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Some New Fractional Trapezium-Type Inequalities for Preinvex Functions

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Abstract: In this paper, authors the present the discovery of an interesting identity regarding trapezium-type integral inequalities. By using the lemma as an auxiliary result, some new estimates with respect to trapezium-type integral inequalities via general fractional integrals are obtained. It is pointed out that some new special cases can be deduced from the main results. Some applications regarding special means for different real numbers are provided as well. The ideas and techniques described in this paper may stimulate further research.

Keywords: trapezium-type integral inequalities; preinvexity; general fractional integrals

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K. The set of integrable functions on the interval $[a_1, a_2]$ is denoted by $L[a_1, a_2]$.

The following inequality obtained by Hermite and Hadamard is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a_1, a_2 \in I$ with $a_1 < a_2$. Then, the following inequality holds:*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1) + f(a_2)}{2}. \tag{1}$$

This inequality (1) is known as Hermite–Hadamard or trapezium inequality. As a result of the rich applications in the field of numerical analysis, this result has attracted many mathematicians attention from all over the world. For other recent results which generalize, improve, and extend the inequality (1) through various classes of convex functions interested readers are referred to References [1–33].
Let us recall some special functions and evoke some basic definitions as follows.

Definition 1 ([34])**.** *A set $S \subseteq \mathbb{R}^n$ is said to be an invex set with respect to the mapping $\eta : S \times S \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$.*

The invex set S is also termed an η-connected set.

Definition 2. *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f : S \longrightarrow [0, +\infty)$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)f(x) + tf(y). \tag{2}$$

Furthermore, let us define a function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty, \tag{3}$$

$$\frac{1}{A} \leq \frac{\varphi(s)}{\varphi(r)} \leq A \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{4}$$

$$\frac{\varphi(r)}{r^2} \leq B \frac{\varphi(s)}{s^2} \text{ for } s \leq r \tag{5}$$

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2} \right| \leq C|r - s| \frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{6}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then φ satisfies (3)–(6), see Reference [35]. Therefore, Sarıkaya and Ertuğral [28] defined the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$${}_{a_1^+}I_\varphi f(x) = \int_{a_1}^{x} \frac{\varphi(x - t)}{x - t} f(t)dt, \quad x > a_1, \tag{7}$$

$${}_{a_2^-}I_\varphi f(x) = \int_{x}^{a_2} \frac{\varphi(t - x)}{t - x} f(t)dt, \quad x < a_2. \tag{8}$$

This fractional integral operators are a new generalization of fractional integrals such as the Riemann–Liouville fractional integral, the k-Riemann–Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, Hadamard fractional integrals, etc. To read more about fractional analysis, see References [10,11,22,27].

Motivated by the above literature, the main objective of this paper is firstly to discover in Section 2 an interesting identity in order to establish some new bounds regarding trapezium-type integral inequalities. Then, using this lemma as an auxiliary result, some new estimates with respect to trapezium-type integral inequalities via general fractional integrals will be obtained. It is pointed out that some new special cases will be deduced from the main results. In Section 3, some applications regarding special means for different real numbers are given. The ideas and techniques described in this paper may stimulate further research in the field of integral inequalities.

2. Main Results

Throughout this study, let $P = [ma_1, a_2]$ with $a_1 < a_2$, $m \in (0,1]$ be an invex subset with respect to $\eta : P \times P \longrightarrow \mathbb{R}$. Additionally, for brevity, we define

$$\Lambda_m^*(t) := \int_0^t \frac{\varphi\left(\eta(a_2, ma_1 + a_2 - x)u\right)}{u} du < \infty, \quad \eta(a_2, ma_1 + a_2 - x) > 0 \tag{9}$$

and

$$\Delta_m^*(t) := \int_0^t \frac{\varphi\left(\eta(ma_1 + a_2 - x, ma_1)u\right)}{u} du < \infty, \quad \eta(ma_1 + a_2 - x, ma_1) > 0. \tag{10}$$

For establishing some new results regarding general fractional integrals we need to prove the following lemma.

