Nonlocal Cauchy Problem via a Fractional Operator Involving Power Kernel in Banach Spaces

Aysel Keten 1, Mehmet Yavuz 1,*, and Dumitru Baleanu 2,3

1 Department of Mathematics-Computer, Faculty of Science, Necmettin Erbakan University, Konya 42090, Turkey; aksen@erbakan.edu.tr
2 Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, Ankara 06530, Turkey; dumitru@cankaya.edu.tr
3 Institute of Space Sciences, Magurele, P.O.Box MG-23, Ro 077125 Bucharest, Romania

* Correspondence: mehmetyavuz@erbakan.edu.tr; Tel.: +90-332-323-8220 (ext. 5548)

Received: 25 April 2019; Accepted: 13 May 2019; Published: 16 May 2019

Abstract: We investigated existence and uniqueness conditions of solutions of a nonlinear differential equation containing the Caputo–Fabrizio operator in Banach spaces. The mentioned derivative has been proposed by using the exponential decay law and hence it removed the computational complexities arising from the singular kernel functions inherit in the conventional fractional derivatives. The method used in this study is based on the Banach contraction mapping principle. Moreover, we gave a numerical example which shows the applicability of the obtained results.

Keywords: existence-uniqueness conditions; nonlocal Cauchy problem; Caputo–Fabrizio fractional derivative; Banach space

1. Introduction and Some Preliminaries

Modeling real-life problems with fractional differential equations (FDE) has a significant role in recent years. Some significant definitions that deal with fractional derivatives have been developed by Coimbra, Davison–Essex, Riesz, Riemann–Liouville, Hadamard, Grunwald–Letnikov, and Caputo [1,2]. Novel solution methods of such problems have been investigated by using these fractional derivative operators [3–9]. Moreover, in the last decades, new fractional derivative operators have been defined by using an exponential kernel called Caputo–Fabrizio (CF) [10] and the Mittag–Leffler kernel called Atangana–Baleanu (AB) [11]. These operators are very efficient for modeling complex nonlinear fractional dynamical systems and solving them. Caputo and Fabrizio have given a different perspective to fractional operators by introducing a new fractional operator without a singular kernel. Actually, if the CF operator was compared with the classical Caputo derivative, it can be seen that the new derivative with an exponential kernel has rapid stabilization in accordance with the memory effect. This definition comes naturally from the constitutive equation relating to the flux and gradient by exponential damping functions. In addition to being a very useful mathematical definition, it is an operator that is highly preferred in terms of physical meaning [12,13].

Some illustrative applications of the CF operator in various fields where the nonlocality appears in real world phenomena and more information about CF operator can be found in [14–18]. On the other hand, the AB fractional operator is defined with Mittag–Leffler function (MLF) and since the MLF is considered a nonlocal function, the kernel of AB derivative is nonlocal. Some researchers have studied this operator by applying it to a physical problem [19], a model of groundwater [20], initial and boundary value problems [21], and comparing it with the Liouville–Caputo fractional operator in terms of the solutions of nonlinear fractional equations [22] and comparing it with the CF derivative operator [23] as well as many others [24–34].
It is well known that the Cauchy problem (CP) consists of a differential equation with initial conditions. Starting with the work of Peano in the 1890s, analysis have had a continuing interest in the Cauchy problem. One of the crucial problems in the theory of Cauchy problems was finding the conditions that guarantee the existence of solutions of CPs.

We considered the following Cauchy problem:

\[ \dot{x}(\xi) = f(\xi, x), \quad x(\xi_0) = x_0, \]  

where \( f : [\xi_0, \xi_0 + a] \times \mathcal{E} \to \mathcal{E}, a > 0 \) and \( \mathcal{E} \) is a Banach space. A survey of the research history of this problem shows that a solution of Equation (1) (a continuous function \( x(\xi) : [\xi_0, \xi_0 + L] \to \mathcal{E} \) such that it satisfies Equation (1)) always exists if \( \text{dim} \mathcal{E} < \infty, L \) is small enough, and \( f(\xi, x) \) is a continuous function (this result is also known as Peano’s theorem), where \( \text{dim} \mathcal{E} \) is the dimension of the Banach space \( \mathcal{E} \). On the other hand, Cauchy problems in infinite-dimensional spaces may have no solutions. That is, there is no guarantee of the validity of Peano’s theorem in infinite-dimensional Banach spaces. Dieudonné [35] provided the first example of a continuous map from an infinitely dimensional Banach space \( E \) for which there is no solution to the related Cauchy problem in Equation (1). Afterwards, Godunov [36] proved that Peano’s theorem is false in every infinite-dimensional Banach space. More precisely, for every infinite-dimensional Banach space \( \mathcal{E}, \xi_0 \in R, u_0 \in \mathcal{E} \), there exists a continuous mapping \( f : R \times \mathcal{E} \to \mathcal{E}, \) such that there exists no solution of Equation (1). Therefore, determining existence and uniqueness conditions (EUC) of solutions of a DE in Banach spaces is important. It is possible to find a few different approaches to EUC of real-life problems defined with non-integer order derivative in the literature. Among them, Balachandran and Trujillo [37] studied the existence of nonlinear FDEs solutions in the Caputo sense in Banach spaces. Lakshmikantham and Devi [38] discussed the general theory of FDEs in Banach spaces. Benchohra and Seba [39] studied the existence of nonlinear FDEs solutions in the Caputo sense in Banach spaces. Wang et al. [40] developed two sufficient conditions for nonlocal controllability for fractional evolution systems. Lv et al. [41] employed about a new existence and uniqueness theorem for solutions of a special equation by using a Caputo fractional derivative in a Banach space.

