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The g-Good-Neighbor Diagnosability of Bubble-Sort Graphs under Preparata, Metze, and Chien's (PMC) Model and Maeng and Malek's (MM)* Model

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Abstract: Diagnosability of a multiprocessor system is an important topic of study. A measure for fault diagnosis of the system restrains that every fault-free node has at least g fault-free neighbor vertices, which is called the g-good-neighbor diagnosability of the system. As a famous topology structure of interconnection networks, the n-dimensional bubble-sort graph B_n has many good properties. In this paper, we prove that (1) the 1-good-neighbor diagnosability of B_n is $2n-3$ under Preparata, Metze, and Chien's (PMC) model for $n \geq 4$ and Maeng and Malek's (MM)* model for $n \geq 5$; (2) the 2-good-neighbor diagnosability of B_n is $4n-9$ under the PMC model and the MM* model for $n \geq 4$; (3) the 3-good-neighbor diagnosability of B_n is $8n-25$ under the PMC model and the MM* model for $n \geq 7$.

Keywords: interconnection network; graph; diagnosability; PMC model; MM* model; bubble-sort graph

1. Introduction

A multiprocessor system and interconnection network (networks for short) have an underlying topology, which is usually presented by a graph, where nodes represent processors and links represent communication links between processors. We use graphs and networks interchangeably. For the system, some processors may fail in the system, so processor fault identification plays an important role in reliable computing. The first step to deal with faults is to identify the faulty processors from the fault-free ones. The identification process is called the diagnosis of the system. A system is said to be t-diagnosable if all faulty processors can be identified without replacement, provided that the number of faulty processors presented does not exceed t. The diagnosability $t(G)$ of a system G is the maximum value of t such that G is t-diagnosable. Several diagnosis models (e.g., Preparata, Metze, and Chien's (PMC) model [1], Barsi, Grandoni, and Maestrini's (BGM) model [2], and Maeng and Malek's (MM) model [3]) have been proposed to investigate the diagnosability of multiprocessor systems. In particular, two of the proposed models, the PMC model and MM model, are well known and widely used. In the PMC model, the diagnosis of the system is achieved through two linked processors testing each other. In the MM model, to diagnose a system, a node sends the same task to two of its neighbor vertices, and then compares their responses. Sengupta and Dahbura [4] proposed a special case of the MM model, called the MM* model, in which each node must test all the pairs of its adjacent nodes. In 2012, Peng et al. [5] proposed a measure for fault diagnosis of the system, namely, the g-good-neighbor diagnosability of the system (which is also called g-good-neighbor conditional diagnosability), which requires that every fault-free node contains at least g fault-free neighbors. In [5], they studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the PMC model. Numerous studies have been investigated under the PMC model and MM model or MM* model, see [1–21].

In this paper, we prove that (1) the diagnosability of n-dimensional bubble-sort graph B_n is $n-1$ under the PMC model for $n \geq 4$; (2) the 1-good-neighbor diagnosability of B_n is $2n-3$ under the PMC model for $n \geq 4$ and the MM* model for $n \geq 5$; (3) the 2-good-neighbor diagnosability of B_n is $4n-9$ under the PMC model and the MM^* model for $n \geq 4$; (4) the 3-good-neighbor diagnosability of B_n is $8n-25$ under the PMC model and the MM* model for $n \geq 7$.

2. Preliminaries

In this section, some definitions and notations needed are introduced for our discussion, then bubble-sort graphs will be introduced.

2.1. Definitions and Notations

A multiprocessor system is modeled as an undirected simple graph $G = (V, E)$, whose vertices (nodes) represent processors and edges (links) represent communication links. Given a nonempty vertex subset V' of V, the induced subgraph by V' in G, denoted by $G[V']$, is a graph, whose vertex set is V' and the edge set is the set of all the edges of G with both endpoints in V'. The degree $d_G(v)$ of a vertex v is the number of edges incident with v. We denote by $\delta(G)$ the minimum degrees of vertices of G. For any vertex v, we define the neighborhood $N_G(v)$ of v in G to be the set of vertices adjacent to v. u is called a neighbor vertex or a neighbor of v for $u \in N_G(v)$. Let $S \subseteq V$. We use $N_G(S)$ to denote the set $\cup_{v\in S} N_G(v) \setminus S$. For neighborhoods and degrees, we will usually omit the subscript for the graph when no confusion arises. A graph G is said to be k-regular if for any vertex v, $d_G(v) = k$. A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y ; such a partition (X, Y) is called a bipartition of the graph, and X and Y its parts. We denote a bipartite graph G with bipartition (X, Y) by $G = (X, Y; E)$. If $G = (X, Y; E)$ is simple and every vertex in X is joined to every vertex in Y, then $G = (X, Y; E)$ is called a complete bipartite graph, denoted by $K_{n,m}$, where $|X| = n$ and $|Y| = m$. Let $G = (V, E)$ be a connected graph. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left. A fault set $F \subseteq V$ is called a g-good-neighbor faulty set if $|N(v) \cap (V \setminus F)| \geq g$ for every vertex v in $V \setminus F$. A g-good-neighbor cut of a graph G is a g-good-neighbor faulty set F such that $G - F$ is disconnected. The minimum cardinality of g-good-neighbor cuts is said to be the g-good-neighbor connectivity of G, denoted by $\kappa^{(g)}(G)$. For graph-theoretical terminology and notation not defined here we follow [22].

2.2. The Bubble-Sort Graph

The bubble-sort graph has been known as a famous topology structure of interconnection networks. In this section, its definition and some useful properties are introduced.

In the permutation $\binom{1\ \ 2\ \ \cdots\ \ n}{p_1\ p_2\ \cdots\ p_n}$, $i \longrightarrow p_i$. For the convenience, we denote the permutation $\binom{1\ \ 2\ \ \cdots\ \ n}{p_1\ p_2\ \cdots\ p_n}$ by $p_1\ p_2\ \ldots\ p_n$. Every permutation can be denoted by a product of cycles [23]. For example, $\binom{1\ 2\ 3}{3\ 1\ 2} = (132)$. Specially, $\binom{1\ 2\ \cdots\ n}{1\ 2\ \cdots\ n} = (1)$. The product $\sigma\tau$ of two permutations is the composition function τ followed by σ, for example, $(12)(13) = (132)$. For terminology and notation not defined here we follow [23].

Let $[n] = \{1, 2, \cdots, n\}$, and let S_n be the symmetric group on $[n]$ containing all permutations $p = p_1 p_2 \cdots p_n$ of $[n]$. It is well known that $\{(i, i+1) : i = 1, 2, \ldots, n-1\}$ is a generating set for S_n. The n-dimensional bubble-sort graph B_n [24] is the graph with vertex set $V(B_n) = S_n$ in which two vertices u, v are adjacent if and only if $u = v(i, i+1)$, $1 \leq i \leq n-1$. It is easy to see from the definition that B_n is a $(n-1)$-regular graph on $n!$ vertices. The graphs B_2, B_3 and B_4 are depicted in Figure 1.

