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The Stability of Parabolic Problems with Nonstandard $p(x,t)$-Growth

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Abstract: In this paper, we study weak solutions to the following nonlinear parabolic partial differential equation $\partial_t u - \operatorname{div} a(x,t,\nabla u) + \lambda(|u|^{p(x,t)-2}u) = 0$ in Ω_T, where $\lambda \geq 0$ and $\partial_t u$ denote the partial derivative of u with respect to the time variable t, while ∇u denotes the one with respect to the space variable x. Moreover, the vector-field $a(x,t,\cdot)$ satisfies certain nonstandard $p(x,t)$-growth and monotonicity conditions. In this manuscript, we establish the existence of a unique weak solution to the corresponding Dirichlet problem. Furthermore, we prove the stability of this solution, i.e., we show that two weak solutions with different initial values are controlled by these initial values.

Keywords: nonlinear parabolic problems; existence theory; variable exponents; stability

MSC: 35K55; 35A01; 35B35

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1. Introduction

The aim of this paper is to establish the existence theory to nonlinear parabolic equations with nonstandard $p(x,t)$-growth of the following form

$$\partial_t u - \operatorname{div} a(x,t,\nabla u) + \lambda(|u|^{p(x,t)-2}u) = 0 \quad \text{in } \Omega_T, \tag{1}$$

where $\lambda \geq 0$ and the vector-field $a(x,t,\cdot)$ satisfy certain $p(x,t)$-growth and monotonicity conditions. More precisely, we will prove that there exists a unique weak solution to the following Dirichlet problem:

$$\begin{cases} \partial_t u - \operatorname{div} a(x,t,\nabla u) = & -\lambda(|u|^{p(x,t)-2}u) \quad \text{in } \Omega_T, \\ u = & 0, \qquad\qquad\qquad\quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = & u_0, \qquad\qquad\qquad\; \text{on } \Omega \times \{0\}. \end{cases} \tag{2}$$

Moreover, we will show that two unique weak solutions u and v of (2) with different initial values $u_0(x), v_0(x) \in L^2(\Omega)$ satisfy the following stability estimate:

$$\int_\Omega |u(x,t) - v(x,t)|^2 \mathrm{d}x \leq \int_\Omega |u_0(x) - v_0(x)|^2 \mathrm{d}x \tag{3}$$

for a.e. $t \in [0,T)$. More precisely, we prove the stability of the unique weak solution to the Dirichlet problem (2) in the sense that the solutions are controlled by the initial value completely, cf. [1–3].

The motivation of this paper contains several aspects. The first one is that in general parabolic problems are important for the modelling of space- and time-dependent problems, e.g., problems from physics or biology. In particular, evolutionary equations and systems can be used to model physical processes, e.g., heat conduction or diffusion processes. One example is the Navier–Stokes equation, the basic equation in fluid mechanics. Furthermore, we want to refer to [4], where fluids in motion are studied. Applications also include climate modelling and climatology (see [5,6]).

The second interesting aspect of this paper is the nonstandard growth setting. Such setting arises for example by studying certain classes of non-Newtonian fluids such as electro-rheological fluids or fluids with viscosity depending on the temperature. Some properties of solutions to systems of such modified Navier–Stokes equation are studied in [7]. In general, electro-rheological fluids are of high technological interest because of their ability to change their mechanical properties under the influence of an exterior electro-magnetic field (see [8–10]). Many electro-rheological fluids are suspensions consisting of solid particles and a carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field (see [11]). Most of the known results concern the stationary case with $p(x)$-growth condition (see, e.g., [8,12,13]). Furthermore, for the restoration in image processing, one also uses some diffusion models with nonstandard growth conditions (please see [14–17]). Moreover, we want to refer to [18–21] for some numerical aspects regarding the numerical approximation of problems related to the parabolic p-Laplacian, the $p(x)$-Laplacian or electro-rheological fluids, respectively. Finally, we would like to mention the papers [22,23], where the stability of solutions with respect to continuous perturbations in the growth exponent $p(x)$ is studied.

In the context of parabolic problems with $p(x,t)$-growth applications are models for flows in porous media [24,25] or nonlinear parabolic obstacle problems [26–30]. Moreover, in the last few years, parabolic problems with $p(x,t)$-growth arouse more and more interest in mathematics (see, e.g., [26–33]). Furthermore, we want to highlight that, in the case of certain parabolic equations with nonstandard growth conditions, several existence results are available (please see [32,34–37]).

The third interesting aspect of the investigation of problems related to (2) is motivated amongst others by the following observation: In [38] (for the case $p(x,t) = \text{const.}$), the authors explained where they studied the asymptotic behaviour of the solution u to the homogeneous case of the following evolutionary p-Laplace equation

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda(|u|^{p-2}u) \quad \text{in } \Omega_T$$

that $v(x) = \lim_{t\to\infty} u(x,t)$ should be a solution to the stationary problem

$$-\operatorname{div}(|Dv|^{p-2}Dv) = \lambda(|v|^{p-2}v) \quad \text{in } \Omega.$$

For this equation, the first eigenvalue is the minimum of the Rayleigh quotient

$$\lambda_1 = \min_v \frac{\int_\Omega |Dv|^p \mathrm{d}x}{\int_\Omega |v|^p \mathrm{d}x},$$

cf. [39] and see also for further details [40]. Similarly, the stationary solution of the appropriate nonstandard $p(x,t)$-problem should be the solution of the corresponding eigenvalue problem of the $p(x)$-Laplacian (please see [41,42]). Therefore, the study of problems related to (1) are also of interest, since these problems are associated with the study of long-term behaviour of solutions and the corresponding eigenvalue problems.