Lemma 1. *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on (ma_1, a_2). If $f' \in L(P)$, then the following identity for generalized fractional integrals holds:*

$$\begin{aligned}
&\left[\frac{1}{2\Lambda_m^*(1)} \times {}_{(ma_1+a_2-x)^+}I_\varphi f\left(ma_1 + a_2 - x + \eta(a_2, ma_1 + a_2 - x)\right)\right. \\
&\left.+\frac{1}{2\Delta_m^*(1)} \times {}_{(ma_1+\eta(ma_1+a_2-x,ma_1))^-}I_\varphi f\left(ma_1\right)\right] \\
&-\frac{f(ma_1 + \eta(ma_1 + a_2 - x, ma_1)) + f(ma_1 + a_2 - x)}{2} \\
= \; &\frac{\eta(a_2, ma_1 + a_2 - x)}{2\Lambda_m^*(1)} \int_0^1 \Lambda_m^*(1-t) f'\left(ma_1 + a_2 - x + t\eta(a_2, ma_1 + a_2 - x)\right) dt \\
&-\frac{\eta(ma_1 + a_2 - x, ma_1)}{2\Delta_m^*(1)} \int_0^1 \Delta_m^*(t) f'\left(ma_1 + t\eta(ma_1 + a_2 - x, ma_1)\right) dt.
\end{aligned} \tag{11}$$

We denote

$$\begin{aligned}
&T_{f,\Lambda_m^*,\Delta_m^*}(x; a_1, a_2) := \frac{\eta(a_2, ma_1 + a_2 - x)}{2\Lambda_m^*(1)} \\
&\times \int_0^1 \Lambda_m^*(1-t) f'\left(ma_1 + a_2 - x + t\eta(a_2, ma_1 + a_2 - x)\right) dt \\
&-\frac{\eta(ma_1 + a_2 - x, ma_1)}{2\Delta_m^*(1)} \int_0^1 \Delta_m^*(t) f'\left(ma_1 + t\eta(ma_1 + a_2 - x, ma_1)\right) dt.
\end{aligned} \tag{12}$$

Proof. Integrating by parts (12), using (9) and (10) and changing the variables of integration, we have

$$
\begin{aligned}
& T_{f,\Lambda_m^*,\Delta_m^*}(x;a_1,a_2) \\
= & \frac{\eta(a_2,ma_1+a_2-x)}{2\Lambda_m^*(1)}\left\{\frac{\Lambda_m^*(1-t)f\left(ma_1+a_2-x+t\eta(a_2,ma_1+a_2-x)\right)}{\eta(a_2,ma_1+a_2-x)}\Bigg|_0^1\right. \\
& +\frac{1}{\eta(a_2,ma_1+a_2-x)} \\
& \left.\times\int_0^1\frac{\varphi\left(\eta(a_2,ma_1+a_2-x)(1-t)\right)}{1-t}f\left(ma_1+a_2-x+t\eta(a_2,ma_1+a_2-x)\right)dt\right\} \\
& -\frac{\eta(ma_1+a_2-x,ma_1)}{2\Delta_m^*(1)}\left\{\frac{\Delta_m^*(t)f\left(ma_1+t\eta(ma_1+a_2-x,ma_1)\right)}{\eta(ma_1+a_2-x,ma_1)}\Bigg|_0^1\right. \\
& -\frac{1}{\eta(ma_1+a_2-x,ma_1)} \\
& \left.\times\int_0^1\frac{\varphi\left(\eta(ma_1+a_2-x,ma_1)t\right)}{t}f\left(ma_1+t\eta(ma_1+a_2-x,ma_1)\right)dt\right\} \\
= & \frac{\eta(a_2,ma_1+a_2-x)}{2\Lambda_m^*(1)}\left\{-\frac{\Lambda_m^*(1)f(ma_1+a_2-x)}{\eta(a_2,ma_1+a_2-x)}+\frac{1}{\eta(a_2,ma_1+a_2-x)}\right. \\
& \left.\times\;{}_{(ma_1+a_2-x)^+}I_\varphi f\left(ma_1+a_2-x+\eta(a_2,ma_1+a_2-x)\right)\right\} \\
& -\frac{\eta(ma_1+a_2-x,ma_1)}{2\Delta_m^*(1)}\left\{\frac{\Delta_m^*(1)f\left(ma_1+\eta(ma_1+a_2-x,ma_1)\right)}{\eta(ma_1+a_2-x,ma_1)}-\frac{1}{\eta(ma_1+a_2-x,ma_1)}\right. \\
& \left.\times\;{}_{(ma_1+\eta(ma_1+a_2-x,ma_1))^-}I_\varphi f\left(ma_1\right)\right\} \\
= & \left[\frac{1}{2\Lambda_m^*(1)}\times\;{}_{(ma_1+a_2-x)^+}I_\varphi f\left(ma_1+a_2-x+\eta(a_2,ma_1+a_2-x)\right)\right. \\
& \left.+\frac{1}{2\Delta_m^*(1)}\times\;{}_{(ma_1+\eta(ma_1+a_2-x,ma_1))^-}I_\varphi f\left(ma_1\right)\right] \\
& -\frac{f(ma_1+\eta(ma_1+a_2-x,ma_1))+f(ma_1+a_2-x)}{2}.
\end{aligned}
$$