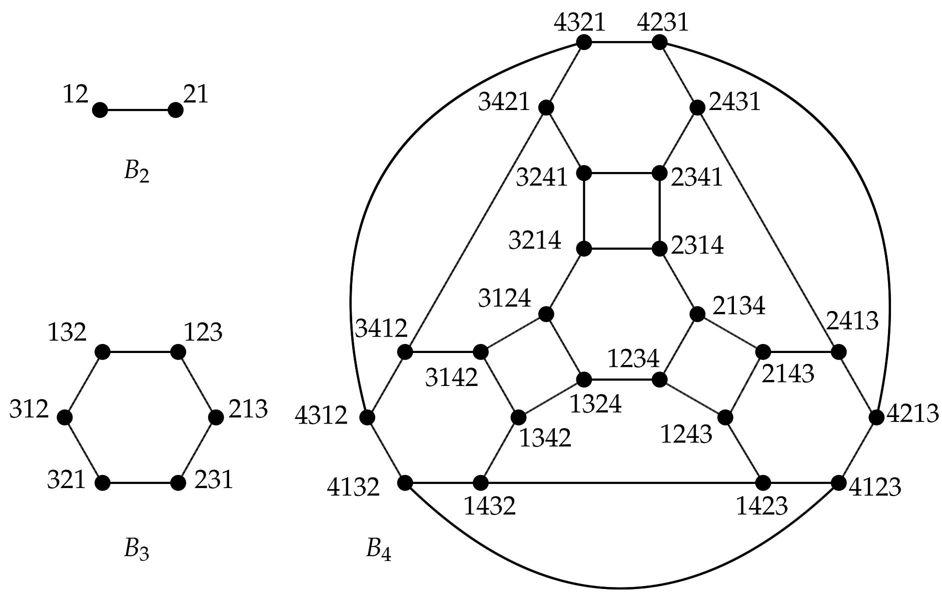

Figure 1. The bubble-sort graphs B_2, B_3 and B_4.

Note that B_n is a subclass of Cayley graphs. B_n has the following useful properties.

Proposition 1. *For any integer $n \geq 2$, B_n is $(n-1)$-regular and vertex transitive.*

Proposition 2. *For any integer $n \geq 2$, B_n is bipartite.*

Proposition 3. *For any integer $n \geq 3$, the girth of B_n is 4.*

Theorem 1 ([23])**.** *Every nonidentity permutation in the symmetric group is uniquely (up to the order of the factors) a product of disjoint cycles, each of which has length of at least 2.*

Proposition 4 ([12])**.** *Let B_n be a bubble-sort graph. If two vertices u, v are adjacent, there is no common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| = 0$. If two vertices u, v are not adjacent, there is at most two common neighbor vertices of these two vertices, i.e., $|N(u) \cap N(v)| \leq 2$.*

Theorem 2 ([7,25,26])**.** $\kappa(B_n) = \kappa^{(0)}(B_n) = n - 1$ *for* $n \geq 2$.

Theorem 3 ([7,25,26])**.** $\kappa^{(1)}(B_n) = 2n - 4$ *for* $n \geq 3$.

Theorem 4 ([7,25,26])**.** $\kappa^{(2)}(B_n) = 4n - 12$ *for* $n \geq 4$.

Theorem 5 ([27])**.** $\kappa^{(3)}(B_n) = 12$ *for* $n = 5$ *and* $\kappa^{(3)}(B_n) = 8n - 32$ *for* $n \geq 6$.

3. The Diagnosability of the Bubble-Sort Graph under the PMC Model

In this section, we shall show the g-good-neighbor diagnosability of the bubble-sort graph under the PMC model for $g = 0, 1, 2, 3$.

Let F_1 and F_2 be two distinct subsets of V for a system $G = (V, E)$. Define the symmetric difference $F_1 \triangle F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. Yuan et al. [20] presented a sufficient and necessary condition for a system to be g-good-neighbor t-diagnosable under the PMC model.

Lemma 1 ([20])**.** *A system $G = (V, E)$ is g-good-neighbor t-diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \backslash (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ (See Figure 2). The g-good-neighbor diagnosability $t_g(G)$ of G is the maximum value of t such that G is g-good-neighbor t-diagnosable under the PMC model.*

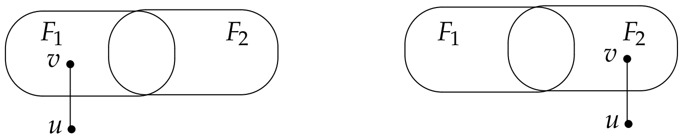

Figure 2. Illustration of a distinguishable pair (F_1, F_2) under Preparata, Metze, and Chien's (PMC) model.

Theorem 6. *The diagnosability of the bubble-sort graph B_n is $n - 1$ under the PMC model when $n \geq 4$.*

Proof. Let $A = \{(1)\}$. Then $|N(A)| = n - 1$. Let $F_1 = N(A)$ and $F_2 = A \cup N(A)$. Then $|F_1| = n - 1$ and $|F_2| = n$. Since $(1) = F_1 \triangle F_2$ and $N_{B_n}((1)) = F_1 \subset F_2$, there is no edge of B_n between $V(B_n) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Lemma 1, we show that B_n is not n-diagnosable under the PMC model. Hence, by the definition of the diagnosability, we have that the diagnosability of B_n is less than n-diagnosable, i.e., $t(B_n) = t_0(B_n) \leq n - 1$.

By the definition of the diagnosability, it is sufficient to show that B_n is $(n - 1)$-diagnosable under the PMC model. By Lemma 1, to prove that B_n is $(n - 1)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(B_n)$ with $u \in V(B_n) \backslash (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of faulty subsets F_1 and F_2 of $V(B_n)$ with $|F_1| \leq n - 1$ and $|F_2| \leq n - 1$. We prove this statement by contradiction. Suppose that there are two distinct faulty subsets F_1 and F_2 of $V(B_n)$ with $|F_1| \leq n - 1$ and $|F_2| \leq n - 1$, but the vertex set pair (F_1, F_2) is not satisfied with the condition in Theorem 1, i.e., there are no edges between $V(B_n) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose $V(B_n) = F_1 \cup F_2$. By the definition of B_n, $|F_1 \cup F_2| = |S_n| = n!$. It is obvious that $n! > 2n - 2$ for $n \geq 4$. Since $n \geq 4$, we have that $n! = |V(B_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| < 2n - 2$, a contradiction. Therefore, $V(B_n) \neq F_1 \cup F_2$. Since there are no edges between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and $|V(B_n) \setminus (F_1 \cup F_2)| \neq 0$ and $|F_1 \triangle F_2| \neq 0$, we have that $F_1 \cap F_2$ is a cut set. By Theorem 2, $|F_1 \cap F_2| \geq n - 1$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq 1 + n - 1 = n$, which contradicts with that $|F_2| \leq n - 1$. So B_n is $(n - 1)$-diagnosable. By the definition of $t(B_n)$, the diagnosability $t(B_n) \geq n - 1$. □

Theorem 7. *The 1-good-neighbor diagnosability of B_n is $2n - 3$ under the PMC model when $n \geq 4$.*

Proof. Let $A = \{(1), (12)\}$. By Proposition 2, $|N(A)| = 2n - 4$. Let $F_1 = N(A)$ and $F_2 = A \cup N(A)$. Then $|F_1| = 2n - 4$ and $|F_2| = 2n - 2$. Let $v \in V(B_n) \backslash (F_1 \cup F_2)$. By Proposition 4, $|N(v) \cap N((1))| \leq 2$ and $|N(v) \cap N((12))| \leq 2$. By Proposition 2, $N(v) \cap N((1)) \neq \emptyset$ and $N(v) \cap N((12)) = \emptyset$ or $N(v) \cap N((12) \neq \emptyset$ and $N(v) \cap N((1)) = \emptyset$. Therefore, $d(v) \geq n - 1 - 2 \geq 1$ $(n \geq 4)$ in $B_n - (F_1 \cup F_2)$ and F_1 is a 1-good-neighbor cut of B_n. Since $\{(1), (12)\} = F_1 \triangle F_2$ and $F_1 \subset F_2$, there is no edge of B_n between $V(B_n) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Lemma 1, we show that B_n is not 1-good-neighbor $(2n - 2)$-diagnosable under the PMC model. Hence, by the definition of the 1-good-neighbor diagnosability, we have that $t_1(B_n) \leq 2n - 3$.