1.1. General Assumptions

In this paper, we consider a bounded domain $\Omega \subset \mathbb{R}^n$ of dimension $n \geq 2$ and we write $\Omega_T := \Omega \times (0,T)$ for the space-time cylinder over Ω of height $T > 0$. Here, u_t or $\partial_t u$, respectively, denote the partial derivative with respect to the time variable t and ∇u denotes the one with respect to the space variable x. Moreover, we denote by $\partial_{\mathcal{P}}\Omega_T = (\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,T))$ the parabolic boundary of Ω_T and we write $z = (x,t)$ for points in $\mathbb{R}^{n+1}$. Furthermore, we consider vector-fields a:

$\Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ that are assumed to be Carathéodory functions—i.e., $a(z, w)$ is measurable in the first argument for every $w \in \mathbb{R}^n$ and continuous in the second one for a.e. $z \in \Omega_T$—and satisfy the following nonstandard growth and monotonicity properties, for some growth exponent $p\colon \Omega_T \to (\frac{2n}{n+2}, \infty)$ and structure constants $0 < \nu \leq 1 \leq L$ and $\mu \in [0, 1]$:

$$|a(z,w)| \leq \; L(1+|w|)^{p(z)-1}, \tag{4}$$

$$(a(z,w) - a(z,w_0)) \cdot (w - w_0) \geq \nu(\mu^2 + |w|^2 + |w_0|^2)^{\frac{p(z)-2}{2}} |w - w_0|^2, \tag{5}$$

for all $z \in \Omega_T$ and $w,\ w_0 \in \mathbb{R}^n$. Furthermore, the growth exponent function $p\colon \Omega_T \to (\frac{2n}{n+2}, \infty)$ satisfies the following conditions: there exist constants $\gamma_1 < \infty$ and $\gamma_2 < \infty$, such that

$$\frac{2n}{n+2} < \gamma_1 \leq p(z) \leq \gamma_2 \quad \text{and} \quad |p(z_1) - p(z_2)| \leq \omega(d_{\mathcal{P}}(z_1, z_2)) \tag{6}$$

hold for any choice of $z_1,\ z_2 \in \Omega_T$, where $\omega\colon\ [0,\infty) \to [0,1]$ denotes a modulus of continuity. More precisely, we assume that $\omega(\cdot)$ is a concave, non-decreasing function with

$$\lim_{\rho \downarrow 0} \omega(\rho) = 0 = \omega(0).$$

Moreover, the parabolic distance is given by $d_{\mathcal{P}}(z_1, z_2)\colon = \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}$ for $z_1 = (x_1, t_1),\ z_2 = (x_2, t_2) \in \mathbb{R}^{n+1}$. In addition, for the modulus of continuity $\omega(\cdot)$, we assume the following weak logarithmic continuity condition

$$\limsup_{\rho \downarrow 0} \omega(\rho) \log\left(\frac{1}{\rho}\right) < +\infty. \tag{7}$$

Finally, we point out that the monotonicity condition (5) implies, by using the growth condition (4) and Young's inequality, the coercivity property

$$a(z,w) \cdot w \geq \frac{\nu}{c(\gamma_1, \gamma_2)} |w|^{p(z)} - c(\gamma_1, \gamma_2, \nu, L) \tag{8}$$

for all $z \in \Omega_T$ and $w \in \mathbb{R}^n$.

1.2. The Function Spaces

The spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote the usual Lebesgue and Sobolev spaces, while the nonstandard $p(z)$-Lebesgue space $L^{p(z)}(\Omega_T, \mathbb{R}^k)$ is defined as the set of those measurable functions $v\colon \Omega_T \to \mathbb{R}^k$ for $k \in \mathbb{N}$, which satisfy $|v|^{p(z)} \in L^1(\Omega_T, \mathbb{R}^k)$, i.e.,

$$L^{p(z)}(\Omega_T, \mathbb{R}^k) : \ = \left\{ v : \Omega_T \to \mathbb{R}^k \text{ is measurable in } \Omega_T : \int_{\Omega_T} |v|^{p(z)} \mathrm{d}z < +\infty \right\}.$$

The set $L^{p(z)}(\Omega_T, \mathbb{R}^k)$ equipped with the Luxemburg norm

$$\|v\|_{L^{p(z)}(\Omega_T)} : \ = \inf\left\{ \delta > 0 : \int_{\Omega_T} \left|\frac{v}{\delta}\right|^{p(z)} \mathrm{d}z \leq 1 \right\}$$

becomes a Banach space. This space is separable and reflexive (see [34,35]). For elements of $L^{p(z)}(\Omega_T, \mathbb{R}^k)$, the generalized Hölder's inequality holds in the following form: if $f \in L^{p(z)}(\Omega_T, \mathbb{R}^k)$ and $g \in L^{p'(z)}(\Omega_T, \mathbb{R}^k)$, where $p'(z) = \frac{p(z)}{p(z)-1}$, we have

$$\left| \int_{\Omega_T} fg \mathrm{d}z \right| \leq \left(\frac{1}{\gamma_1} + \frac{\gamma_2 - 1}{\gamma_2} \right) \|f\|_{L^{p(z)}(\Omega_T)} \|g\|_{L^{p'(z)}(\Omega_T)} \tag{9}$$

(see also [35]). Moreover, the norm $\|\cdot\|_{L^{p(z)}(\Omega_T)}$ can be estimated as follows

$$-1+\|v\|^{\gamma_1}_{L^{p(z)}(\Omega_T)} \le \int_{\Omega_T} |v|^{p(z)}\mathrm{d}z \le \|v\|^{\gamma_2}_{L^{p(z)}(\Omega_T)} + 1. \tag{10}$$