This completes the proof of the lemma. □

Remark 1. *Taking* $m = 1$, $\eta(ma_1 + a_2 - x, ma_1) = (ma_1 + a_2 - x) - ma_1$ *and* $\eta(a_2, ma_1 + a_2 - x) = a_2 - (ma_1 + a_2 - x)$ *in Lemma 1, we get*

$$T_{f,\Lambda_1^*,\Delta_1^*}(x; a_1, a_2) = \left[\frac{1}{2\Lambda_1^*(1)} \times {}_{(a_1+a_2-x)^+}I_\varphi f(a_2) + \frac{1}{2\Delta_1^*(1)} \times {}_{(a_1+a_2-x)^-}I_\varphi f(a_1)\right] - f(a_1 + a_2 - x). \tag{13}$$

Theorem 2. *Suppose that* $m \in (0, 1]$ *is a fixed number. Let* $f : P \longrightarrow \mathbb{R}$ *be a differentiable mapping on* (ma_1, a_2). *If* $|f'|^q$ *is preinvex on* P *for* $q > 1$ *and* $p^{-1} + q^{-1} = 1$, *then the following inequality for generalized fractional integrals holds:*

$$\begin{aligned} &\left|T_{f,\Lambda_m^*,\Delta_m^*}(x; a_1, a_2)\right| \\ \leq\ & \frac{\eta(a_2, ma_1 + a_2 - x)}{2\Lambda_m^*(1)} \sqrt[p]{B_{\Lambda_m^*}(p)} \sqrt[q]{\frac{|f'(ma_1 + a_2 - x)|^q + |f'(a_2)|^q}{2}} \\ &+ \frac{\eta(ma_1 + a_2 - x, ma_1)}{2\Delta_m^*(1)} \sqrt[p]{C_{\Delta_m^*}(p)} \sqrt[q]{\frac{|f'(ma_1)|^q + |f'(ma_1 + a_2 - x)|^q}{2}}, \end{aligned} \tag{14}$$

where

$$B_{\Lambda_m^*}(p) := \int_0^1 \left[\Lambda_m^*(1-t)\right]^p dt, \quad C_{\Delta_m^*}(p) := \int_0^1 \left[\Delta_m^*(t)\right]^p dt. \tag{15}$$

Proof. From Lemma 1, preinvexity of $|f'|^q$, Hölder inequality, and the properties of the modulus, we have

$$\begin{aligned} &\left|T_{f,\Lambda_m^*,\Delta_m^*}(x; a_1, a_2)\right| \\ \leq\ & \frac{\eta(a_2, ma_1 + a_2 - x)}{2\Lambda_m^*(1)} \int_0^1 \Lambda_m^*(1-t)\left|f'\left(ma_1 + a_2 - x + t\eta(a_2, ma_1 + a_2 - x)\right)\right| dt \\ &+ \frac{\eta(ma_1 + a_2 - x, ma_1)}{2\Delta_m^*(1)} \int_0^1 \Delta_m^*(t)\left|f'\left(ma_1 + t\eta(ma_1 + a_2 - x, ma_1)\right)\right| dt \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\eta(a_2, ma_1 + a_2 - x)}{2\Lambda_m^*(1)} \left(\int_0^1 \left[\Lambda_m^*(1-t) \right]^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 \left| f'\left(ma_1 + a_2 - x + t\eta(a_2, ma_1 + a_2 - x)\right) \right|^q dt \right)^{\frac{1}{q}} \\
&\quad + \frac{\eta(ma_1 + a_2 - x, ma_1)}{2\Delta_m^*(1)} \left(\int_0^1 \left[\Delta_m^*(t) \right]^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 \left| f'\left(ma_1 + t\eta(ma_1 + a_2 - x, ma_1)\right) \right|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{\eta(a_2, ma_1 + a_2 - x)}{2\Lambda_m^*(1)} \sqrt[p]{B_{\Lambda_m^*}(p)} \left(\int_0^1 \left[(1-t)\left|f'(ma_1 + a_2 - x)\right|^q + t\left|f'(a_2)\right|^q \right] dt \right)^{\frac{1}{q}} \\
&\quad + \frac{\eta(ma_1 + a_2 - x, ma_1)}{2\Delta_m^*(1)} \sqrt[p]{C_{\Delta_m^*}(p)} \left(\int_0^1 \left[(1-t)\left|f'(ma_1)\right|^q + t\left|f'(ma_1 + a_2 - x)\right|^q \right] dt \right)^{\frac{1}{q}} \\
&= \frac{\eta(a_2, ma_1 + a_2 - x)}{2\Lambda_m^*(1)} \sqrt[p]{B_{\Lambda_m^*}(p)} \sqrt[q]{\frac{|f'(ma_1 + a_2 - x)|^q + |f'(a_2)|^q}{2}} \\
&\quad + \frac{\eta(ma_1 + a_2 - x, ma_1)}{2\Delta_m^*(1)} \sqrt[p]{C_{\Delta_m^*}(p)} \sqrt[q]{\frac{|f'(ma_1)|^q + |f'(ma_1 + a_2 - x)|^q}{2}}.
\end{aligned}
$$

