fractal and fractional

Article

On the Fractal Langevin Equation

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Abstract: In this paper, fractal stochastic Langevin equations are suggested, providing a mathematical model for random walks on the middle-τ Cantor set. The fractal mean square displacement of different random walks on the middle-τ Cantor set are presented. Fractal under-damped and over-damped Langevin equations, fractal scaled Brownian motion, and ultra-slow fractal scaled Brownian motion are suggested and the corresponding fractal mean square displacements are obtained. The results are plotted to show the details.

Keywords: local fractal calculus; fractal mean square displacement; middle-τ Cantor sets; fractal Langevin equations

1. Introduction

In the last decade, analysis on fractals has been developed by many researchers [1–8]. Harmonic analysis is used to define integrals and derivatives on fractal sets [4]. Probability theory is used to define Laplacians on fractals [9]. Fractional spaces are mapped to continuous real space in order to define differential equations on fractals [10–13]. Fractional calculus is applied in fractal spaces to explain anomalous diffusion [14–18]. Time-fractional continuum models with short memory are studied to model the evolution law for the damage variable for hyperelastic materials [19].

In a seminal paper, generalized standard calculus is formulated to define derivatives and integrals on totally disconnected fractal sets and fractal curves [20–23]. Recently, an extension of fractal calculus for the fractals embedding in 2D is formulated [24].

Mean square displacements of random walks having power law are modeled utilizing F^{α}-calculus to provide applications in statistical mechanics [23,25]. The over-damped Langevin equation is investigated, which describes dynamics of Brownian particles in the long time limit. The anomalous diffusion of particles in free cooling granular gases is modeled in [26].

In this paper, we suggest fractal under-damped and over-damped Langevin equations, fractal scaled Brownian motion, and ultra-slow fractal scaled Brownian motion. Using stochastic fractal differential equations, the fractal mean square displacement is derived, which leads to a new hierarchy of random walks.

The outline of the paper is as follows: In Section 2, we review basic tools. We define the fractal Langevin equation with different coefficients and work out the mean square displacement for under-damped and over-damped Langevin equations in Section 3. In Section 4, we present fractal ultra-slow and scaled Brownian motion and their fractal mean displacements. Finally, we conclude our results in Section 5.

2. Basic Tools

In this section, we give a short review of local generalized Riemman calculus on fractal middle-τ Cantor set.

2.1. Middle-τ Cantor Set

The middle-τ Cantor set created by following stages:

(I) Delete an open interval of length $0 < \tau < 1$ from the middle of the $I = [0, 1]$.

$$C_1^\tau = [0, \frac{1}{2}(1-\tau)] \cup [\frac{1}{2}(1+\tau), 1] \tag{1}$$

(II) Remove disjoint open intervals of length τ from the remaining sections of step I.

$$C_2^\tau = [0, \frac{1}{4}(1-\tau)^2] \cup [\frac{1}{4}(1-\tau^2), \frac{1}{2}(1-\tau)] \cup [\frac{1}{2}(1+\tau) + \frac{1}{2}((1+\tau) + \frac{1}{2}(1-\tau)^2)] \cup [\frac{1}{2}(1+\tau)(1+\frac{1}{2}(1-\tau)), 1]. \tag{2}$$

$$\vdots$$

(III) Pick up disjoint open intervals of length τ from the remaining sections of previous step, and so on ad infinitum.

$$C^\tau = \bigcap_{k=1}^{\infty} C_k^\tau \tag{3}$$

The Lebesgue measures of middle-τ Cantor sets are zero and their Hausdorff dimensions are given by

$$\dim_H(C^\tau) = \frac{\log 2}{\log 2 - \log(1-\tau)}, \tag{4}$$

where $H(C^\tau)$ is the Hausdorff measure [27].