Notice that we will use also the abbreviation $p(\cdot)$ for the exponent $p(z)$. Next, we introduce nonstandard Sobolev spaces for fixed $t \in (0,T)$. From assumption (6), we know that $p(\cdot,t)$ satisfies

$$|p(x_1,t)-p(x_2,t)| \le \omega(|x_1-x_2|)$$

for any choice of $x_1,\ x_2 \in \Omega$ and for every $t \in (0,T)$. Then, we define for every fixed $t \in (0,T)$ the Banach space $W^{1,p(\cdot,t)}(\Omega)$ as

$$W^{1,p(\cdot,t)}(\Omega) := \{u \in L^{p(\cdot,t)}(\Omega,\mathbb{R}) \mid \nabla u \in L^{p(\cdot,t)}(\Omega,\mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{W^{1,p(\cdot,t)}(\Omega)} := \|u\|_{L^{p(\cdot,t)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot,t)}(\Omega)}.$$

In addition, we define $W_0^{1,p(\cdot,t)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot,t)}(\Omega)$ and we denote by $W^{1,p(\cdot,t)}(\Omega)'$ its dual. For every $t \in (0,T)$, the inclusion $W_0^{1,p(\cdot,t)}(\Omega) \subset W_0^{1,\gamma_1}(\Omega)$ holds true.

Furthermore, we consider more general nonstandard Sobolev spaces without fixed t. By $W_g^{p(\cdot)}(\Omega_T)$, we denote the Banach space

$$W_g^{p(\cdot)}(\Omega_T): \ = \left\{u \in [g+L^1(0,T;W_0^{1,1}(\Omega))] \cap L^{p(\cdot)}(\Omega_T) \mid \nabla u \in L^{p(\cdot)}(\Omega_T,\mathbb{R}^n)\right\}$$

equipped by the norm

$$\|u\|_{W^{p(\cdot)}(\Omega_T)}: \ = \|u\|_{L^{p(\cdot)}(\Omega_T)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega_T)}.$$

If $g=0$, we write $W_0^{p(\cdot)}(\Omega_T)$ instead of $W_g^{p(\cdot)}(\Omega_T)$. Here, it is worth mentioning that the notion $(u-g) \in W_0^{p(\cdot)}(\Omega_T)$ or $u \in g + W_0^{p(\cdot)}(\Omega_T)$, respectively, indicate that u agrees with g on the lateral boundary of the cylinder Ω_T, i.e., $u \in W_g^{p(\cdot)}(\Omega_T)$.

Our next aim is to introduce the dual space of $W_0^{p(\cdot)}(\Omega_T)$. Therefore, we denote by $W^{p(\cdot)}(\Omega_T)'$ the dual of the space $W_0^{p(\cdot)}(\Omega_T)$. Assume that $v \in W^{p(\cdot)}(\Omega_T)'$. Then, there exist functions $v_i \in L^{p'(\cdot)}(\Omega_T)$, $i=0,1,...,n$, such that

$$\langle\langle v,w\rangle\rangle_{\Omega_T} = \int_{\Omega_T}\left(v_0 w + \sum_{i=1}^n v_i D_i w\right)\mathrm{d}z \tag{11}$$

for all $w \in W_0^{p(\cdot)}(\Omega_T)$. Furthermore, if $v \in W^{p(\cdot)}(\Omega_T)'$, we define the norm

$$\|v\|_{W^{p(\cdot)}(\Omega_T)'} = \sup\{\langle\langle v,w\rangle\rangle_{\Omega_T} | w \in W_0^{p(\cdot)}(\Omega_T),\ \|w\|_{W_0^{p(\cdot)}(\Omega_T)} \le 1\}.$$

Notice that whenever (11) holds, we can write $v = v_0 - \sum_{i=1}^n D_i v_i$, where $D_i v_i$ has to be interpreted as a distributional derivate. By

$$w \in W(\Omega_T): \ = \left\{w \in W^{p(\cdot)}(\Omega_T) | w_t \in W^{p(\cdot)},(\Omega_T)'\right\}$$

we mean that there exists $w_t \in W^{p(\cdot)}(\Omega_T)'$, such that

$$\langle\langle w_t, \varphi \rangle\rangle_{\Omega_T} = -\int_{\Omega_T} w \cdot \varphi_t \mathrm{d}z \quad \text{for all } \varphi \in C_0^\infty(\Omega_T)$$

(see also [34]). The previous equality makes sense due to the inclusions

$$W^{p(\cdot)}(\Omega_T) \hookrightarrow L^2(\Omega_T) \cong (L^2(\Omega_T))' \hookrightarrow W^{p(\cdot)}(\Omega_T)',$$

which allow us to identify w as an element of $W^{p(\cdot)}(\Omega_T)'$.

Finally, we are in the situation to give the definition of a weak solution to the parabolic nonstandard growth equation (1):

Definition 1. *We identify a function $u \in L^1(\Omega_T)$ as a weak solution of the parabolic equation (1), if and only if $u \in C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$ and*

$$\int_{\Omega_T} [u \cdot \varphi_t - a(z, \nabla u) \cdot \nabla \varphi] \, \mathrm{d}z = \lambda \int_{\Omega_T} |u|^{p(x,t)-2} u \cdot \varphi \mathrm{d}z \tag{12}$$

holds, whenever $\varphi \in C_0^\infty(\Omega_T)$.

Remark 1. *In this paper, we consider certain initial value problems. Therefore, we should also mention the meaning when referring to an initial condition of the type $u(\cdot, 0) = u_0$ a.e. on Ω. Here, we shall always mean*

$$\frac{1}{h} \int_0^h \int_\Omega |u - u_0|^2 \mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } h \downarrow 0. \tag{13}$$

In particular, when $u \in C^0([0,T]; L^2(\Omega))$, then (13) is obviously equivalent with saying $u(\cdot, 0) = u_0$.