The proof of this theorem is complete. □

We point out some special cases of Theorem 2.

Corollary 1. *Taking* $m = 1$, $\eta(ma_1 + a_2 - x, ma_1) = (ma_1 + a_2 - x) - ma_1$ *and* $\eta(a_2, ma_1 + a_2 - x) = a_2 - (ma_1 + a_2 - x)$ *in Theorem 2, we get*

$$
\begin{aligned}
\left|T_{f,\Lambda_1^*,\Delta_1^*}(x; a_1, a_2)\right| &\leq \frac{(x - a_1)}{2\Lambda_1^*(1)} \sqrt[p]{B_{\Lambda_1^*}(p)} \sqrt[q]{\frac{|f'(a_1 + a_2 - x)|^q + |f'(a_2)|^q}{2}} \\
&\quad + \frac{(a_2 - x)}{2\Delta_1^*(1)} \sqrt[p]{C_{\Delta_1^*}(p)} \sqrt[q]{\frac{|f'(a_1)|^q + |f'(a_1 + a_2 - x)|^q}{2}}.
\end{aligned} \tag{16}
$$

Corollary 2. *Taking* $x = \dfrac{a_1 + a_2}{2}$ *in Corollary 1, we get*

$$
\begin{aligned}
\left| T_{f,\Lambda_1^*,\Delta_1^*}\left(\frac{a_1 + a_2}{2}; a_1, a_2\right) \right| &\leq \frac{(a_2 - a_1)}{4\sqrt[q]{2}\Lambda_1^*(1)} \left\{ \sqrt[p]{B_{\Lambda_1^*}(p)} \sqrt[q]{\left| f'\left(\frac{a_1 + a_2}{2}\right) \right|^q + |f'(a_2)|^q} \right. \\
&\quad \left. + \sqrt[p]{C_{\Delta_1^*}(p)} \sqrt[q]{|f'(a_1)|^q + \left| f'\left(\frac{a_1 + a_2}{2}\right) \right|^q} \right\}.
\end{aligned} \tag{17}
$$

Corollary 3. *Taking* $p = q = 2$ *in Theorem 2, we get*

$$
\begin{aligned}
\left|T_{f,\Lambda_m^*,\Delta_m^*}(x; a_1, a_2)\right| &\leq \frac{\eta(a_2, ma_1 + a_2 - x)}{2\Lambda_m^*(1)} \sqrt{B_{\Lambda_m^*}(2)} \sqrt{\frac{|f'(ma_1 + a_2 - x)|^2 + |f'(a_2)|^2}{2}} \\
&\quad + \frac{\eta(ma_1 + a_2 - x, ma_1)}{2\Delta_m^*(1)} \sqrt{C_{\Delta_m^*}(2)} \sqrt{\frac{|f'(ma_1)|^2 + |f'(ma_1 + a_2 - x)|^2}{2}}.
\end{aligned} \tag{18}
$$

Corollary 4. *Taking $\varphi(t) = t$ in Theorem 2, we get*

$$\left|T_{f,\Lambda_m^*,\Delta_m^*}(x;a_1,a_2)\right| \leq \frac{\eta(a_2, ma_1+a_2-x)}{2\sqrt[p]{p+1}}\sqrt[q]{\frac{|f'(ma_1+a_2-x)|^q+|f'(a_2)|^q}{2}} + \frac{\eta(ma_1+a_2-x, ma_1)}{2\sqrt[p]{p+1}}\sqrt[q]{\frac{|f'(ma_1)|^q+|f'(ma_1+a_2-x)|^q}{2}}. \tag{19}$$