By the definition of the 1-good-neighbor diagnosability, it is sufficient to show that B_n is 1-good-neighbor $(2n - 3)$-diagnosable. By Lemma 1, to prove that B_n is 1-good-neighbor $(2n - 3)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(B_n)$ with $u \in V(B_n) \backslash (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of 1-good-neighbor faulty subsets F_1 and F_2 of $V(B_n)$ with $|F_1| \leq 2n - 3$ and $|F_2| \leq 2n - 3$.

We prove this statement by contradiction. Suppose that there are two distinct 1-good-neighbor faulty subsets F_1 and F_2 of $V(B_n)$ with $|F_1| \leq 2n - 3$ and $|F_2| \leq 2n - 3$, but the vertex set pair (F_1, F_2)

is not satisfied with the condition in Lemma 1, i.e., there are no edges between $V(B_n)\backslash(F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose $V(B_n) = F_1 \cup F_2$. Since $n \geq 4$, we have that $n! = |V(B_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(2n-3) = 4n-6$, a contradiction. Therefore, $V(B_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a 1-good-neighbor faulty set, $B_n - F_1$ has two parts $B_n - F_1 - F_2$ and $B_n[F_2 \setminus F_1]$ (for convenience). Thus, $\delta(B_n - F_1 - F_2) \geq 1$ and $\delta(B_n[F_2 \setminus F_1]) \geq 1$. Similarly, $\delta(B_n[F_1 \setminus F_2]) \geq 1$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a 1-good-neighbor faulty set. When $F_1 \setminus F_2 = \emptyset$, $F_1 \cap F_2 = F_1$ is also a 1-good-neighbor faulty set. Since there are no edges between $V(B_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 1-good-neighbor cut. By Theorem 3, $|F_1 \cap F_2| \geq 2n-4$. Note that $|F_2 \backslash F_1| \geq 2$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq 2 + 2n - 4 = 2n-2$, which contradicts with that $|F_2| \leq 2n-3$. So B_n is 1-good-neighbor $(2n-3)$-diagnosable. By the definition of $t_1(B_n)$, $t_1(B_n) \geq 2n-3$. □

Lemma 2. *Let* $A = \{(1), (12), (34), (12)(34)\}$. *If* $n \geq 4$, $F_1 = N_{B_n}(A)$, $F_2 = A \cup N_{B_n}(A)$, *then* $|F_1| = 4n-12$, $|F_2| = 4n-8$, $\delta(B_n - F_1) \geq 2$, *and* $\delta(B_n - F_2) \geq 2$.

Proof. By $A = \{(1), (12), (34), (12)(34)\}$, we have that $B_n[A]$ is a 4-cycle. By Propositions 3 and 4, $|N_{B_n}(A)| = 4n-12$. Thus from calculating, we have $|F_1| = 4n-12$, $|F_2| = |A| + |F_1| = 4n-8$.

Let $v \in V(B_n)\backslash F_2$ and $|N(v) \cap N(A)| \neq 0$ and $w \in N(v) \cap N(A)$. Let $u \in A$ and $uw \in E(B_n)$. By Proposition 1, let $u = (1)$. Then $w = (ab) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}\backslash\{(12), (34)\}$. By Proposition 2, there is no $u' \in \{(12), (34)\}$ such that $|N(u') \cap N(v)| \geq 1$. Therefore, we consider only $u' \in \{(1), (12)(34)\}$. We discuss the following cases.

Case 1. $v = (ab)(cd)$ and $\{a, b\} \cap \{c, d\} = \emptyset$, $(cd) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}\backslash\{(ab)\}$.

If $(cd) \in \{(12), (34)\}$, then a contradiction to $v \in V(B_n)\backslash F_2$. Therefore, $(cd) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}\backslash\{(ab), (12), (34)\}$. In this case, $|N(v) \cap N(u)| = 2$. Consider $(12)(34)(xy)$ and $(ab)(cd)(uv)$, $(xy) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}\backslash\{(12), (34)\}$. Suppose $\{x, y\} \cap \{1, 2, 3, 4\} = \emptyset$. Since $(ab), (cd) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}\backslash\{(12), (34)\}$, $(12)(34)(xy) \neq (ab)(cd)(uv)$. If $(xy) = (23)$, then $(12)(34)(23) = (1243)$. If $(uv) \neq (12)$, then, $1 \to 1$ in $(ab)(cd)(uv)$, $(12)(34)(23) = (1243) \neq (ab)(cd)(uv)$. If $(uv) = (12)$, then, $2 \to 1$ in $(ab)(cd)(uv)$, $(12)(34)(23) = (1243) \neq (ab)(cd)(uv)$. If $(xy) = (45)$, then $(12)(34)(45) = (12)(345)$. If $(uv) \neq (12)$, then, $1 \to 1$ in $(ab)(cd)(uv)$, $(12)(34)(45) = (12)(345) \neq (ab)(cd)(uv)$. If $(uv) = (12)$, then, $3 \to 3$ or $3 \to 2$ in $(ab)(cd)(uv)$, $(12)(34)(45) = (12)(345) \neq (ab)(cd)(uv)$. Therefore, $|N(v) \cap N(A)| \leq 2$.

Case 2. $v = (ab)(cd)$ and $\{a, b\} \cap \{c, d\} \neq \emptyset$, $(cd) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}\backslash\{(ab)\}$.

Without loss of generality, let $v = (ab)(bd) = (abd)$. Let $w' \in N(v) \setminus \{w\}$. Then $w' = (ab)(bd)(uv)$, $(uv) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}\backslash\{(cd)\}$. If $(uv) = (ab)$, then $w' = (ab)(bd)(uv) = (ad)$. Note $(ad) \notin \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}$. Then $|N(v) \cap N(u)| = 1$. Suppose $(uv) \neq (ab)$. Consider $(12)(34)(xy)$ and $(ab)(cd)(uv)$, $(xy) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}\backslash\{(12), (34)\}$. If $\{x, y\} \cap \{1, 2, 3, 4\} = \emptyset$, then, by Theorem 1, $(12)(34)(xy) \neq w' = (ab)(bd)(uv)$. If $(xy) = (23)$, then $(12)(34)(23) = (1243)$. If $(uv) = (12)$, then, $2 \to 1$ in $(ab)(cd)(uv)$, $(12)(34)(23) = (1243) \neq (ab)(cd)(uv)$. If $(uv) \neq (12)$, then, $1 \to 1$ or $1 \to 3$ in $(ab)(cd)(uv)$, $(12)(34)(23) = (1243) \neq (ab)(cd)(uv)$. If $(xy) = (45)$, then $(12)(34)(45) = (12)(345)$. If $(uv) \neq (12)$, then, $1 \to 1$ in $(ab)(cd)(uv)$, $(12)(34)(45) = (12)(345) \neq (ab)(cd)(uv)$. If $(uv) = (12)$, then, $3 \to 3$ or $3 \to 2$ in $(ab)(cd)(uv)$, $(12)(34)(45) = (12)(345) \neq (ab)(cd)(uv)$. Therefore, $|N(v) \cap N(A)| \leq 2$.