1.3. Statement of the Result and Plan of the Paper

In the following, we mention our main result and we briefly describe the strategy of the proof to these results and the novelties of the paper. We start with some useful and important preliminary results (see Section 2). In Section 3, we prove the existence of a unique weak solution to (2) and we investigate its stability. The approach to prove the existence of weak solutions to the Dirichlet problem is to construct a solution, which solves the problem (2). We start by constructing a sequence of the Galerkin's approximations, where the limit of this sequence is equal to the solution in (2). Then, we show that this approximate solution converges to a general solution. Finally, we will use this existence result to derive the desired stability estimate (3). This yields the following.

Theorem 1. *Let $\lambda \geq 0$, $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain and $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies (6) and (7). Then, suppose that the vector-field a: $\Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and satisfies the growth condition (4) and the monotonicity condition (5). Moreover, let $u_0 \in L^2(\Omega)$. Then, there exists a unique weak solution $u \in C^0([0,T]; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ with $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$ of (2) and this solution satisfies the following estimate:*

$$\sup_{0 \leq t \leq T} \int_\Omega |u(\cdot, t)|^2 \mathrm{d}x + \int_{\Omega_T} |u|^{p(\cdot)} \mathrm{d}z + \int_{\Omega_T} |\nabla u|^{p(\cdot)} \mathrm{d}z \leq c \left(\|u_0\|_{L^2(\Omega)}^2 + |\Omega_T| \right), \tag{14}$$

with $u(\cdot, 0) = u_0$ and a constant $c = c(\gamma_1, \gamma_2, \nu, L)$ if $\lambda \geq 1$ or $\lambda = 0$ and $c = c(\gamma_1, \gamma_2, \nu, L, \frac{1}{\lambda})$ if $\lambda \in (0,1)$. Furthermore, for two weak solutions $u, v \in C^0([0,T]; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ with $\partial_t u, \partial_t v \in W^{p(\cdot)}(\Omega_T)'$ and different initial values $u_0, v_0 \in L^2(\Omega)$ (i.e., $u_0 \neq v_0$) of (2) the stability estimate (3), i.e.,

$$\|u(x,t) - v(x,t)\|_{L^2(\Omega)}^2 \leq \|u_0 - v_0\|_{L^2(\Omega)}^2$$

holds true for a.e. $t \in [0, T)$.

Remark 2. *Please note that we can conclude from (3) and Hölder's inequality that*

$$\fint_\Omega |u(x,t) - v(x,t)|\mathrm{d}x \leq \left(\fint_\Omega |u_0(x) - v_0(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}}$$

for a.e. $t \in [0, T)$*—where* $\fint_\Omega f \mathrm{d}x := \frac{1}{|\Omega|} \int_\Omega f \mathrm{d}x$*—or*

$$\|u(x,t) - v(x,t)\|_{L^1(\Omega)} \leq |\Omega|^{\frac{1}{2}} \|u_0(x) - v_0(x)\|_{L^2(\Omega)}$$

for a.e. $t \in [0, T)$*, respectively.*

Remark 3. *Moreover, we want to emphasise that we can also prove the existence of a unique weak solution to (2), if we assume that* $a(\cdot)$ *satisfies the growth condition (4), coercivity condition (8) and the monotonicity condition* $(a(z, w) - a(z, w_0)) \cdot (w - w_0) \geq 0$ *for all* $z \in \Omega_T$ *and* $w,\ w_0 \in \mathbb{R}^n$*. Furthermore, the existence of solutions to the initial value problem (2) can be extended to general boundary value problems and, moreover, we are also able to prove the statement of Theorem 1 if we consider further inhomogeneities on the right-hand side of (1), i.e.,* $f - \mathrm{div}(|F|^{p(x,t)-2}F)$ *satisfying* $f \in L^{\gamma_1'}(\Omega_T)$ *and* $F \in L^{p(x,t)}(\Omega_T)$ *(please see the approach in [32]).*

2. Preliminaries

In the following, we will refer to some useful tools, which we will need for our proof. First of all, we refer to two lemmas, which are useful tools when dealing with p-growth problems. To this aim, we define a function $V_{\mu,\mathfrak{p}} : \mathbb{R}^k \to \mathbb{R}^k$ by

$$V_{\mu,\mathfrak{p}}(A) := (\mu^2 + |A|^2)^{\frac{\mathfrak{p}}{2}} A$$

for $A \in \mathbb{R}^k$, $\mathfrak{p} > -1$ and $\mu \geq 0$. Moreover, we cite the following lemma from ([43], Lemma 2.1), which is established for the case $\mathfrak{p} \geq 0$ in [44] and in the case $0 > \mathfrak{p} > -1$ in [43].

Lemma 1. *Suppose that* $\mu \geq 0$*. Then, there exists a positive constant* c*, depending on* $\mathfrak{p} > -1$*, such that for all* $A, B \in \mathbb{R}^k$ *with* $A \neq B$*, we have*

$$c^{-1}(\mu^2 + |A|^2 + |B|^2)^{\frac{\mathfrak{p}}{2}} |A - B| \leq |V_{\mu,\mathfrak{p}}(A) - V_{\mu,\mathfrak{p}}(B)| \leq c(\mu^2 + |A|^2 + |B|^2)^{\frac{\mathfrak{p}}{2}} |A - B|.$$

Since $p(\cdot) > \frac{2n}{n+2}$, we are able to choose $\mathfrak{p} = p(\cdot) - 2 > -1$. Then, choosing $\mu = 0$ and $k = n \geq 2$ we consider $V(A) = |A|^{p(\cdot)-2}A$. This allows us to conclude from Lemma 1 ((cf. [45], Lemma 2.2) in the case $p(\cdot) > 2$ and ([46], Lemma 2) in the case $1 < p(\cdot) < 2$) the following lemma.