Corollary 5. *Taking $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2, we get*

$$\left|T_{f,\Lambda_m^*,\Delta_m^*}(x;a_1,a_2)\right| \leq \frac{\eta(a_2, ma_1+a_2-x)}{2\sqrt[p]{p\alpha+1}}\sqrt[q]{\frac{|f'(ma_1+a_2-x)|^q+|f'(a_2)|^q}{2}} + \frac{\eta(ma_1+a_2-x, ma_1)}{2\sqrt[p]{p\alpha+1}}\sqrt[q]{\frac{|f'(ma_1)|^q+|f'(ma_1+a_2-x)|^q}{2}}. \tag{20}$$

Corollary 6. *Taking $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2, we get*

$$\left|T_{f,\Lambda_m^*,\Delta_m^*}(x;a_1,a_2)\right| \leq \frac{\eta(a_2, ma_1+a_2-x)}{2\sqrt[p]{\frac{p\alpha}{k}+1}}\sqrt[q]{\frac{|f'(ma_1+a_2-x)|^q+|f'(a_2)|^q}{2}} + \frac{\eta(ma_1+a_2-x, ma_1)}{2\sqrt[p]{\frac{p\alpha}{k}+1}}\sqrt[q]{\frac{|f'(ma_1)|^q+|f'(ma_1+a_2-x)|^q}{2}}. \tag{21}$$

Theorem 3. *Suppose that $m \in (0,1]$ is a fixed number. Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on (ma_1, a_2). If $|f'|^q$ is preinvex on P for $q \geq 1$, then the following inequality for generalized fractional integrals holds:*

$$\begin{aligned} &\left|T_{f,\Lambda_m^*,\Delta_m^*}(x;a_1,a_2)\right| \\ \leq\ & \frac{\eta(a_2, ma_1+a_2-x)}{2\Lambda_m^*(1)}\left(B_{\Lambda_m^*}(1)\right)^{1-\frac{1}{q}}\sqrt[q]{D_{\Lambda_m^*}|f'(ma_1+a_2-x)|^q + E_{\Lambda_m^*}|f'(a_2)|^q} \\ &+\frac{\eta(ma_1+a_2-x, ma_1)}{2\Delta_m^*(1)}\left(C_{\Delta_m^*}(1)\right)^{1-\frac{1}{q}}\sqrt[q]{F_{\Delta_m^*}|f'(ma_1)|^q + G_{\Delta_m^*}|f'(ma_1+a_2-x)|^q}, \end{aligned} \tag{22}$$

where

$$D_{\Lambda_m^*} := \int_0^1 t\Lambda_m^*(t)dt, \quad E_{\Lambda_m^*} := \int_0^1 t\Lambda_m^*(1-t)dt, \tag{23}$$

$$F_{\Delta_m^*} := \int_0^1 (1-t)\Delta_m^*(t)dt, \quad G_{\Delta_m^*} := \int_0^1 t\Delta_m^*(t)dt, \tag{24}$$

and $B_{\Lambda_m^*}(1)$, $C_{\Delta_m^*}(1)$ are defined as in Theorem 2.

Proof. From Lemma 1, the preinvexity of $|f'|^q$, the power mean inequality, and the properties of the modulus, we have

$$\begin{aligned}
& \left|T_{f,\Lambda_m^*,\Delta_m^*}(x;a_1,a_2)\right| \\
\leq\ & \frac{\eta(a_2,ma_1+a_2-x)}{2\Lambda_m^*(1)}\int_0^1 \Lambda_m^*(1-t)\left|f'\left(ma_1+a_2-x+t\eta(a_2,ma_1+a_2-x)\right)\right|dt \\
& +\frac{\eta(ma_1+a_2-x,ma_1)}{2\Delta_m^*(1)}\int_0^1 \Delta_m^*(t)\left|f'\left(ma_1+t\eta(ma_1+a_2-x,ma_1)\right)\right|dt
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\eta(a_2,ma_1+a_2-x)}{2\Lambda_m^*(1)}\left(\int_0^1 \Lambda_m^*(1-t)dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 \Lambda_m^*(1-t)\left|f'\left(ma_1+a_2-x+t\eta(a_2,ma_1+a_2-x)\right)\right|^q dt\right)^{\frac{1}{q}} \\
& +\frac{\eta(ma_1+a_2-x,ma_1)}{2\Delta_m^*(1)}\left(\int_0^1 \Delta_m^*(t)dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 \Delta_m^*(t)\left|f'\left(ma_1+t\eta(ma_1+a_2-x,ma_1)\right)\right|dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{\eta(a_2,ma_1+a_2-x)}{2\Lambda_m^*(1)}\left(B_{\Lambda_m^*}(1)\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 \Lambda_m^*(1-t)\left[(1-t)\left|f'(ma_1+a_2-x)\right|^q+t\left|f'(a_2)\right|^q\right]dt\right)^{\frac{1}{q}} \\
& +\frac{\eta(ma_1+a_2-x,ma_1)}{2\Delta_m^*(1)}\left(C_{\Delta_m^*}(1)\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 \Delta_m^*(t)\left[(1-t)\left|f'(ma_1)\right|^q+t\left|f'(ma_1+a_2-x)\right|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{\eta(a_2,ma_1+a_2-x)}{2\Lambda_m^*(1)}\left(B_{\Lambda_m^*}(1)\right)^{1-\frac{1}{q}} \\
& \times\sqrt[q]{D_{\Lambda_m^*}|f'(ma_1+a_2-x)|^q+E_{\Lambda_m^*}|f'(a_2)|^q} \\
& +\frac{\eta(ma_1+a_2-x,ma_1)}{2\Delta_m^*(1)}\left(C_{\Delta_m^*}(1)\right)^{1-\frac{1}{q}}\sqrt[q]{F_{\Delta_m^*}|f'(ma_1)|^q+G_{\Delta_m^*}|f'(ma_1+a_2-x)|^q}.
\end{aligned}$$