By Cases 1 and 2, $d(v) \geq n-1-2 \geq 2$ $(n \geq 5)$ in $B_n - (F_1 \cup F_2)$ and F_1 is a 2-good-neighbor cut of B_n. When $n = 4$, it is easy to verify that F_1 is a 2-good-neighbor cut of B_n. □

Lemma 3. *Let* $n \geq 4$. *Then the 2-good-neighbor diagnosability* $t_2(B_n) \leq 4n-9$ *under the PMC model.*

Proof. Let A be defined in Lemma 2, and let $F_1 = N_{B_n}(A)$, $F_2 = A \cup N_{B_n}(A)$. By Lemma 2, $|F_1| = 4n - 12$, $|F_2| = 4n - 8$, $\delta(B_n - F_1) \geq 2$ and $\delta(B_n - F_2) \geq 2$. Therefore, F_1 and F_2 are both 2-good-neighbor faulty sets of B_n with $|F_1| = 4n - 12$ and $|F_2| = 4n - 8$. Since $A = F_1 \triangle F_2$ and $N_{B_n}(A) = F_1 \subset F_2$, there is no edge of B_n between $V(B_n) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Lemma 1, we show that B_n is not 2-good-neighbor $(4n-8)$-diagnosable under the PMC model. Hence, by the definition of 2-good-neighbor diagnosability, we conclude that the 2-good-neighbor diagnosability of B_n is less than $4n - 8$, i.e., $t_2(B_n) \leq 4n - 9$. □

Lemma 4. *Let H be a subgraph of B_n such that $\delta(H) = 2$. Then $|V(H)| \geq 4$.*

By the definition of B_n, we have Lemma 4.

Lemma 5. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability $t_2(B_n) \geq 4n - 9$ under the PMC model.*

Proof. By the definition of 2-good-neighbor diagnosability, it is sufficient to show that B_n is 2-good-neighbor $(4n-9)$-diagnosable. By Theorem 1, to prove B_n is 2-good-neighbor $(4n-9)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(B_n)$ with $u \in V(B_n) \backslash (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of 2-good-neighbor faulty subsets F_1 and F_2 of $V(B_n)$ with $|F_1| \leq 4n - 9$ and $|F_2| \leq 4n - 9$.

We prove this statement by contradiction. Suppose that there are two distinct 2-good-neighbor faulty subsets F_1 and F_2 of $V(B_n)$ with $|F_1| \leq 4n - 9$ and $|F_2| \leq 4n - 9$, but the vertex set pair (F_1, F_2) is not satisfied with the condition in Lemma 1, i.e., there are no edges between $V(B_n) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose $V(B_n) = F_1 \cup F_2$. By the definition of B_n, $|F_1 \cup F_2| = |S_n| = n!$. It is obvious that $n! > 8n - 18$ for $n \geq 4$. Since $n \geq 4$, we have that $n! = |V(B_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(4n-9) = 8n - 18$, a contradiction. Therefore, $V(B_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a 2-good-neighbor faulty set, $B_n - F_1$ has two parts $B_n - F_1 - F_2$ and $B_n[F_2 \setminus F_1]$. Thus, $\delta(B_n - F_1 - F_2) \geq 2$ and $\delta(B_n[F_2 \setminus F_1]) \geq 2$. Similarly, $\delta(B_n[F_1 \setminus F_2]) \geq 2$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a 2-good-neighbor faulty set. When $F_1 \setminus F_2 = \emptyset$, $F_1 \cap F_2 = F_1$ is also a 2-good-neighbor faulty set. Since there are no edges between $V(B_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-good-neighbor cut. Since $n \geq 4$, by Theorem 4, $|F_1 \cap F_2| \geq 4n - 12$. By Lemma 4, $|F_2 \backslash F_1| \geq 4$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq 4 + 8n - 22 = 8n - 18$, which contradicts with that $|F_2| \leq 4n - 9$. So B_n is 2-good-neighbor $(8n-19)$-diagnosable. By the definition of $t_2(B_n)$, $t_2(B_n) \geq 4n - 9$. □

Combining Lemmas 3 and 5, we have the following theorem.

Theorem 8. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability of the bubble-sort graph B_n under the PMC model is $4n - 9$.*

Lemma 6. *Let $A = \{(1), (12), (34), (56), (12)(34), (12)(56), (34)(56), (12)(34)(56)\}$. If $n \geq 7$, $F_1 = N_{B_n}(A)$, $F_2 = A \cup N_{B_n}(A)$, then $|F_1| = 8n - 32$, $|F_2| = 8n - 24$, $\delta(B_n - F_1) \geq 3$ and $\delta(B_n - F_2) \geq 3$.*

Proof. By $A = \{(1), (12), (34), (56), (12)(34), (12)(56), (34)(56), (12)(34)(56)\}$, we have that $B_n[A]$ is 3-regular and $|A| = 8$.

Claim 1. $(N(u) \cap N(v)) \backslash A = \emptyset$ for $u, v \in A$.

By Proposition 1, let $u = (1)$. By Proposition 2, we consider only $v \in \{(12)(34), (12)(56), (34)(56)\}$. Since $|N(u) \cap N(v)| = 2$, by Proposition 4, we have $(N(u) \cap N(v)) \backslash A = \emptyset$. The proof of Claim 1 is complete.

By Claim 1, $|N_{B_n}(A)| = 8n - 32$. Thus from calculating, we have $|F_1| = 8n - 32$, $|F_2| = |A| + |F_1| = 8n - 24$. Let $v \in V(B_n) \backslash F_2$ and $|N(v) \cap N(A)| \neq 0$ and $w \in N(v) \cap N(A)$. Let $u \in A$

and $uw \in E(B_n)$. By Proposition 1, let $u = (1)$. Then $w = (ab) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(12), (34), (56)\}$. By Proposition 2, there is no $u' \in \{(12), (34), (56), (12)(34)(56)\}$ such that $|N(u') \cap N(v)| \geq 1$. Therefore, we consider only $u' \in \{(1), (12)(34), (12)(56), (34)(56)\}$.

Claim 2. $|N(A) \cap N(v)| \leq 2$.

Let $v \in V(B_n) \backslash F_2$. We discuss the following cases.

Case 1. $v = (ab)(cd)$ and $\{a, b\} \cap \{c, d\} = \emptyset$, $(cd) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(ab)\}$.

If $(cd) \in \{(12), (34), (56)\}$, then a contradiction to $v \in V(B_n) \backslash F_2$. Therefore, $(cd) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(ab), (12), (34), (56)\}$. Consider $(ab)(cd)(uv)$. If $(uv) = (ab)$, then $|N(v) \cap N(u)| = 2$. Let $(uv) \neq (ab)$.