2.2. Local Fractal Calculus

If C^τ is middle-τ Cantor set, then the flag function is defined by [20,21,23],

$$\mathbf{F}(C^\tau, \mathbf{J}) = \begin{cases} 1 & \text{if} \quad C^\tau \cap \mathbf{J} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

where $\mathbf{J} = [d_1, d_2]$. Then, $\mathcal{P}^\alpha[C^\tau, \mathbf{N}]$ is given in [20,21,23] by

$$\mathcal{P}^\alpha[C^\tau, \mathbf{N}] = \sum_{i=1}^{n} \Gamma(\alpha+1)(t_i - t_{i-1})^\alpha \mathbf{F}(C^\tau, [t_{i-1}, t_i]).$$

where $\mathbf{N}_{[d_1,d_2]} = \{d_1 = t_0, t_1, t_2, \ldots, t_n = d_2\}$ is a subdivisions of $\mathbf{J}$.

The **mass function** $\gamma^\alpha(C^\tau, d_1, d_2)$ is defined in [20,21,23] by

$$\gamma^\alpha(C^\tau, b_1, b_2) = \lim_{\delta \to 0} \mathcal{P}_\delta^\alpha[C^\tau, \mathbf{N}] = \lim_{\delta \to 0} \left(\inf_{\mathbf{N}_{[d_1,d_2]}:|\mathbf{N}| \leq \delta} \mathcal{P}^\alpha[C^\tau, \mathbf{N}] \right), \tag{6}$$

where infimum is taken over all subdivisions $\mathbf{N}$ of $[d_1, d_2]$ satisfying $|\mathbf{N}| := \max_{1 \leq i \leq n}(t_i - t_{i-1}) \leq \delta$.

The **integral staircase function** $S_{C^\tau}^\alpha(t)$ is defined in [20,21] by

$$S_{C^\tau}^\alpha(t) = \begin{cases} \gamma^\alpha(C^\tau, d_0, t) & \text{if} \quad t \geq d_0 \\ -\gamma^\alpha(C^\tau, d_0, t) & \text{otherwise,} \end{cases} \tag{7}$$

where d_0 is an arbitrary and fixed real number.

The γ**-dimension** of a set $C^\tau \cap [d_1, d_2]$ is defined as

$$\begin{aligned} \dim_\gamma(C^\tau \cap [d_1, d_2]) &= \inf\{\alpha : \gamma^\alpha(C^\epsilon, d_1, d_2) = 0\} \\ &= \sup\{\alpha : \gamma^\alpha(C^\tau, d_1, d_2) = \infty\}. \end{aligned} \quad (8)$$

The C^τ**-limit** of a function $g : C^\tau \to \Re$ is given by

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0 \ \ z \in C^\tau \ \text{ and } \ |z - t| < \delta \Rightarrow |g(z) - l| < \epsilon. \quad (9)$$

If l exists, then we have

$$l = C^\tau\text{-}\lim_{z \to t} g(z). \quad (10)$$

The C^τ**-continuity** of a function $g : C^\tau \to \Re$ is defined by

$$g(t) = C^\tau\text{-}\lim_{z \to t} g(z). \quad (11)$$

The C^τ**-derivative** of $f(t)$ at t is defined [20]

$$D^\alpha_{C^\tau} f(t) = \begin{cases} C^\tau\text{-}\lim\limits_{y \to t} \frac{f(y) - f(t)}{S^\alpha_{C^\tau}(y) - S^\alpha_{C^\tau}(t)} & \text{if, } \ t \in C^\tau, \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

if the limit exists.

In Figure 1, we plot middle-τ Cantor set, sharacteristic function, staircase function and γ-dimension for middle-τ Cantor set with $\tau = 5/7$. (The red line indicates upper bound of staircase function $\Gamma(1.35)t^{0.35}$).

The C^τ**-integral** of $k(t)$ on $\mathbf{J} = [d_1, d_2]$ is defined in [20,21,23] and approximately given by

$$\int_{d_1}^{d_2} k(t) d^\alpha_{C^\tau} t \approx \sum_{i=1}^{n} k(t_i)(S^\alpha_{C^\tau}(t_i) - S^\alpha_{C^\tau}(t_{i-1})). \quad (13)$$

For more details, we refer the reader to [20,21].