The proof of this theorem is complete. □

We point out some special cases of Theorem 3.

Corollary 7. *Taking $m=1$, $\eta(ma_1+a_2-x,ma_1)=(ma_1+a_2-x)-ma_1$ and $\eta(a_2,ma_1+a_2-x)=a_2-(ma_1+a_2-x)$ in Theorem 3, we get*

$$\begin{aligned}
& \left|T_{f,\Lambda_m^*,\Delta_m^*}(x;a_1,a_2)\right| \\
\leq\ & \frac{(x-a_1)}{2\Lambda_1^*(1)}\left(B_{\Lambda_1^*}(1)\right)^{1-\frac{1}{q}}\sqrt[q]{D_{\Lambda_1^*}|f'(a_1+a_2-x)|^q+E_{\Lambda_1^*}|f'(a_2)|^q} \\
& +\frac{(a_2-x)}{2\Delta_1^*(1)}\left(C_{\Delta_1^*}(1)\right)^{1-\frac{1}{q}}\sqrt[q]{F_{\Delta_1^*}|f'(a_1)|^q+G_{\Delta_1^*}|f'(a_1+a_2-x)|^q}.
\end{aligned} \tag{25}$$

Corollary 8. *Taking* $x = \dfrac{a_1 + a_2}{2}$ *in Corollary 7, we get*

$$
\begin{aligned}
&\left| T_{f,\Lambda_1^*,\Delta_1^*}\left(\frac{a_1+a_2}{2}; a_1, a_2\right) \right| \\
\leq\ & \frac{(a_2 - a_1)}{4\Lambda_1^*(1)} \left\{ \left(B_{\Lambda_1^*}(1)\right)^{1-\frac{1}{q}} \sqrt[q]{D_{\Lambda_1^*}\left| f'\left(\frac{a_1+a_2}{2}\right)\right|^q + E_{\Lambda_1^*}|f'(a_2)|^q} \right. \\
&\left. + \left(C_{\Delta_1^*}(1)\right)^{1-\frac{1}{q}} \sqrt[q]{F_{\Delta_1^*}|f'(a_1)|^q + G_{\Delta_1^*}\left| f'\left(\frac{a_1+a_2}{2}\right)\right|^q} \right\}.
\end{aligned} \tag{26}
$$

Corollary 9. *Taking* $q = 1$ *in Theorem 3, we get*

$$
\begin{aligned}
&\left| T_{f,\Lambda_m^*,\Delta_m^*}(x; a_1, a_2) \right| \\
\leq\ & \frac{\eta(a_2, ma_1 + a_2 - x)}{2\Lambda_m^*(1)} \left[D_{\Lambda_m^*}|f'(ma_1 + a_2 - x)| + E_{\Lambda_m^*}|f'(a_2)| \right] \\
&+ \frac{\eta(ma_1 + a_2 - x, ma_1)}{2\Delta_m^*(1)} \left[F_{\Delta_m^*}|f'(ma_1)| + G_{\Delta_m^*}|f'(ma_1 + a_2 - x)| \right].
\end{aligned} \tag{27}
$$