Lemma 2. *There exists a constant* $c := c(n, \gamma_1, \gamma_2)$*, such that for any* $A, B \in \mathbb{R}^n$*, there holds*

$$(|A|^2 + |B|^2)^{\frac{p(\cdot)-2}{2}} |A - B|^2 \leq c\,(V(A) - V(B)) \cdot (A - B),$$

where $A \neq B$.

Finally, we need the following Theorem ([32], Theorem 1.3), since this Theorem implies the strong convergence in $p(z)$-Lebesgue spaces and therefore, it is important for our existence result.

Theorem 2. *Let* $\Omega \subset \mathbb{R}^n$ *an open, bounded Lipschitz domain with* $n \geq 2$ *and* $p(\cdot) > \frac{2n}{n+2}$ *satisfying (6) and (7). Furthermore, define* $\hat{p}(\cdot) := \max\{2, p(\cdot)\}$*. Then, the inclusion* $W(\Omega_T) \hookrightarrow L^{\hat{p}(\cdot)}(\Omega_T)$ *is compact.*

3. Proof of the Main Result

First of all, we will prove the existence of a unique weak solution to the Dirichlet problem (2). Then, we are able to derive the desired stability estimate (3) immediately. The proof reads as follows:

Proof of Theorem 1. We start by constructing a sequence of the Galerkin's approximations, where the limit of this sequence is equal to the solution in (2). Therefore, we consider $\{\phi_i(x)\}_{i=1}^{\infty} \subset W_0^{1,\gamma_2}(\Omega)$, which is an orthonormal basis in $L^2(\Omega)$. Since $W_0^{1,\gamma_2}(\Omega)$ is separable, it is a span of a countable set of linearly independent functions $\{\phi_k\} \subset W_0^{1,\gamma_2}(\Omega)$. Moreover, we have the dense embedding $W_0^{1,\gamma_2}(\Omega) \subset L^2(\Omega)$ for any $\gamma_2 > \frac{2n}{n+2}$ (see, e.g., [47,48]). Thus, without loss of generality, we may assume that this system forms an orthonormal basis of $L^2(\Omega)$. Now, we fix a positive integer m and define the approximate solution to (2) as follows:

$$u^{(m)}(z) : \ = \sum_{i=1}^{m} c_i^{(m)}(t)\phi_i(x),$$

where the coefficients $c_i^{(m)}(t)$ are defined via the identity

$$\int_\Omega \left(u_t^{(m)}\phi_i(x) + a(x,t,\nabla u^{(m)})D\phi_i(x) + \lambda|u^{(m)}|^{p(x,t)-2}u^{(m)}\phi_i(x) \right) \mathrm{d}x = 0, \tag{15}$$

for $i = 0, ..., m$ and $t \in (0, T)$ with the initial condition

$$c_i^{(m)}(0) = \int_\Omega u_0\phi_i \mathrm{d}x, \quad i = 1, \ldots, m. \tag{16}$$

Then, Equation (15) together with the initial condition (16) generates a system of m ordinary differential equations

$$\begin{cases} \left(c_i^{(m)}\right)'(t) & = F_i\left(t, c_1^{(m)}(t), \ldots, c_m^{(m)}(t)\right), \\ c_i^{(m)}(0) & = \int_\Omega u_0\phi_i \mathrm{d}x, \quad i = 1, \ldots, m, \end{cases} \tag{17}$$

since $\{\phi_i(x)\}$ is orthonormal in $L^2(\Omega)$. By ([49], Theorem 1.44, p. 25), we know that there is, for every finite system (17), a solution $c_i^{(m)}(t)$, $i = 1, \ldots, m$ on the interval $(0, T_m)$ for some $T_m > 0$. Therefore, we multiply Equation (15) by the coefficients $c_i^{(m)}(t)$, $i = 1, \ldots, m$. Then, we need a priori estimates that permit us to extend the solution to the whole domain $(0, T)$. Thus, we integrate the equation over $(0, \tau)$ for an arbitrarily $\tau \in (0, T_m)$. Next, we sum the resulting equation over $i = 1, ..., m$. Therefore, it follows

$$\int_{\Omega_\tau} \partial_t u^{(m)} \cdot u^{(m)} + a(z, \nabla u^{(m)}) \cdot \nabla u^{(m)} + \lambda|u^{(m)}|^{p(x,t)-2}u^{(m)} \cdot u^{(m)} \mathrm{d}z = 0 \tag{18}$$

for a.e. $\tau \in (0, T_m)$. Furthermore, we use

$$\int_{\Omega_\tau} \partial_t u^{(m)} \cdot u^{(m)} \mathrm{d}z \ \geq \ \frac{1}{2}\int_\Omega |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega |u_0|^2 \mathrm{d}x$$

for a.e. $\tau \in (0, T_m)$, since $u_0 \in L^2(\Omega)$, $\{\phi_i\}_{i=1}^{\infty} \subset L^2(\Omega)$ and $\|u^{(m)}(\cdot, 0)\|_{L^2(\Omega)}^2 \leq \|u_0\|_{L^2(\Omega)}^2$, cf. [32]. Then, we derive at

$$\frac{1}{2}\int_\Omega |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \int_{\Omega_\tau} a(z, \nabla u^{(m)}) \cdot \nabla u^{(m)} + \lambda|u^{(m)}|^{p(\cdot)} \mathrm{d}z \leq \frac{1}{2}\|u_0\|_{L^2(\Omega)}^2 \tag{19}$$