The **characteristic function of the middle-**τ **Cantor set** is defined in [23] by

$$\chi_{C^\tau}(\alpha, t) = \begin{cases} \frac{1}{\Gamma(\alpha+1)}, & t \in C^\tau, \\ 0, & otherwise. \end{cases} \quad (14)$$

The **delta function on middle-**τ **Cantor set**, which is called fractal Gaussian noise, is defined by

$$\delta^\gamma_{C^\tau}(t - t_1) = \begin{cases} \infty, & t = t_1 \\ 0, & t \neq t_1. \end{cases}$$

and

$$\int_{-\infty}^{+\infty} g(t) \delta^\gamma_{C^\tau}(t - t_1) d^\alpha_{C^\tau} t = g(t_1).$$

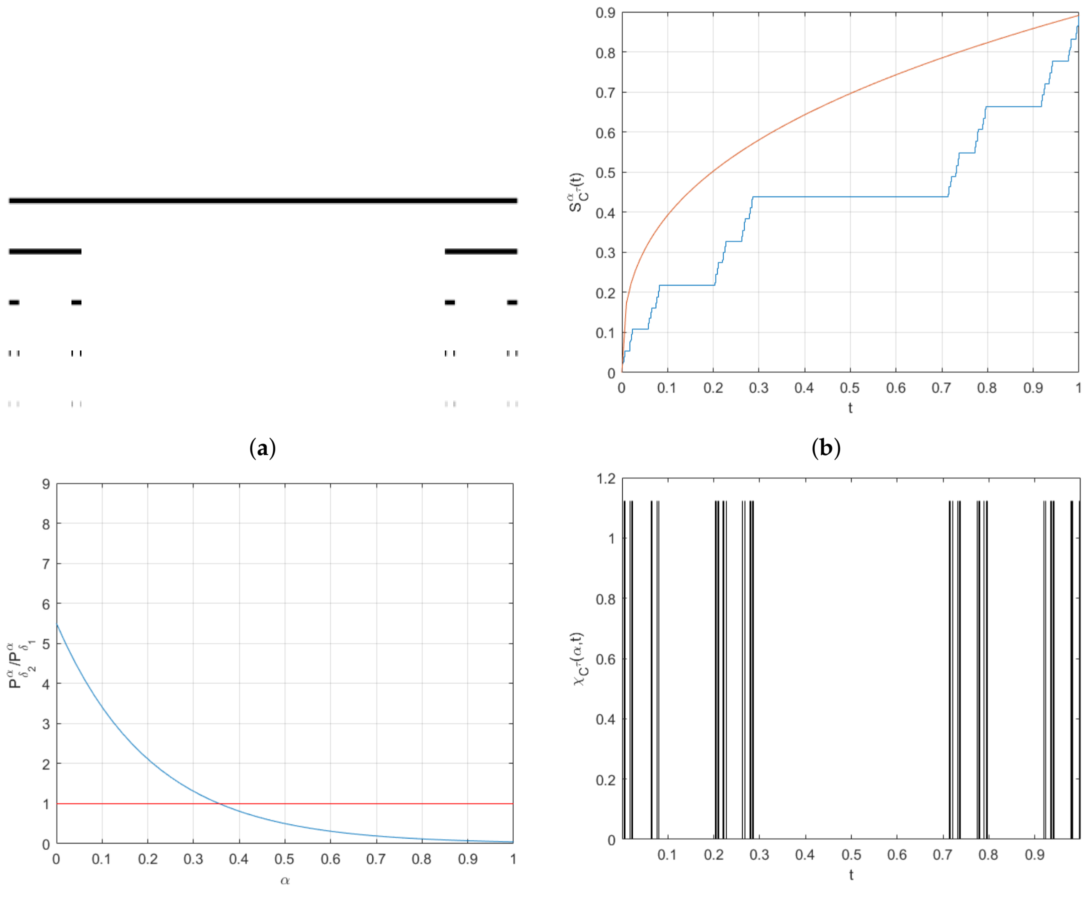

Figure 1. Figures for Section 2: (**a**) middle-τ Cantor set with $\tau = 5/7$; (**b**) staircase function of middle-τ Cantor set with $\tau = 5/7$; (**c**) the γ-dimension gives $\alpha = 0.35$ to middle-τ Cantor set with $\tau = 5/7$; and (**d**) characteristic function for middle-τ Cantor set with $\tau = 5/7$.

3. Fractal Langevin Equation with Different Coefficients

In this section, we study over-damped and under-damped Langevin equations.

3.1. Fractal Over-Damped Langevin Equation

Consider over-damped fractal Langevin equation