Consider $(12)(34)(xy)$ and $(ab)(cd)(uv)$, $(xy) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(12), (34), (56)\}$. Suppose $\{x, y\} \cap \{1, 2, 3, 4\} = \emptyset$. Since $(ab), (cd) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(12), (34), (56)\}$, $(12)(34)(xy) \neq (ab)(cd)(uv)$. If $(xy) = (23)$, then $(12)(34)(23) = (1243)$. If $(uv) \neq (12)$, then, $1 \to 1$ in $(ab)(cd)(uv)$, $(12)(34)(23) = (1243) \neq (ab)(cd)(uv)$. If $(uv) = (12)$, then, $2 \to 1$ in $(ab)(cd)(uv)$, $(12)(34)(23) = (1243) \neq (ab)(cd)(uv)$. If $(xy) = (45)$, then $(12)(34)(45) = (12)(345)$. If $(uv) \neq (12)$, then, $1 \to 1$ in $(ab)(cd)(uv)$, $(12)(34)(45) = (12)(345) \neq (ab)(cd)(uv)$. If $(uv) = (12)$, then, $3 \to 3$ or $3 \to 2$ in $(ab)(cd)(uv)$, $(12)(34)(45) = (12)(345) \neq (ab)(cd)(uv)$.

Consider $(34)(56)(xy)$ and $(ab)(cd)(uv)$, $(xy) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(12), (34), (56)\}$. Suppose $\{x, y\} \cap \{3, 4, 5, 6\} = \emptyset$. Since $(ab), (cd) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(12), (34), (56)\}$, $(34)(56)(xy) \neq (ab)(cd)(uv)$. If $(xy) = (23)$, then $(34)(23)(56) = (243)(56)$. If $(uv) = (23)$, then $2 \to 3$ or $2 \to 2$ in $(ab)(cd)(uv)$, $(34)(23)(56) = (243)(56) \neq (ab)(cd)(uv)$. If $(uv) \neq (23)$, then $(uv) = (12)$ or (34) or (uv) $(u, v \geq 4)$. When $(uv) = (12)$, $2 \to 1$ in $(ab)(cd)(uv)$, $(34)(23)(56) = (243)(56) \neq (ab)(cd)(uv)$. When $(uv) = (34)$, $2 \to 2$ or $2 \to 3$ in $(ab)(cd)(uv)$, $(34)(23)(56) = (243)(56) \neq (ab)(cd)(uv)$. When $u, v \geq 4$, $2 \to 2$ or $2 \to 3$ in $(ab)(cd)(uv)$, $(34)(23)(56) = (243)(56) \neq (ab)(cd)(uv)$.

Similarly, consider $(12)(56)(xy)$ and $(ab)(cd)(uv)$. We have $(12)(56)(xy) \neq (ab)(cd)(uv)$. Therefore, $|N(v) \cap N(A)| \leq 2$.

Case 2. $v = (ab)(cd)$ and $\{a, b\} \cap \{c, d\} \neq \emptyset$, $(cd) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(ab)\}$.

Without loss of generality, let $v = (ab)(bd) = (abd)$. Let $w' \in N(v) \setminus \{w\}$. Then $w' = (ab)(bd)(uv)$, $(uv) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(cd)\}$. If $(uv) = (ab)$, then $w' = (ab)(bd)(uv) = (ad)$. Note $(ad) \notin \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\}$. Then $|N(v) \cap N(u)| = 1$. Suppose $(uv) \neq (ab)$. Consider $(12)(34)(xy)$ and $(ab)(cd)(uv)$, $(xy) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(12), (34), (56)\}$. If $\{x, y\} \cap \{1, 2, 3, 4\} = \emptyset$, then, by Theorem 1, $(12)(34)(xy) \neq w' = (ab)(bd)(uv)$. If $(xy) = (23)$, then $(12)(34)(23) = (1243)$. If $(uv) = (12)$, then, $2 \to 1$ in $(ab)(cd)(uv)$, $(12)(34)(23) = (1243) \neq (ab)(cd)(uv)$. If $(uv) \neq (12)$, then, $1 \to 1$ or $1 \to 3$ in $(ab)(cd)(uv)$, $(12)(34)(23) = (1243) \neq (ab)(cd)(uv)$. If $(xy) = (45)$, then $(12)(34)(45) = (12)(345)$. If $(uv) \neq (12)$, then, $1 \to 1$ or $1 \to 3$ in $(ab)(cd)(uv)$, $(12)(34)(45) = (12)(345) \neq (ab)(cd)(uv)$. If $(uv) = (12)$, then, $3 \to 3$ or $3 \to 2$ in $(ab)(cd)(uv)$, $(12)(34)(45) = (12)(345) \neq (ab)(cd)(uv)$.

Consider $(34)(56)(xy)$ and $(ab)(cd)(uv)$, $(xy) \in \{(i, i+1) : i = 1, 2, 3, \ldots, n-1\} \backslash \{(12), (34), (56)\}$. If $\{x, y\} \cap \{3, 4, 5, 6\} = \emptyset$, then, by Theorem 1, $(34)(56)(xy) \neq w' = (ab)(bd)(uv)$. If $(xy) = (23)$, then $(34)(56)(xy) = (243)(56)$. If $(uv) = (12)$, then, $2 \to 1$ in $(ab)(cd)(uv)$, $(34)(56)(xy) \neq (ab)(cd)(uv)$. If $(uv) \neq (12)$, then, $1 \to 1$ or $1 \to 3$ in $(ab)(cd)(uv)$, $(34)(56)(xy) \neq (ab)(cd)(uv)$. If $(xy) = (45)$, then $(34)(56)(xy) = (3465)$. If $(uv) \neq (12)$, then, $1 \to 1$ or $1 \to 3$ in $(ab)(cd)(uv)$, $(34)(56)(xy) \neq (ab)(cd)(uv)$. If $(uv) = (12)$, then, $3 \to 3$ or $3 \to 2$ in $(ab)(cd)(uv)$, $(34)(56)(xy) \neq (ab)(cd)(uv)$. If $(xy) = (67)$, then $(34)(56)(xy) = (34)(567)$. If $(uv) \neq (12)$, then, $1 \to 1$ or $1 \to 3$ in $(ab)(cd)(uv)$, $(34)(56)(xy) \neq (ab)(cd)(uv)$. If $(uv) = (12)$, then, $3 \to 3$ or $3 \to 2$ in $(ab)(cd)(uv)$, $(34)(56)(xy) \neq (ab)(cd)(uv)$.

Similarly, consider $(12)(56)(xy)$ and $(ab)(cd)(uv)$. We have $(12)(56)(xy) \neq (ab)(cd)(uv)$. Therefore, $|N(v) \cap N(A)| \leq 2$. The proof of Claim 2 is complete.