for a.e. $\tau \in (0, T_m)$. Using the coercivity condition (8) on the left-hand side of (19), this yields

$$\frac{1}{2}\int_{\Omega} |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \frac{\nu}{c(\gamma_1,\gamma_2)}\int_{\Omega_\tau} |\nabla u^{(m)}|^{p(\cdot)} + \lambda |u^{(m)}|^{p(\cdot)} \mathrm{d}z \le c\left(\|u_0\|^2_{L^2(\Omega)} + |\Omega_T|\right),$$

where $c = c(\gamma_1, \gamma_2, \nu, L)$. This estimate holds for a.e. $\tau \in (0, T_m)$. Therefore, we have shown that $u^{(m)}$ is uniformly bounded in $W^{p(\cdot)}(\Omega_{T_m})$ and $L^\infty(0, T_m; L^2(\Omega))$ independently of m. Thus, the solution of system (17) can be continued to the maximal interval $(0, T)$ and we have

$$\sup_{0\le\tau\le T}\int_{\Omega} |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \int_{\Omega_T} |\nabla u^{(m)}|^{p(\cdot)} + |u^{(m)}|^{p(\cdot)} \mathrm{d}z \le c\left(\|u_0\|^2_{L^2(\Omega)} + |\Omega_T|\right). \tag{20}$$

Please notice that, if $\lambda \ge 1$, we can estimate the left-hand side of the second last inequality from below by choosing $\lambda = 1$, while if $\lambda = 0$ the term depending on λ disappears. If $\lambda \in (0, 1)$, we first of all divide the second last equation by λ, then the constant c depends on λ, i.e., $c = c(\gamma_1, \gamma_2, \nu, L, \frac{1}{\lambda})$, and, finally, we estimate the left-hand side of the resulting estimate from below by using $\frac{1}{\lambda} \ge 1$.

Next, we want to derive a uniform bound for $\partial_t u^{(m)}$ in $W^{p(\cdot)}(\Omega_T)'$. Therefore, we define a subspace of the set of admissible test functions

$$\mathcal{W}_m(\Omega_T) :\ = \left\{\eta :\ \eta = \sum_{i=1}^m d_i \phi_i,\ d_i \in C^1([0,T])\right\} \subset W_0^{p(\cdot)}(\Omega_T).$$

Then, we choose a test function

$$\varphi(z) = \sum_{i=1}^m d_i(t)\phi_i(x) \in \mathcal{W}_m(\Omega_T) \quad \text{with } d_i(0) = d_i(T) = 0.$$

Note that $\partial_t \varphi$ exists, since the coefficients $d_i(t)$ lie in $C^1([0, T])$. Moreover, we know that $C^1([0, T], W_0^{1,\gamma_2}(\Omega_T)) \subset W_0^{p(\cdot)}(\Omega_T)$, and, therefore, we have also $\varphi \in W_0^{p(\cdot)}(\Omega_T)$. Thus, we can conclude by the definition of $u^{(m)}$ and (15) that

$$-\int_{\Omega_T} u^{(m)}\varphi_t \mathrm{d}z = \int_{\Omega_T} u_t^{(m)}\varphi \mathrm{d}z = -\int_{\Omega_T} a(z, \nabla u^{(m)}) \cdot \nabla\varphi + \lambda |u^{(m)}|^{p(\cdot)-2}u^{(m)} \cdot \varphi \mathrm{d}z.$$

Then, we derive by utilizing the growth condition (4) and the generalized Hölder's inequality (9) the following estimate

$$\begin{aligned}
\left|\int_{\Omega_T} u^{(m)}\varphi_t \mathrm{d}z\right| &\le \int_{\Omega_T}\left(|a(z,\nabla u^{(m)})| + \lambda|u^{(m)}|^{p(\cdot)-1}\right)\cdot(|\nabla\varphi| + |\varphi|)\,\mathrm{d}z \\
&\le \theta\int_{\Omega_T}\left(|a(z,\nabla u^{(m)})| + |u^{(m)}|^{p(\cdot)-1}\right)\cdot(|\nabla\varphi| + |\varphi|)\,\mathrm{d}z \\
&\le c\left[\|(1 + |\nabla u^{(m)}|^{p(\cdot)-1} + |u^{(m)}|^{p(\cdot)-1})\|_{L^{p'(\cdot)}(\Omega_T)}\right] \times \|\varphi\|_{W^{p(\cdot)}(\Omega_T)},
\end{aligned}$$

where $c = c(\gamma_1, \gamma_2, L, \theta)$ with $\theta :\ = \max\{1, \lambda\}$. Using (10) and (20), we have for every $\varphi \in \mathcal{W}_m(\Omega_T) \subset W_0^{p(\cdot)}(\Omega_T)$ and any m that

$$\left|\int_{\Omega_T} u^{(m)}\varphi_t \mathrm{d}z\right| \le c\|\varphi\|_{W^{p(\cdot)}(\Omega_T)}$$

with a constant $c = c(\gamma_1, \gamma_2, \nu, L, \theta, \|u_0\|_{L^2}, |\Omega_T|)$, where c is independent of m. This shows that $u_t^{(m)} \in W^{p(\cdot)}(\Omega_T)'$ with

$$\|u_t^{(m)}\|_{W^{p(\cdot)}(\Omega_T)'} \le c(\gamma_1, \gamma_2, \nu, L, \theta, \|u_0\|_{L^2}, |\Omega_T|).$$