$$D^{\gamma}_{C^{\tau},t}x(t) = \sqrt{2D_0}\,\zeta(t), \quad K = C^{\tau}, \tag{15}$$

where D_0 ($\mathrm{m}^2\ \mathrm{s}^{-\gamma/\alpha}$) is coefficient of diffusion and $\zeta(t)$ is

$$\begin{aligned} <\zeta(t_2)\zeta(t_1)> &= \delta^{\gamma}_{K}(t_2 - t_1), \\ <\zeta(t)> &= 0. \end{aligned} \tag{16}$$

The fractal mean square displacement (FMSD) of random walk corresponding to Equation (15) is given by

$$< S^{\alpha}_{K}(x)^2 >= 2D_0 S^{\gamma}_{K}(t), \tag{17}$$

where α and γ are fractal space and time dimensions, respectively. Using upper bound of staircase function, namely

$$S^{\gamma}_{K}(t) < t^{\gamma}, \tag{18}$$

By substituting Equation (18) into Equation (17), we obtain

$$< x(t)^2 >\approx 2D_0 t^{\gamma/\alpha}. \tag{19}$$

In Figure 2, we plot Equation (19), in which the red, blue, and green lines are to super-, normal- and sub-diffusion, respectively.

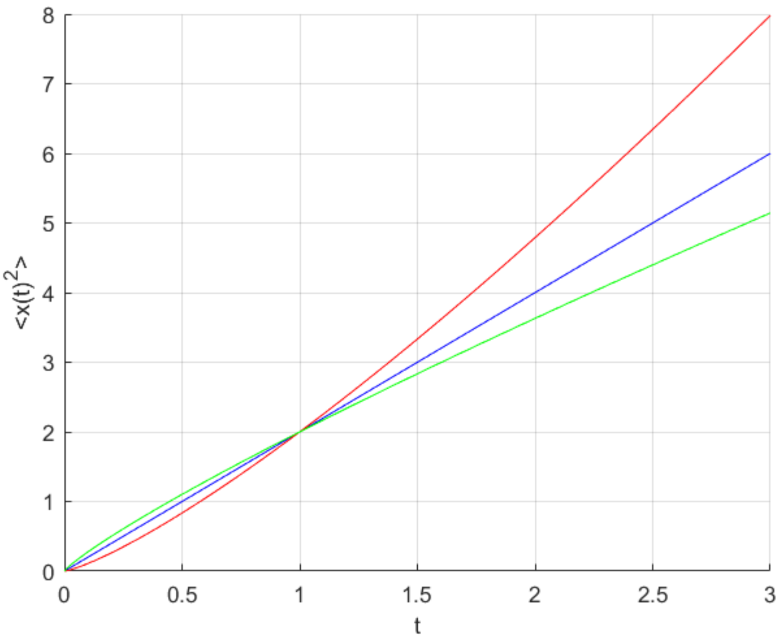

Figure 2. Graph of FMSD of over-damped Langevin equation setting $\gamma = 1$, $\alpha = 1$ (blue), $\gamma = 0.63$, $\alpha = 0.5$ (red), and $\gamma = 0.43$, $\alpha = 0.5$ (green).