In this study, after giving preliminary material, we obtained EUC of solutions of the following Cauchy problem with nonlocal initial conditions (nonlocal Cauchy problem) which includes the Caputo–Fabrizio operator in a Banach space \( \mathcal{E} \). Let us consider:

\[ \begin{align*}
  \mathcal{C}^{\text{F}}D_{\xi}^{\alpha}\omega(\mu) &= T\omega(\mu) + h(\mu, \omega(\mu)), \quad 0 \leq \mu \leq 1, \\
  \omega(0) &= \int_{0}^{1}g(\xi)\omega(\xi)d\xi,
\end{align*} \]

where \( \mathcal{C}^{\text{F}}D_{\xi}^{\alpha} \) is the Caputo–Fabrizio derivative of order \( \alpha \in (0, 1) \), \( g : [0, 1] \to [0, 1] \) is a continuous function, and \( h : [0, 1] \times \mathcal{E} \to \mathcal{E} \) is a given function, \( T : \mathcal{E} \to \mathcal{E} \) is a given operator satisfying some assumptions that will be specified in Section 3.

2. Preliminaries

We begin by introducing some notations and basic terminology. \( \mathcal{E} \) will always represent real Banach spaces. We will denote with the notation \( L(\mathcal{E}, \mathcal{E}) \) the Banach space of all linear bounded operators from \( \mathcal{E} \) to \( \mathcal{E} \). We also show with \( C(J, \mathcal{E}) \), the complete space of all continuous functions from \( J = [0, 1] \) to \( \mathcal{E} \), with the norm \( \| \omega \| = \max_{\xi \in J} \| \omega(\xi) \| \). Let \( c_0 \) be the Banach space of all null sequences with norm \( \| \omega \| = \sup_{n \in N} \| \omega_n \| \). Let \( L^1([0, 1], \mathcal{E}) \) be the Banach space of measurable functions \( x : [0, 1] \to \mathcal{E} \) which are Lebesgue integrable, equipped with norm \( \| x \| = \int_{0}^{1} \| x(s) \| ds \). Let \( \beta = \int_{0}^{1}g(\xi)d\xi \) and \( R_+ = [0, +\infty) \). A function \( w \in C(J, \mathcal{E}) \) which satisfies Equations (2) and (3) is called as a solution of these equations.
Definition 1. Let \( \alpha \in (0, 1) \), \( b > 0 \), and \( \omega \in C^1(0, b) \). The Caputo–Fabrizio fractional derivative \( CF D_\mu^\alpha \omega(\mu) \) of order \( \alpha \) is defined by:
\[
CF D_\mu^\alpha \omega(\mu) = \frac{(2 - \alpha)M(\alpha)}{(2 - \alpha)M(\alpha) - \alpha} \frac{1}{2(1 - \alpha)} \int_0^\mu e^{-\frac{1}{\beta}(\mu - \xi)} \omega'(\xi)d\xi, \quad \mu \geq 0,
\]
where \( M(\alpha) \) is a normalization constant depending on \( \alpha \) [10,14].