By Claim 2, $d(v) \geq n-1-3 \geq 3$ $(n \geq 7)$ in $B_n - (F_1 \cup F_2)$ and F_1 is a 3-good-neighbor cut of B_n. □

Lemma 7. *Let $n \geq 7$. Then the 3-good-neighbor diagnosability $t_3(B_n) \leq 8n-25$ under the PMC model.*

Proof. Let A be defined in Lemma 6, and let $F_1 = N_{B_n}(A)$, $F_2 = A \cup N_{B_n}(A)$. By Lemma 6, $|F_1| = 8n-32$, $|F_2| = 8n-24$, $\delta(B_n - F_1) \geq 3$ and $\delta(B_n - F_2) \geq 3$. Therefore, F_1 and F_2 are both 3-good-neighbor faulty sets of B_n with $|F_1| = 8n-32$ and $|F_2| = 8n-24$. Since $A = F_1 \bigtriangleup F_2$ and $N_{B_n}(A) = F_1 \subset F_2$, there is no edge of B_n between $V(B_n)\setminus(F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$. By Lemma 1, we can deduce that B_n is not 3-good-neighbor $(8n-24)$-diagnosable under the PMC model. Hence, by the definition of 3-good-neighbor diagnosability, we conclude that the 2-good-neighbor diagnosability of B_n is less than $8n-24$, i.e., $t_2(B_n) \leq 8n-25$. □

Lemma 8. *Let H be a subgraph of B_n such that $\delta(H) = 3$. Then $|V(H)| \geq 8$.*

Proof. Note that there is no subgraph $K_{3,3}$ of B_n. Suppose, on the contrary, that there is a subgraph H' of B_n such that $\delta(H') \geq 3$ and $|V(H')| = 7$. Since B_n is bipartite, let $V(H') = (U, W)$ and $|U| = 3$, $|W| = 4$. By Proposition 1, let $W = \{(1), x, y, z\}$ and $U = \{a, b, c\}$. Since $\delta(H') \geq 3$, $N(x) \cap N(y) = \{a, b, c\}$, a contradiction to Proposition 4. Therefore, $|V(H)| \geq 8$. □

Lemma 9. *Let $n \geq 7$. Then the 3-good-neighbor diagnosability $t_3(B_n) \geq 8n-25$ under the PMC model.*

Proof. By the definition of 3-good-neighbor diagnosability, it is sufficient to show that B_n is 3-good-neighbor $(8n-25)$-diagnosable. By Lemma 1, to prove B_n is 3-good-neighbor $(8n-25)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(B_n)$ with $u \in V(B_n)\setminus(F_1 \cup F_2)$ and $v \in F_1 \bigtriangleup F_2$ for each distinct pair of 3-good-neighbor faulty subsets F_1 and F_2 of $V(B_n)$ with $|F_1| \leq 8n-25$ and $|F_2| \leq 8n-25$.

We prove this statement by contradiction. Suppose that there are two distinct 3-good-neighbor faulty subsets F_1 and F_2 of $V(B_n)$ with $|F_1| \leq 8n-25$ and $|F_2| \leq 8n-25$, but the vertex set pair (F_1, F_2) is not satisfied with the condition in Lemma 1, i.e., there are no edges between $V(B_n)\setminus(F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose $V(B_n) = F_1 \cup F_2$. By the definition of B_n, $|F_1 \cup F_2| = |S_n| = n!$. It is obvious that $n! > 16n-50$ for $n \geq 7$. Since $n \geq 7$, we have that $n! = |V(B_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(8n-25) = 16n-50$, a contradiction. Therefore, $V(B_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$, and F_1 is a 3-good-neighbor faulty set, $B_n - F_1$ has two parts $B_n - F_1 - F_2$ and $B_n[F_2 \setminus F_1]$. Thus, $\delta(B_n - F_1 - F_2) \geq 3$ and $\delta(B_n[F_2 \setminus F_1]) \geq 3$. Similarly, $\delta(B_n[F_1 \setminus F_2]) \geq 3$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a 3-good-neighbor faulty set. When $F_1 \setminus F_2 = \emptyset$, $F_1 \cap F_2 = F_1$ is also a 3-good-neighbor faulty set. Since there are no edges between $V(B_n - F_1 - F_2)$ and $F_1 \bigtriangleup F_2$, $F_1 \cap F_2$ is a 3-good-neighbor cut. Since $n \geq 7$, by Theorem 5, $|F_1 \cap F_2| \geq 8n-32$. By Lemma 8, $|F_2 \setminus F_1| \geq 8$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 8 + 8n - 32 = 8n-24$, which contradicts with that $|F_2| \leq 8n-25$. So B_n is 3-good-neighbor $(8n-25)$-diagnosable. By the definition of $t_3(B_n)$, $t_3(B_n) \geq 8n-25$. □

Combining Lemmas 7 and 9, we have the following theorem.

Theorem 9. *Let $n \geq 7$. Then the 3-good-neighbor diagnosability of the bubble-sort graph B_n under the PMC model is $8n-25$.*

4. The Diagnosability of the Bubble-Sort Graph B_n under the MM* Model

Before discussing the diagnosability of the bubble-sort graph B_n under the MM* model, we first give an existing result.

Lemma 10 ([4,20]). *A system $G = (V, E)$ is g-good-neighbor t-diagnosable under the MM* model if and only if for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ satisfies one of the following conditions. (1) There are two vertices $u, w \in V \setminus (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \triangle F_2$ such that $uw \in E$ and $vw \in E$. (2) There are two vertices $u, v \in F_1 \setminus F_2$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$. (3) There are two vertices $u, v \in F_2 \setminus F_1$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$ (See Figure 3). The g-good-neighbor diagnosability $t_g(G)$ of G is the maximum value of t such that G is g-good-neighbor t-diagnosable under the MM* model.*

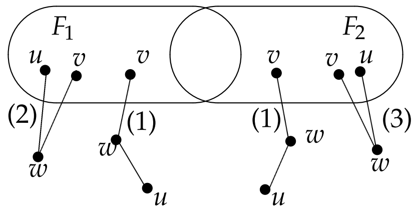

Figure 3. Illustration of a distinguishable pair (F_1, F_2) under Maeng and Malek's (MM)* model.

Theorem 10 ([12]). *The diagnosability $t(G) = t_0(G)$ of B_n is $n - 1$ under the MM* model when $n \geq 4$.*

A component of a graph G is odd according as it has an odd number of vertices. We denote by $o(G)$ the number of odd component of G.

Lemma 11 ([22]). *A graph $G = (V, E)$ has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subseteq V$.*

Lemma 12 ([22]). *Let $k \geq 0$ be an integer. Then every k-regular bipartite graph has k edge-disjoint perfect matchings.*

Since the bubble-sort graph is a regular bipartite graph, we have the following corollary by Lemma 12.

Corollary 1. *The bubble-sort graph has a perfect matching.*

Lemma 13. *Let $n \geq 4$. Then the 1-good-neighbor diagnosability of the bubble-sort graph B_n under the MM* model is less than or equal to $2n - 3$, i.e., $t_1(B_n) \leq 2n - 3$.*

Proof. Let $u = (1)$ and $v = (12)$. Then u is adjacent to v. Let $F_1 = N(\{u, v\})$ and $F_2 = F_1 \cup \{u, v\}$. By Proposition 2, $|F_1| = 2n - 4$, $|F_2| = 2n - 2$. Let $w \in V(B_n) \setminus (F_1 \cup F_2)$. By Proposition 4, $|N(w) \cap N((1))| \leq 2$ and $|N(w) \cap N((12))| \leq 2$. By Proposition 2, if $N(w) \cap N((1)) \neq \emptyset$, then $N(w) \cap N((12)) = \emptyset$ or if $N(w) \cap N((12) \neq \emptyset$, then $N(w) \cap N((1)) = \emptyset$. Therefore, $d(v) \geq n - 1 - 2 \geq 1$ ($n \geq 4$) in $B_n - (F_1 \cup F_2)$ and F_1 is a 1-good-neighbor cut of B_n. Since $\{(1), (12)\} = F_1 \triangle F_2$ and $F_1 \subset F_2$, there is no edge of B_n between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Lemma 10, we show that B_n is not 1-good-neighbor $(2n - 2)$-diagnosable under the MM* model. Hence, by the definition of the 1-good-neighbor diagnosability, we have that $t_1(B_n) \leq 2n - 3$. □