3.2. Fractal Under-Damped Langevin Equation

Let us consider the fractal under-damped Langevin equation as follows

$$(D^{\gamma}_{C^{\tau},t})^2 x(t) + \gamma_0 D^{\gamma}_{C^{\tau},t} x(t) = \sqrt{2D_0}\, \gamma_0 \zeta(t), \tag{20}$$

where γ_0 ($s^{-\gamma/\alpha}$) and $D_0 = T/m\gamma_0$ are called fractal friction coefficient and fractal diffusion constant, respectively [26]. Let $D^{\gamma}_{K,t} x(t) = v_K(t)$, then, by Equation (20), we obtain

$$< v_K(t_1) v_K(t_2) >= \frac{T}{m} \exp\left(-\gamma_0 |S^{\gamma}_K(t_2) - S^{\gamma}_K(t_1)|\right). \tag{21}$$

Using Equation (21), we get

$$< S^{\alpha}_K(x)^2 >= 2D_0 S^{\gamma}_K(t) + \frac{2D_0}{\gamma_0}(e^{-\gamma_0 S^{\gamma}_K(t)} - 1). \tag{22}$$

which is named FMSD of the fractal under-damped Langevin equation. Utilizing upper bound of $S^{\gamma}_K(t) < t^{\gamma}$, we obtain

$$< x(t)^2 >\approx \left[2D_0 t^{\gamma} + \frac{2D_0}{\gamma_0}(e^{-\gamma_0 t^{\gamma}} - 1)\right]^{1/\alpha}. \tag{23}$$

Replacing the short time $t \ll \frac{1}{\gamma_0}$ into Equation (22), we obtain

$$< S^{\alpha}_K(x)^2 >\simeq D_0 \gamma_0 S^{\gamma}_K(t)^2. \tag{24}$$

By substituting long time $t \gg \frac{1}{\gamma_0}$ into Equation (22), one arrives at Equation (17). In Figure 3, we sketch Equation (23).

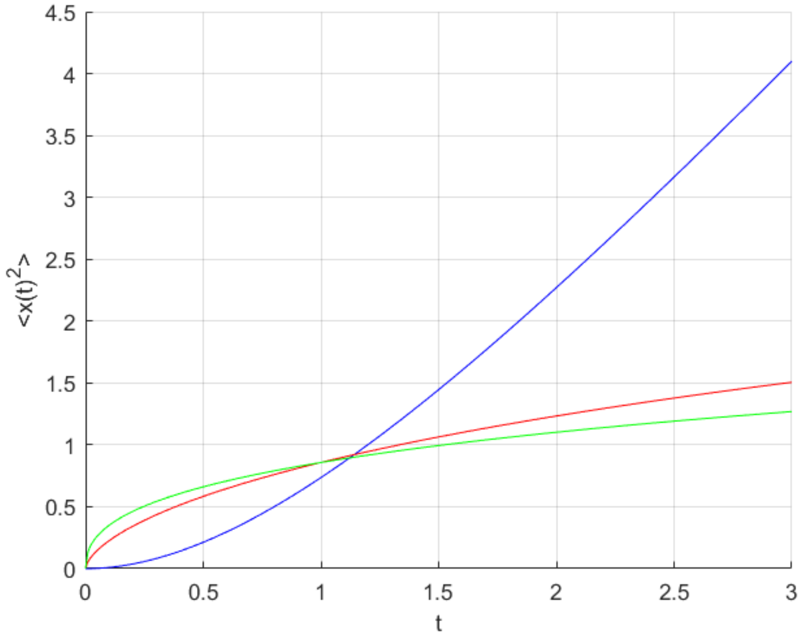

Figure 3. Graph of FMSD of under-damped Langevin equation setting $\gamma = 1,\ \alpha = 1$ (blue), $\gamma = 0.63,\ \alpha = 0.5$ (red), and $\gamma = 0.43,\ \alpha = 0.5$ (green).

