > \frac{1}{2}.$$

This completes the proof. □

As a special case, if $q \to 1^-$, then $f \in S^*(\frac{1}{2})$ and $a_m = O(1)$.

Theorem 4. *Let*

$$F(z) = D_q^n f(z) = z + \sum_{m=2}^{\infty} A_m z^m,$$

where $D_q^n f(z)$ *is given by* (5) *and* $f \in S_q^*(n, \alpha)$, $\alpha = \frac{1}{1+q}$. *Then,*

$$a_m = O(1)\left(\frac{[m-1,q]!}{[m,q][n+1,q]_{m-1}}\right) \cdot m^{qq_1}, \quad qq_1 > \frac{1}{2},$$

where $O(1)$ *is a constant which depends only on* q.

The proof follows easily by using Lemma 1 and the definition that $D_q^n f \in ST_q$.

As a special case, we observe that $D_q^0 f = f$ and we have

$$a_m = \frac{O(1) \cdot m^{qq_1}}{[m,q]}.$$

For $q \to 1^-$, it yields the result $a_m = O(1)$.

3. Conclusions

In this paper, we have used q-calculus to define and study some new sub-classes of analytic functions involving the Ruscheweyh derivative.. Some interesting inclusion and subordination properties of these new classes have been derived. Some special cases have been discussed as applications of our main results. Applications of the q-Ruscheweyh differential operator in the real world will be an interesting and encouraging future study for researchers.

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