Corollary 10. *Taking* $\varphi(t) = t$ *in Theorem 3, we get*

$$
\begin{aligned}
&\left| T_{f,\Lambda_m^*,\Delta_m^*}(x; a_1, a_2) \right| \\
\leq\ & \frac{\eta(a_2, ma_1 + a_2 - x)}{4\sqrt[q]{3}} \sqrt[q]{2|f'(ma_1 + a_2 - x)|^q + |f'(a_2)|^q} \\
&+ \frac{\eta(ma_1 + a_2 - x, ma_1)}{4\sqrt[q]{3}} \times \sqrt[q]{|f'(ma_1)|^q + 2|f'(ma_1 + a_2 - x)|^q}.
\end{aligned} \tag{28}
$$

Corollary 11. *Taking* $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ *in Theorem 3, we get*

$$
\begin{aligned}
&\left| T_{f,\Lambda_m^*,\Delta_m^*}(x; a_1, a_2) \right| \\
\leq\ & \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \frac{\eta^{\frac{\alpha+q-1}{q}}(a_2, ma_1 + a_2 - x)}{\Gamma(\alpha+1)} \sqrt[q]{\frac{|f'(ma_1 + a_2 - x)|^q}{\alpha+2} + \beta(2, \alpha+1)|f'(a_2)|^q} \\
&+ \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \frac{\eta^{\frac{\alpha+q-1}{q}}(ma_1 + a_2 - x, ma_1)}{\Gamma(\alpha+1)} \times \sqrt[q]{\beta(2, \alpha+1)|f'(ma_1)|^q + \frac{|f'(ma_1 + a_2 - x)|^q}{\alpha+2}}.
\end{aligned} \tag{29}
$$

Corollary 12. *Taking* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem 3, we get*

$$
\begin{aligned}
&\left| T_{f,\Lambda_m^*,\Delta_m^*}(x; a_1, a_2) \right| \\
\leq\ & \left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}} \frac{\eta^{\frac{\frac{\alpha}{k}+q-1}{q}}(a_2, ma_1 + a_2 - x)}{\Gamma_k(\alpha+k)} \sqrt[q]{\frac{|f'(ma_1 + a_2 - x)|^q}{\frac{\alpha}{k}+2} + \beta\left(2, \frac{\alpha}{k}+1\right)|f'(a_2)|^q} \\
&+ \left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}} \frac{\eta^{\frac{\frac{\alpha}{k}+q-1}{q}}(ma_1 + a_2 - x, ma_1)}{\Gamma_k(\alpha+k)} \sqrt[q]{\beta\left(2, \frac{\alpha}{k}+1\right)|f'(ma_1)|^q + \frac{|f'(ma_1 + a_2 - x)|^q}{\frac{\alpha}{k}+2}}.
\end{aligned} \tag{30}
$$

3. Applications to Special Means

Consider the following special means for different real numbers α, β, and $\alpha\beta \neq 0$, as follows:

1. The arithmetic mean:
$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2},$$
2. The harmonic mean:
$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}},$$
3. The logarithmic mean:
$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln|\beta| - \ln|\alpha|},$$
4. The generalized log-mean:
$$L_n := L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)}\right]^{\frac{1}{n}}; \;\; n \in \mathbb{Z} \setminus \{-1, 0\}.$$

It is well known that L_n is monotonic nondecreasing over $n \in \mathbb{Z}$ with $L_{-1} := L$. In particular, we have the following inequality $H \leq L \leq A$. Now, using the theory results in Section 2, we give some applications regarding special means for different real numbers.

Proposition 1. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$. Then, for $r \geq 2\,(r \in \mathbb{Z})$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds:*

$$\begin{aligned} &\left| \frac{L_r\left(ma_1 + a_2 - x, ma_1 + a_2 - x + \eta(a_2, ma_1 + a_2 - x)\right) + L_r\left(ma_1, ma_1 + \eta(ma_1 + a_2 - x, ma_1)\right)}{2} \right. \\ &\left. - A\left((ma_1 + \eta(ma_1 + a_2 - x, ma_1))^r, (ma_1 + a_2 - x)^r\right) \right| \\ \leq\; & \frac{r\eta(a_2, ma_1 + a_2 - x)}{2\sqrt[p]{p+1}} \times \sqrt[q]{A\left(|ma_1 + a_2 - x|^{q(r-1)}, |a_2|^{q(r-1)}\right)} \\ &+ \frac{r\eta(ma_1 + a_2 - x, ma_1)}{2\sqrt[p]{p+1}} \times \sqrt[q]{A\left(|ma_1|^{q(r-1)}, |ma_1 + a_2 - x|^{q(r-1)}\right)}. \end{aligned} \tag{31}$$

Proof. Applying Corollary 4 for $f(t) = t^r$, one can obtain the result immediately. □