4. Fractal Scaled Brownian Motion

The fractal stochastic Langevin equation, which is a model of fractal scaled Brownian motion, is given by

$$D^{\gamma}_{C^{\tau},t}x(t) = \sqrt{2D(t)}\,\zeta(t), \;\; t \in C^{\tau}, \tag{25}$$

with the condition in Equation (16) and

$$D(t) = D_0(1 + \frac{t}{\tau_0})^{\nu-1}, \; t \in K, \tag{26}$$

where $\nu \in [0,2]$ and τ_0 (s) are constant. Using Equation (26), we obtain FMSD in the following form

$$\begin{aligned} < S^{\alpha}_{K}(x)^2 > &= 2\int_0^t D(t')d^{\gamma}_{K}t' = \frac{2D_0\tau_0}{\nu}\left([1 + \frac{S^{\gamma}_{K}(t)}{\tau_0}]^{\nu} - 1\right), \\ < x(t)^2 > &\approx \frac{2D_0\tau_0}{\nu}\left([1 + \frac{t^{\gamma}}{\tau_0}]^{\nu} - 1\right)^{1/\alpha}. \end{aligned} \tag{27}$$

In Figure 4, we plot Equation (27). Replacing the short time $t \ll \tau_0$ into Equation (27), we have

$$< S^{\alpha}_{K}(x)^2 > \approx 2D_0\, S^{\gamma}_{K}(t), \tag{28}$$

and substituting the long time $t \gg \tau_0$ into Equation (27), we obtain

$$< S^{\alpha}_{K}(x)^2 > \approx S^{\gamma}_{K}(t)^{\nu}. \tag{29}$$

Consequently, Equation (29) covers both sub-and super-diffusive processes. Ultra-slow fractal scaled Brownian motion is obtained substituting $\nu = 0$ into Equation (26), and we have

$$D(t) = D_0(1 + \frac{t}{\tau_0})^{-1},\ t \in K. \tag{30}$$

In view of Equations (30) and (25), we thus obtain FMSD as follows:

$$< S_K^{\alpha}(x)^2 > \ = \ 2D_0\tau_0 \log\left(1 + \frac{S_K^{\gamma}(t)}{S_K^{\gamma}(\tau_0)}\right), \tag{31}$$

$$< x(t)^2 > \ \approx \ 2D_0\tau_0 \log\left(1 + \frac{t^{\gamma}}{\tau_0^{\gamma}}\right). \tag{32}$$