Therefore, we have a uniform bound of $u_t^{(m)}$ in $W^{p(\cdot)}(\Omega_T)'$ and it follows that

$$\begin{cases} u^{(m)} \in W_0^{p(\cdot)}(\Omega_T) \subseteq L^{\gamma_1}(0,T;W_0^{1,\gamma_1}(\Omega)), \\ u_t^{(m)} \in W^{p(\cdot)}(\Omega_T)' \subseteq L^{\gamma_2'}(0,T;W^{-1,\gamma_2'}(\Omega)) \end{cases}$$

are bounded. This implies the following weak convergences for the sequence $\left\{u^{(m)}\right\}$ (up to a subsequence):

$$\begin{cases} u^{(m)} \rightharpoonup^* u \text{ weakly* in } L^\infty(0,T;L^2(\Omega)), \\ \nabla u^{(m)} \rightharpoonup \nabla u \text{ weakly in } L^{p(\cdot)}(\Omega_T,\mathbb{R}^n), \\ u_t^{(m)} \rightharpoonup u_t \text{ weakly in } W^{p(\cdot)}(\Omega_T)'. \end{cases}$$

Moreover, by Theorem 2, we can conclude that the sequence $\left\{u^{(m)}\right\}$ (up to a subsequence) converges strongly in $L^{\hat{p}(\cdot)}(\Omega_T)$ with $\hat{p}(\cdot) := \max\{2, p(\cdot)\}$ to some function $u \in W(\Omega_T)$. Thus, we get the desired convergences

$$\begin{cases} u^{(m)} \to u \text{ strongly in } L^{\hat{p}(\cdot)}(\Omega_T), \\ u^{(m)} \to u \text{ a.e. in } \Omega_T \end{cases}$$

for the sequence $\left\{u^{(m)}\right\}$ (up to a subsequence).

Furthermore, the growth assumption of $a(z,\cdot)$ and the estimate (20) imply that the sequence $\left\{a(z,\nabla u^{(m)})\right\}_{m\in\mathbb{N}}$ is bounded in $L^{p'(\cdot)}(\Omega_T,\mathbb{R}^n)$. Consequently, after passing to a subsequence once more, we can find a limit map $A_0 \in L^{p'(\cdot)}(\Omega_T,\mathbb{R}^n)$ with

$$a(z,\nabla u^{(m)}) \to A_0 \quad \text{as } m \to \infty. \tag{21}$$

Our next aim is to show that $A_0 = a(z,\nabla u)$ for almost every $z \in \Omega_T$. First of all, we should mention that each of $u^{(m)}$ satisfies the identity (15) with a test function $\varphi \in \mathcal{W}_m(\Omega_T)$. This follows by the method of construction (see [36]). Then, we fix an arbitrary $m \in \mathbb{N}$. Thus, we have for every $s \le m$ the following equation

$$-\int_{\Omega_T} u_t^{(m)}\varphi + a(z,\nabla u^{(m)})\nabla\varphi + \lambda(|u^{(m)}|^{p(x,t)-2}u^{(m)})\varphi \mathrm{d}z = 0$$

for all test functions $\varphi \in \mathcal{W}_s(\Omega_T)$. Passing to the limit $m \to \infty$, we can conclude that, for all test functions $\varphi \in \mathcal{W}_s(\Omega_T)$, we have

$$-\int_{\Omega_T} u_t\varphi + A_0\nabla\varphi + \lambda(|u|^{p(x,t)-2}u)\varphi \mathrm{d}z = 0 \tag{22}$$

with an arbitrary $s \in \mathbb{N}$, by the convergence from above. Therefore, it follows that the identity (22) holds for every $\varphi \in W_0^{p(\cdot)}(\Omega_T)$. According to monotonicity assumption (5), we know that for every $w \in \mathcal{W}_s(\Omega_T)$ and every $s \le m$, the following holds

$$\int_{\Omega_T} [a(z,\nabla u^{(m)}) - a(z,\nabla w)]\nabla(u^{(m)} - w)\mathrm{d}z \ge 0. \tag{23}$$

Moreover, it follows from (15) the conclusion from above and the choice of an admissible test function $\varphi = u^{(m)} - w$ with $w \in \mathcal{W}_s(\Omega_T)$ that

$$-\int_{\Omega_T} u_t^{(m)}\varphi + a(z,\nabla u^{(m)})\nabla\varphi + \lambda(|u^{(m)}|^{p(x,t)-2}u^{(m)})\varphi \mathrm{d}z = 0. \tag{24}$$

Adding (23) and (24), we then have

$$-\int_{\Omega_T} u_t^{(m)}\varphi + a(z,\nabla u^{(m)})\nabla\varphi + \lambda(|u^{(m)}|^{p(x,t)-2}u^{(m)})\varphi \mathrm{d}z + \int_{\Omega_T}[a(z,\nabla u^{(m)}) - a(z,\nabla w)]\nabla\varphi \mathrm{d}z \geq 0$$

with a test function $\varphi = u^{(m)} - w$. This yields

$$-\int_{\Omega_T} u_t^{(m)}(u^{(m)} - w) + a(z,\nabla w)\nabla(u^{(m)} - w) + \lambda(|u^{(m)}|^{p(x,t)-2}u^{(m)})(u^{(m)} - w)\mathrm{d}z \geq 0.$$