Lemma 14. *Let $n \geq 5$. Then the 1-good-neighbor diagnosability of the bubble-sort graph B_n under the MM* model is more than or equal to $2n - 3$, i.e., $t_1(B_n) \geq 2n - 3$.*

Proof. By the definition of 1-good-neighbor diagnosability, it is sufficient to show that B_n is 1-good-neighbor $(2n-3)$-diagnosable. By Lemma 10, suppose, on the contrary, that there are two distinct 1-good-neighbor faulty subsets F_1 and F_2 of B_n with $|F_1| \leq 2n-3$ and $|F_2| \leq 2n-3$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 10. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similarly to the discussion on $V(B_n) \neq F_1 \cup F_2$ in Theorem 3, we have $V(B_n) \neq F_1 \cup F_2$.

Claim 1. $B_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $B_n - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a 1-good-neighbor faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 10, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Thus, there is just a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Assume $F_1 \setminus F_2 = \emptyset$. Then $F_1 \subseteq F_2$. Since F_2 is a 1-good-neighbor faulty set, $B_n - F_2 = B_n - F_1 - F_2$ has no isolated vertex, a contradiction. Therefore, let $F_1 \setminus F_2 \neq \emptyset$ as follows. Similarly, we can show that there is just a vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w. Let $W \subseteq V(B_n) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $B_n[V(B_n) \setminus (F_1 \cup F_2)]$, and let H be the subgraph induced by the vertex set $V(B_n) \setminus (F_1 \cup F_2 \cup W)$. Then for any $w \in W$, there are $(n-3)$ neighbors in $F_1 \cap F_2$. By Corollary 1, B_n has a perfect matching. By Lemma 11, $|W| \leq o(G - (F_1 \cup F_2)) \leq |F_1 \cup F_2| \leq |F_1| + |F_2| - |F_1 \cap F_2| \leq 2(2n-3) - (n-3) = 3n-3$. Assume $V(H) = \emptyset$. Note that $n! = |V(B_n)| = |F_1 \cup F_2| + |W| \leq 2(3n-3) = 6n-6$. This is a contradiction to $n \geq 5$. So $V(H) \neq \emptyset$. Since the vertex set pair (F_1, F_2) is not satisfied with the condition (1) of Theorem 10, and any vertex of $V(H)$ is not isolated in H, we induce that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Thus, $F_1 \cap F_2$ is a vertex cut of B_n and $\delta(B_n - (F_1 \cap F_2)) \geq 1$, i.e., $F_1 \cap F_2$ is a 1-good-neighbor cut of B_n. By Theorem 3, $|F_1 \cap F_2| \geq 2n-4$. Because $|F_1| \leq 2n-3$ and $|F_2| \leq 2n-3$, and neither $F_1 \setminus F_2$ nor $F_2 \setminus F_1$ is empty, we have $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$. Let $F_1 \setminus F_2 = \{v_1\}$ and $F_2 \setminus F_1 = \{v_2\}$. Then for any vertex $w \in W$, w is adjacent to v_1 and v_2. According to Proposition 4, there are at most three common neighbors for any pair of vertices in B_n, it follows that there are at most two isolated vertices in $B_n - F_1 - F_2$, i.e., $|W| \leq 2$.

Suppose that there is exactly one isolated vertex v in $B_n - F_1 - F_2$. Let v_1 and v_2 be adjacent to v. Then $N_{B_n}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$. Note that B_n has no 3-cycle. Thus, $N_{B_n}(v_1) \setminus \{v\} \subseteq F_1 \cap F_2$, $N_{B_n}(v_2) \setminus \{v\} \subseteq F_1 \cap F_2$, $|(N_{B_n}(v) \setminus \{v_1, v_2\}) \cap (N_{B_n}(v_1) \setminus \{v\})| = 0$ and $|(N_{B_n}(v) \setminus \{v_1, v_2\}) \cap (N_{B_n}(v_2) \setminus \{v\})| = 0$ and $|[N_{B_n}(v_1) \setminus \{v\}] \cap [N_{B_n}(v_2) \setminus \{v\}]| \leq 1$. Thus, $|F_1 \cap F_2| \geq |N_{B_n}(v) \setminus \{v_1, v_2\}| + |N_{B_n}(v_1) \setminus \{v\}| + |N_{B_n}(v_2) \setminus \{v\}| \geq (n-1-2) + (n-1-1) + (n-1-1) - 1 = 3n-8$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 3n - 8 = 3n - 7 > 2n - 3 \ (n \geq 5)$, which contradicts $|F_2| \leq 2n-3$.

Suppose that there are exactly two isolated vertices v and w in $B_n - F_1 - F_2$. Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{B_n}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{B_n}(w) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{B_n}(v_1) \setminus \{v, w\} \subseteq F_1 \cap F_2$, $N_{B_n}(v_2) \setminus \{v, w\} \subseteq F_1 \cap F_2$, $|(N_{B_n}(v) \setminus \{v_1, v_2\}) \cap (N_{B_n}(v_1) \setminus \{v, w\})| = 0$ and $|(N_{B_n}(v) \setminus \{v_1, v_2\}) \cap (N_{B_n}(v_2) \setminus \{v, w\})| = 0$. $|(N_{B_n}(w) \setminus \{v_1, v_2\}) \cap (N_{B_n}(v_1) \setminus \{v, w\})| = 0$, $|(N_{B_n}(w) \setminus \{v_1, v_2\}) \cap (N_{B_n}(v_2) \setminus \{v, w\})| = 0$ and $|[N_{B_n}(v_1) \setminus \{v, w\}] \cap [N_{B_n}(v_2) \setminus \{v, w\}]| = 0$. By Proposition 4, there are at most two common neighbors for any pair of vertices in B_n. Thus, it follows that $|(N_{B_n}(v) \setminus \{v_1, v_2\}) \cap (N_{B_n}(w) \setminus \{v_1, v_2\})| = 0$. Thus, $|F_1 \cap F_2| \geq |N_{B_n}(v) \setminus \{v_1, v_2\}| + |N_{B_n}(w) \setminus \{v_1, v_2\}| + |N_{B_n}(v_1) \setminus \{v, w\}| + |N_{B_n}(v_2) \setminus \{v, w\}| = (n-1-2) + (n-1-2) + (n-1-2) + (n-1-2) = 4n-12$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 4n - 12 = 4n - 11 > 2n - 3 \ (n \geq 5)$, which contradicts $|F_2| \leq 2n-3$. The proof of Claim 1 is complete.