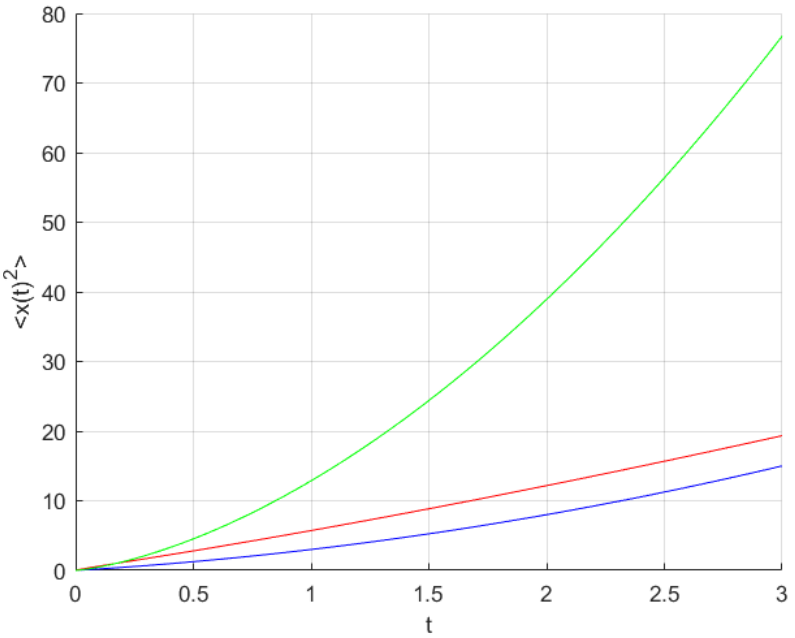

Figure 4. Graph of FMSD of scaled Brownian motion equation setting $\gamma = 1, \alpha = 1,\ \nu = 2$ (blue), $\gamma = 0.5, \alpha = 0.63,\ \nu = 2$ (red), and $\gamma = 0.5, \alpha = 0.43,\ \nu = 2$ (green).

Substituting long times $t \gg \tau_0$ into Equation (31) leads to

$$< S_K^{\alpha}(x)^2 > \sim \log(S_K^{\gamma}(t)). \tag{33}$$

By Equation (33) and recalling Equation (18), we get

$$< x(t)^2 > \approx (\log(t^{\gamma}))^{1/\alpha}. \tag{34}$$

Remark 1. *These results switch to the known result for the ordinary or scaled Brownian motion and classical Langevin equation by choosing* $\gamma = \alpha = 1$.

In Figure 5, we sketch Equation (34).

Remark 2. *We derive equations using the conjugacy of* C^{τ}*-calculus with the standard calculus [21,26].*

Remark 3. *Figures 2–5 show super-, normal and sub-diffusion for the models.*

Remark 4. *In Figures 2–5, we set* $\gamma_0 = \tau_0 = D_0 = 1$.

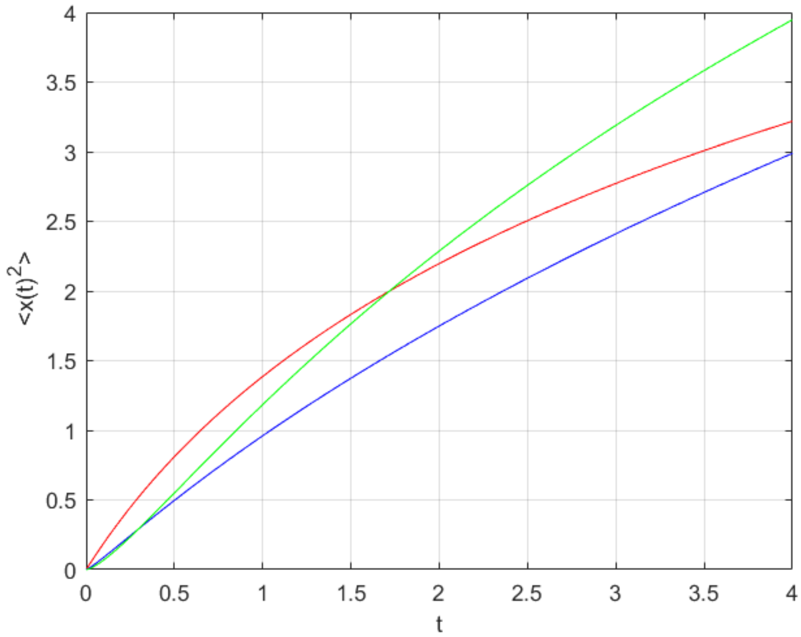

Figure 5. Graph of FMSD of ultra-slow fractal scaled Brownian motion setting $\gamma = 1,\ \alpha = 1$ (red), $\gamma = 0.63,\ \alpha = 0.5$ (blue), and $\gamma = 1,\ \alpha = 0.7$ (green).

5. Conclusions

In this work, we have studied fractal scaled Brownian motion, the fractal under-damped Langevin equation, and the fractal over-damped Langevin equation. The stochastic Langevin equations with different diffusion coefficients are considered to give different fractal mean square displacements. The results obtained in this manuscript are generalizations of the known results for the ordinary Langevin equation and scaled Brownian motion. Moreover, we obtain different conditions that are related to the dimensions of space and time.

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