Definition 2. Let \( \alpha \in (0, 1) \). The Caputo–Fabrizio fractional integral \( CF I_\mu^\alpha \omega(\mu) \) of order \( \alpha \), is given by [14]:
\[
CF I_\mu^\alpha \omega(\mu) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} \omega(\mu) + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^\mu \omega(\xi)d\xi, \quad \mu \geq 0.
\]

Remark 1. Imposing [14]:
\[
\frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} + \frac{2\alpha}{(2 - \alpha)M(\alpha)} = 1,
\]
it is obtained an explicit formula for \( M(\alpha) \),
\[
M(\alpha) = \frac{2}{2 - \alpha}.
\]

Remark 2. Let \( \alpha \in (0, 1) \). Then we have [14,42]:
\[
CF D_\mu^\alpha(CF I_\mu^\alpha \omega(\mu)) = \omega(\mu) \quad \text{and} \quad CF I_\mu^\alpha(CF D_\mu^\alpha \omega(\mu)) = \omega(\mu) - \omega(0).
\]

3. Main Results

Now, we are ready to prove the uniqueness and existence of the solutions for Equations (2) and (3) under the following hypotheses:

(I) \( T \in L_c(E, E) \).

(II) \( h \in C(J \times E, E) \) and there exist a \( p_h \in L^1([0,1], \mathbb{R}^+) \) such that \( \|h(\mu, v)\| \leq p_h(\mu)\|v\| \) for \( \mu \in J \) and each \( v \in E \).

(III) \( H : J \to E, H(\cdot) = h(\cdot, \omega(\cdot)) \) is a differentiable function, for any \( \omega \in C^1(J, E) \).

(IV) There exists a constant \( L \) such that \( \|T\| + L < 1 - \beta \) and:
\[
\|h(\xi, v) - h(\xi, \vartheta)\| \leq L\|v - \vartheta\| \quad \text{for every} \quad v, \vartheta \in E.
\]

Lemma 1. If the conditions (I), (II), and (III) are satisfied then Equations (2) and (3) are equivalent to the following equation:
\[
\omega(\mu) = a_\alpha [T\omega(\mu) + h(\mu, \omega(\mu))] + b_\alpha \int_0^\mu [T\omega(\xi) + h(\xi, \omega(\xi))]d\xi \\
+ \frac{a_\alpha}{1 - \beta} \int_0^1 G(\xi) [T\omega(\xi) + h(\xi, \omega(\xi))]d\xi + \frac{b_\alpha}{1 - \beta} \int_0^1 [T\omega(\xi) + h(\xi, \omega(\xi))]G(\xi)d\xi,
\]
where \( G(\xi) = \int_\xi^1 G(\xi)d\xi \), \( a_\alpha = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} \) and \( b_\alpha = \frac{2\alpha}{(2 - \alpha)M(\alpha)} \).

Proof. By considering Remark 1 and Equations (2) and (3), we get:
\[
\omega(\mu) = a_\alpha [T\omega(\mu) + h(\mu, \omega(\mu))] + b_\alpha \int_0^\mu [T\omega(\xi) + h(\xi, \omega(\xi))]d\xi + \omega(0).
\]
So, by using the initial condition in Equation (3) we get:

\[
\omega(0) = \int_0^1 g(\xi)\omega(\xi)d\xi \\
= \int_0^1 g(\xi) \left[ a_\alpha[T\omega(\xi) + g(\xi, \omega(\xi))] + b_\alpha \int_0^\xi [T\omega(\varphi) + h(\varphi, \omega(\varphi))]d\varphi + \omega(0) \right] d\xi
\]

\[
= \omega(0) \int_0^1 g(\xi)d\xi + \int_0^1 g(\xi) \left[ a_\alpha[T\omega(\xi) + h(\xi, \omega(t))] + b_\alpha \int_0^\xi [T\omega(\varphi) + h(\varphi, \omega(\varphi))]d\varphi \right] d\xi
\]

\[
= \frac{a_\alpha}{1-\beta} \int_0^1 g(\xi)[T\omega(\xi) + h(\xi, \omega(\xi))]d\xi + \frac{b_\alpha}{1-\beta} \int_0^1 [T\omega(\varphi) + h(\varphi, \omega(\varphi))] G(\varphi)d\varphi.
\]

Therefore,

\[
\omega(0) = \frac{a_\alpha}{1-\beta} \int_0^1 g(\xi)[T\omega(\xi) + h(\xi, \omega(\xi))]d\xi + \frac{b_\alpha}{1-\beta} \int_0^1 [T\omega(\varphi) + h(\varphi, \omega(\varphi))] G(\varphi)d\varphi.
\]

So, substituting \( \omega(0) \) in Equation (10) we obtain:

\[
\omega(\mu) = a_\alpha[T\omega(\mu) + h(\mu, \omega(\mu))] + b_\alpha \int_0^\mu [T\omega(\tau) + h(\tau, \omega(\tau))]d\tau
\]

\[
+ \frac{a_\alpha}{1-\beta} \int_0^1 g(\xi)[T\omega(\xi) + h(\xi, \omega(\xi))]d\xi + \frac{b_\alpha}{1-\beta} \int_0^1 [T\omega(\varphi) + h(\varphi, \omega(\varphi))] G(\varphi)d\varphi.
\]