Let $u \in V(B_n) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor in $B_n - F_1 - F_2$. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 10, by the condition (1) of Lemma 10, for any pair of adjacent vertices $u, w \in V(B_n) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(B_n)$ and $vw \in E(B_n)$. It follows that u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u, there is no edge between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a 1-good-neighbor faulty set, $\delta_{B_n}([F_2 \setminus F_1]) \geq 1$ and hence $|F_2 \setminus F_1| \geq 2$. Since both F_1 and F_2 are 1-good-neighbor faulty sets, and there is no edge between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 1-good-neighbor cut of B_n. By Theorem 3, $|F_1 \cap F_2| \geq 2n-4$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 2 + 2n - 4 = 2n-2$, which contradicts with that $|F_2| \leq 2n-3$. So B_n is 1-good-neighbor $(2n-3)$-diagnosable. By the definition of $t_1(B_n)$, $t_1(B_n) \geq 3n-4$. □

Combining Lemmas 13 and 14, we have the following theorem.

Theorem 11. *Let $n \geq 5$. Then the 1-good-neighbor diagnosability of the bubble-sort graph B_n under the MM* model is $2n-3$.*

Lemma 15. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability $t_2(B_n) \leq 4n-9$ under the MM* model.*

Proof. Let A, F_1 and F_2 be defined in Lemma 2. By the Lemma 2, $F_1 = N_{B_n}(A)$, $F_2 = A \cup N_{B_n}(A)$, then $|F_1| = 4n-12$, $|F_2| = 4n-8$, $\delta(B_n - F_1) \geq 2$, and $\delta(B_n - F_2) \geq 2$. So both F_1 and F_2 are 2-good-neighbor faulty sets. By the definitions of F_1 and F_2, $F_1 \triangle F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(B_n) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 are not satisfied with any one condition in Lemma 10, and B_n is not 2-good-neighbor $(4n-8)$-diagnosable. Hence, $t_2(B_n) \leq 4n-9$. The proof is complete. □

Lemma 16. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability $t_2(B_n) \geq 4n-9$ under the MM* model.*

Proof. By the definition of 2-good-neighbor diagnosability, it is sufficient to show that B_n is 2-good-neighbor $(4n-9)$-diagnosable. By Lemma 10, suppose, on the contrary, that there are two distinct 2-good-neighbor faulty subsets F_1 and F_2 of B_n with $|F_1| \leq 4n-9$ and $|F_2| \leq 4n-9$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 10. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similarly to the discussion on $V(B_n) \neq F_1 \cup F_2$ in Lemma 5, we have $V(B_n) \neq F_1 \cup F_2$.

Claim 1. $B_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $B_n - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a 2-good neighbor faulty set, there are two vertices $u, v \in F_2 \setminus F_1$ such that u and v are adjacent to w. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 10, this is a contradiction. Therefore, $BS_n - F_1 - F_2$ has no isolated vertex. The proof of Claim 1 is complete.

Let $u \in V(B_n) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor in $B_n - F_1 - F_2$. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 10, by the condition (1) of Lemma 10, for any pair of adjacent vertices $u, w \in V(B_n) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(B_n)$ and $vw \in E(B_n)$. It follows that u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u, there is no edge between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a 2-good-neighbor faulty set, $\delta_{B_n}([F_2 \setminus F_1]) \geq 2$. By Lemma 4, $|F_2 \setminus F_1| \geq 4$. Since both F_1 and F_2 are 2-good-neighbor faulty sets, and there is no edge between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-good-neighbor cut of B_n. By Theorem 4, we have $|F_1 \cap F_2| \geq 4n-12$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 4 + (4n-12) = 4n-8$, which contradicts $|F_2| \leq 4n-9$. Therefore, B_n is 2-good-neighbor $(4n-9)$-diagnosable and $t_2(B_n) \geq 4n-9$. The proof is complete. □

Combining Lemmas 15 and 16, we have the following theorem.

Theorem 12. *Let $n \geq 4$. Then the 2-good-neighbor diagnosability of the bubble-sort star graph B_n under the MM* model is $4n-9$.*

We point out that B_4 is the least bubble-sort graph satisfying the three sufficient conditions in Lemma 10. Because B_3 is a cycle with six vertices which is isomorphic to the 3-dimensional star graph, by [21] B_3 is not 2-diagnosable.

Lemma 17. *Let $n \geq 7$. Then the 3-good-neighbor diagnosability $t_3(B_n) \leq 8n - 25$ under the MM* model.*

Proof. Let A, F_1 and F_2 be defined in Lemma 6. By the Lemma 6, $F_1 = N_{B_n}(A)$, $F_2 = A \cup N_{B_n}(A)$, then $|F_1| = 8n - 32$, $|F_2| = 8n - 24$, $\delta(B_n - F_1) \geq 3$, and $\delta(B_n - F_2) \geq 3$. So both F_1 and F_2 are 3-good-neighbor faulty sets. By the definitions of F_1 and F_2, $F_1 \triangle F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(B_n) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 are not satisfied with any one condition in Lemma 10, and B_n is not 3-good-neighbor $(8n - 24)$-diagnosable. Hence, $t_2(B_n) \leq 8n - 25$. The proof is complete. $\square$

Lemma 18. *Let $n \geq 7$. Then the 3-good-neighbor diagnosability $t_3(B_n) \geq 8n - 25$ under the MM* model.*

Proof. By the definition of 3-good-neighbor diagnosability, it is sufficient to show that B_n is 3-good-neighbor $(8n - 25)$-diagnosable. By Lemma 10, suppose, on the contrary, that there are two distinct 3-good-neighbor faulty subsets F_1 and F_2 of B_n with $|F_1| \leq 8n - 25$ and $|F_2| \leq 8n - 25$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 10. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similarly to the discussion on $V(B_n) \neq F_1 \cup F_2$ in Lemma 9, we have $V(B_n) \neq F_1 \cup F_2$.

Claim 1. $B_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $B_n - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a 3-good neighbor faulty set, there are three vertices $u, v \in F_2 \setminus F_1$ such that u, v and x are adjacent to w. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Lemma 10, this is a contradiction. Therefore, $BS_n - F_1 - F_2$ has no isolated vertex. The proof of Claim 1 is complete.

Let $u \in V(B_n) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor in $B_n - F_1 - F_2$. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 10, by the condition (1) of Lemma 10, for any pair of adjacent vertices $u, w \in V(B_n) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(B_n)$ and $vw \in E(B_n)$. It follows that u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u, there is no edge between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a 3-good-neighbor faulty set, $\delta_{B_n}([F_2 \setminus F_1]) \geq 3$. By Lemma 8, $|F_2 \setminus F_1| \geq 8$. Since both F_1 and F_2 are 3-good-neighbor faulty sets, and there is no edge between $V(B_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 3-good-neighbor cut of B_n. By Theorem 5, we have $|F_1 \cap F_2| \geq 8n - 32$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 8 + (8n - 32) = 8n - 24$, which contradicts $|F_2| \leq 8n - 25$. Therefore, B_n is 3-good-neighbor $(8n - 25)$-diagnosable and $t_3(B_n) \geq 8n - 25$. The proof is complete. $\square$

Combining Lemmas 17 and 18, we have the following theorem.

Theorem 13. *Let $n \geq 7$. Then the 3-good-neighbor diagnosability of the bubble-sort graph B_n under the MM* model is $8n - 25$.*

5. Conclusions

In this paper, we investigate the problem of g-good-neighbor diagnosability of the n-dimensional bubble-sort graph B_n under the PMC model and MM* model and show g-good-neighbor diagnosability of B_n is $2^g(n - g) - 1$ under the PMC model for $g = 0, 1, 2, 3$ and the MM* model for $g = 0, 1, 2, 3$, respectively. The work will help engineers to develop more different networks.

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