Proposition 2. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$. Then, for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds:*

$$\begin{aligned} &\left| \frac{1}{2L\left(|ma_1 + a_2 - x|, |ma_1 + a_2 - x + \eta(a_2, ma_1 + a_2 - x)|\right)} \right. \\ &\left. + \frac{1}{2L\left(|ma_1|, |ma_1 + \eta(ma_1 + a_2 - x, ma_1)|\right)} - \frac{1}{H\left(ma_1 + \eta(ma_1 + a_2 - x, ma_1), ma_1 + a_2 - x\right)} \right| \\ \leq\ & \frac{\eta(a_2, ma_1 + a_2 - x)}{2\sqrt[p]{p+1}} \sqrt[q]{\frac{1}{H\left(|ma_1 + a_2 - x|^{2q}, |a_2|^{2q}\right)}} \\ &+ \frac{\eta(ma_1 + a_2 - x, ma_1)}{2\sqrt[p]{p+1}} \sqrt[q]{\frac{1}{H\left(|ma_1|^{2q}, |ma_1 + a_2 - x|^{2q}\right)}}. \end{aligned} \tag{32}$$

Proof. Applying Corollary 4 for $f(t) = \frac{1}{t}$, one can obtain the result immediately. □

Proposition 3. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$. Then, for $r \geq 2$ $(r \in \mathbb{Z})$ and $q \geq 1$, the following inequality holds:*

$$\begin{aligned} &\left| \frac{L_r\left(ma_1 + a_2 - x, ma_1 + a_2 - x + \eta(a_2, ma_1 + a_2 - x)\right) + L_r\left(ma_1, ma_1 + \eta(ma_1 + a_2 - x, ma_1)\right)}{2} \right. \\ &\left. - A\left((ma_1 + \eta(ma_1 + a_2 - x, ma_1))^r, (ma_1 + a_2 - x)^r\right) \right| \\ \leq\ & \sqrt[q]{\frac{2}{3}} \frac{\eta(a_2, ma_1 + a_2 - x)}{4} \sqrt[q]{A\left(2|ma_1 + a_2 - x|^{q(r-1)}, |a_2|^{q(r-1)}\right)} \\ &+ \sqrt[q]{\frac{2}{3}} \frac{\eta(ma_1 + a_2 - x, ma_1)}{4} \sqrt[q]{A\left(|ma_1|^{q(r-1)}, 2|ma_1 + a_2 - x|^{q(r-1)}\right)}. \end{aligned} \tag{33}$$

Proof. Applying Corollary 10 for $f(t) = t^r$, one can obtain the result immediately. □

Proposition 4. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$. Then, for $q \geq 1$, the following inequality holds:*

$$\begin{aligned} &\left| \frac{1}{2L\left(|ma_1 + a_2 - x|, |ma_1 + a_2 - x + \eta(a_2, ma_1 + a_2 - x)|\right)} \right. \\ &+ \frac{1}{2L\left(|ma_1|, |ma_1 + \eta(ma_1 + a_2 - x, ma_1)|\right)} \\ &\left. - \frac{1}{H\left(ma_1 + \eta(ma_1 + a_2 - x, ma_1), ma_1 + a_2 - x\right)} \right| \\ \leq\ & \frac{\eta(a_2, ma_1 + a_2 - x)}{4\sqrt[q]{3}} \sqrt[q]{\frac{4}{H\left(|ma_1 + a_2 - x|^{2q}, 2|a_2|^{2q}\right)}} \\ &+ \frac{\eta(ma_1 + a_2 - x, ma_1)}{4\sqrt[q]{3}} \sqrt[q]{\frac{4}{H\left(2|ma_1|^{2q}, |ma_1 + a_2 - x|^{2q}\right)}}. \end{aligned} \tag{34}$$

Proof. Applying Corollary 10 for $f(t) = \frac{1}{t}$, one can obtain the result immediately. □

Remark 2. *Applying Theorems 2 and 3 for the appropriate choices of function* $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\varphi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ *for* $\alpha \in (0,1)$, *such that* $|f'|^q$ *is preinvex, we can deduce some new general fractional integral inequalities using the above special means. The details are left to the interested reader.*

Remark 3. *Also, in Remark 2, if we choose* $\eta(y,x) = y - x$, $\forall\, x, y \in P$, *we can deduce some new general fractional integral inequalities for convex functions using above special means. The details are left to the interested reader.*

4. Conclusions

On the basis of a new identity regarding trapezium-type integral inequalities, some new trapezium-type integral inequalities via generalized fractional integral operators are established. Some special cases are consider that are derived from the main results. Furthermore, some applications regarding special means of real numbers are given.

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