Conversely if \( \omega \) is a solution of Equation (10), then for every \( \mu \in J \), according to Remark 2, we have:

\[
{}^{CF}D_\mu^\alpha \omega(\mu) = {}^{CF}D_\mu^\alpha \left( a_\alpha[T\omega(\mu) + h(\mu, \omega(\mu))] + b_\alpha \int_0^\mu [T\omega(\varphi) + h(\varphi, \omega(\varphi))]d\varphi \right)
\]

\[
+ \frac{a_\alpha}{1-\beta} \int_0^1 g(\xi)[T\omega(\xi) + h(\xi, \omega(\xi))]d\xi + \frac{b_\alpha}{1-\beta} \int_0^1 [T\omega(\varphi) + h(\varphi, \omega(\varphi))] G(\varphi)d\varphi.
\]

It is obvious that \( \omega(0) = \int_0^1 g(\xi)\omega(\xi)d\xi \). Therefore, the proof is completed. \( \square \)

**Theorem 1.** If the conditions (I), (II), (III), and (IV) are satisfied then the nonlocal Cauchy problem in Equations (2) and (3) has a unique solution on \( C(J, E) \).

**Proof.** We now consider the operator:

\[
\psi : C(J, E) \rightarrow C(J, E)
\]
defined by:

\[
(\psi \omega)(\mu) = a_n [T \omega(\mu) + h(\mu, \omega(\mu))] + b_n \int_{0}^{\mu} [T \omega(q) + h(q, \omega(q))] dq \\
+ \frac{a_n}{1 - \beta} \int_{0}^{1} g(\xi) [T \omega(\xi) + h(\xi, \omega(\xi))] d\xi + \frac{b_n}{1 - \beta} \int_{0}^{1} [T \omega(q) + h(q, \omega(q))] G(q) dq.
\]

We will show that the operator \( \psi \) is well defined via assumptions. For this aim, we must prove that \( \psi(\omega) \in C(J, \mathcal{E}) \) for every \( \omega \in C(J, \mathcal{E}) \).

Let \( \mu_1, \mu_2 \in J, \mu_1 < \mu_2 \). We deduce that:

\[
\| (\psi \omega)\mu_1 - (\psi \omega)\mu_2 \| = \| a_n [T(\omega(\mu_1) - \omega(\mu_2)) + h(\mu_1, \omega(\mu_1)) - h(\mu_2, \omega(\mu_2))] \\
+ b_n \int_{\mu_1}^{\mu_2} [T(\omega(q) + h(q, \omega(q))] dq \|
\leq a_n \| T \| \| \omega(\mu_1) - \omega(\mu_2) \| + a_n \| h(\mu_1, \omega(\mu_1)) - h(\mu_2, \omega(\mu_2)) \|
+ b_n \int_{\mu_1}^{\mu_2} \| T(\omega(q) + h(q, \omega(q)) - h(q, 0) + h(q, 0) \| dq
\leq a_n \| T \| \| \omega(\mu_1) - \omega(\mu_2) \| + a_n \| h(\mu_1, \omega(\mu_1)) - h(\mu_2, \omega(\mu_1)) \|
+ a_n \| L \| \| \omega(\mu_1) - \omega(\mu_2) \| + b_n \| T \| \| \omega \| + L \| \omega \| + \max_{\omega \in \mathcal{E}} \| h(\mu, 0) \| (\mu_2 - \mu_1)
\leq a_n \{ 1 - \beta \} \| \omega(\mu_1) - \omega(\mu_2) \| + a_n \| h(\mu_1, \omega(\mu_1)) - h(\mu_2, \omega(\mu_1)) \|
+ b_n \{ 1 - \beta \} \| \omega \| + \max_{\omega \in \mathcal{E}} \| h(\mu, 0) \| (\mu_2 - \mu_1).
\]

As \( \mu_1 \to \mu_2 \), the right-hand side of the above inequality tends to zero. Thus, \( \psi \) is well defined.

We must show that \( \psi \) is a contracting mapping. For this, let \( \omega, \omega_2 \in C(J, \mathcal{E}) \), and \( \mu \in J \). Then, we have:

\[
\| (\psi \omega_1)\mu - (\psi \omega_2)\mu \| \leq a_n \| T \| \| \omega_1(\mu) - \omega_2(\mu) \| + a_n \| h(\mu, \omega_1(\mu)) - h(\mu, \omega_2(\mu)) \|
+ b_n \{ (T(\| T \| + L) \| \omega_1(\mu) - \omega_2(\mu) \| + b_n \{ (T(\| T \| + L) \| \omega_1 - \omega_2 \|
+ a_n \| \| T \| + L \| \| \omega_1 - \omega_2 \| \| + b_n \| (T(\| T \| + L) \| \omega_1 - \omega_2 \|
\leq \| T \| + L \| \omega_1 - \omega_2 \|.
\]

Since \( 0 < \frac{\| T \| + L}{1 - \beta} < 1 \), then \( \psi \) is a contraction mapping and therefore there exists a unique fixed point \( \omega \in C(J, \mathcal{E}) \) such that \( \psi(\omega(\mu)) = \omega(\mu) \). Any fixed point of \( \psi \) is the solution of Equations (2) and (3).

\[\square\]

Example 1. Let us consider the infinite system of scalar fractional functional differential equations:

\[
{\text{CF}} D_{\nu}^{\alpha} \omega_n(\mu) = \frac{\omega_n(\mu)}{50.2^n} + \frac{\mu \sin \alpha_n(\mu) - \omega_n(\mu)}{e^\mu - 1 + 50.2^n}, \quad \mu \in J, \quad \alpha \in (0, 1),
\]

\[
\omega_n(0) = \int_{0}^{1} \frac{1}{4} \omega_n(\mu) d\mu, \quad n = 1, 2, 3, ...
\]

(11)
Let $E$ be the Banach space $c_0$. Then the infinite system Equation (11) can be regarded as a problem of for Equations (2) and (3) in $E$. In this case:

$$\omega = (\omega_1, \omega_2, \omega_3, \ldots), \quad T(\omega) := \left( \frac{\omega_n}{50.2^n} \right)_{n=1}^{\infty},$$

$$h(\mu, \omega) := (h_1(\mu, \omega), h_2(\mu, \omega), h_3(\mu, \omega), \ldots) \text{ in which } h_n(\mu, \omega) = \frac{\mu \sin \omega_n - \omega_n}{\mu^{1/1} - 1 + 50.2^n}$$

and $g(\mu) = \frac{1}{4}$.

Therefore, it is obvious that $T \in L(E, E)$ and $h \in C(J \times E, E)$. On the other hand, for every $\mu \in J$ and $\omega \in E$, we have:

$$\|h(\mu, \omega)\| = \sup_{n \in \mathbb{N}} \|h_n(\mu, \omega)\| = \sup_{n \in \mathbb{N}} \left| \frac{\mu \sin \omega_n - \omega_n}{\mu^{1/1} - 1 + 50.2^n} \right| \leq \frac{\mu |\sin \omega_n| + |\omega_n|}{\mu^{1/1} - 1 + 50.2^n} = \frac{1}{2} \|\omega\|,$$

$$p_h(\mu) = 2e^{-\mu} \in L^1([0, 1], R^+).$$

Moreover, for every $\mu \in J$, $H(\mu) = h(\mu, \omega(\mu))$ is a differentiable function and:

$$\|h(\mu, \omega) - h(\mu, \varpi)\| = \sup_{n \in \mathbb{N}} \left| \frac{\mu \sin \omega_n - \omega_n}{\mu^{1/1} - 1 + 50.2^n} - \frac{\mu \sin \varpi_n - \varpi_n}{\mu^{1/1} - 1 + 50.2^n} \right| \leq \frac{1}{100} \sup_{n \in \mathbb{N}} |\mu \sin \omega_n - \omega_n - \mu \sin \varpi_n + \varpi_n| \leq \frac{2}{100} \|\omega - \varpi\| \text{ for all } \omega, \varpi \in E.$$

Since $L = \frac{2}{100}$, $\beta = \frac{3}{4}$ then we obtain that:

$$\|T\| + L \leq \frac{3}{100} < 1 - \beta = \frac{3}{4}.$$

Thus, by Theorem 1, we can show that the infinite system in Equation (11) has a unique solution.

4. Conclusions

In the present study, the existence and uniqueness conditions of the special type nonlinear fractional differential equations were obtained in the Caputo–Fabrizio fractional derivative sense. These conditions were constructed in Banach spaces via the Banach contraction principle mapping. Moreover, the applicability and the effectiveness of the results were confirmed with an illustrative numerical example.

Author Contributions: A.K. and M.Y. conceived the manuscripts, obtained the solution and wrote the paper. D.B. analyzed the paper; all authors read and approved the final manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.
References


27. Dos Santos, M. Fractional Prabhakar Derivative in Diffusion Equation with Non-Static Stochastic Resetting. Physics 2019, 1, 40–58. [CrossRef]