Then, we test Equation (22) with $\varphi = u^{(m)} - w$, subtract the resulting equation from the last estimate and finally pass to the limit $m \to \infty$, yielding

$$-\int_{\Omega_T}[A_0 - a(z,\nabla w)]\nabla(u - w)\mathrm{d}z \geq 0$$

for all $w \in \mathcal{W}_s(\Omega_T)$. Since $\mathcal{W}_s(\Omega_T) \subset W_0^{p(\cdot)}(\Omega_T)$ is dense, we are allowed to choose $w \in W_0^{p(\cdot)}(\Omega_T)$. Hence, we choose $w = u \pm \varepsilon\zeta$ with an arbitrary $\zeta \in W_0^{p(\cdot)}(\Omega_T)$. This yields

$$-\varepsilon\int_{\Omega_T}[A_0 - a(z,\nabla(u \pm \varepsilon\zeta))]\nabla\zeta \mathrm{d}z \geq 0.$$

Then, passing to the limit $\varepsilon \downarrow 0$, we can conclude that

$$\int_{\Omega_T}[A_0 - a(z,\nabla u)]\nabla\zeta \mathrm{d}z = 0$$

for all $\zeta \in W_0^{p(\cdot)}(\Omega_T)$. This shows that

$$A_0 = a(z,\nabla u) \quad \text{for almost every } z \in \Omega_T.$$

Moreover, we have to show that $u(\cdot,0) = u_0$. First of all, we should mention that we get from (22) and the integration by parts the following equation

$$\int_{\Omega_T} u\varphi_t - a(z,\nabla u)\nabla\varphi - \lambda(|u|^{p(x,t)-2}u)\varphi \mathrm{d}z = \int_\Omega (u\cdot\varphi)(\cdot,0)\mathrm{d}x$$

for all $\varphi \in W_0^{p(\cdot)}(\Omega_T)$ with $\varphi(\cdot,T) = 0$. Moreover, we can conclude from (24)—similar to the previous estimate—that

$$\int_{\Omega_T} u^{(m)}\varphi_t - a(z,\nabla u^{(m)})\nabla\varphi - \lambda(|u^{(m)}|^{p(x,t)-2}u^{(m)})\varphi \mathrm{d}z = \int_\Omega (u^{(m)}\cdot\varphi)(\cdot,0)\mathrm{d}x$$

for all $\varphi \in W_0^{p(\cdot)}(\Omega_T)$ with $\varphi(\cdot,T) = 0$. Passing to the limit $m \to \infty$ and using the convergences from above, we get

$$\int_{\Omega_T} u\varphi_t - a(z,\nabla u)\nabla\varphi - \lambda(|u|^{p(x,t)-2}u)\varphi \mathrm{d}z = \int_\Omega u_0\cdot\varphi(\cdot,0)\mathrm{d}x,$$

where $u^{(m)}(\cdot,0) \to u_0$ as $m \to \infty$, since

$$u^{(m)}(\cdot,0) = \sum_{i=1}^{m} c_i^{(m)}(0)\phi_i(x) = \sum_{i=1}^{m} \int_\Omega u_0\phi_i(x)\mathrm{d}x\phi_i(x) \to \sum_{i=1}^{\infty} \int_\Omega u_0\phi_i(x)\mathrm{d}x\phi_i(x) = u_0 \quad \text{as } m \to \infty.$$

Furthermore, $\varphi(\cdot,0)$ is arbitrary. Therefore, we can conclude that $u(\cdot,0) = u_0$. This shows that there exists a weak solution to the Dirichlet problem (2).

Next, we prove the uniqueness of the weak solution. Therefore, we assume that there exist two weak solutions u and $u_* \in C^0([0,T];L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ with $\partial_t u, \partial_t u_* \in W^{p(\cdot)}(\Omega_T)'$ of the Dirichlet problem (2). Thus, we have the following weak formulations

$$\int_{\Omega_T} [u \cdot \varphi_t - a(z,\nabla u)\cdot\nabla\varphi]\,\mathrm{d}z = \lambda \int_{\Omega_T} |u|^{p(x,t)-2}u\cdot\varphi\mathrm{d}z$$

and

$$\int_{\Omega_T} [u_* \cdot \varphi_t - a(z,\nabla u_*)\cdot\nabla\varphi]\,\mathrm{d}z = \lambda \int_{\Omega_T} |u_*|^{p(x,t)-2}u_*\cdot\varphi\mathrm{d}z$$

with the admissible test function $\varphi = u - u_* \in W_0^{p(\cdot)}(\Omega_T)$, since $W_0^{p(\cdot)}(\Omega_T)'$ is the dual of $W_0^{p(\cdot)}(\Omega_T)$. Hence, we can conclude by subtracting the second equation from the first one that

$$\begin{aligned}\int_{\Omega_T} (u-u_*)(u-u_*)_t - (a(z,\nabla u) - a(z,\nabla u_*))\nabla(u-u_*)\mathrm{d}z \\ = \int_{\Omega_T} \lambda(|u|^{p(x,t)-2}u - |u_*|^{p(x,t)-2}u_*)(u-u_*)\mathrm{d}z.\end{aligned}$$

Using the monotonicity condition (5) and Lemma 2, we derive at

$$0 \geq \int_{\Omega_T} (u-u_*)\cdot(u-u_*)_t\mathrm{d}z = \frac{1}{2}\int_{\Omega_T} \partial_t(u-u_*)^2\mathrm{d}z.$$

Therefore, we have that $0 \geq \frac{1}{2}\|u(t) - u_*(t)\|^2_{L^2(\Omega)} \geq 0$ for every $t \in (0,T)$, since $u(\cdot,0) = u_*(\cdot,0) = u_0$.

Finally, we prove the stability of the weak solution to the Dirichlet problem (2). To this aim, we consider the unique weak solution $u \in C^0([0,T];L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ with $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$ to (2) and the unique weak solution $v \in C^0([0,T];L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ with $\partial_t v \in W^{p(\cdot)}(\Omega_T)'$ to

$$\begin{cases} \partial_t v - \operatorname{div} a(z,\nabla v) = & -\lambda(|v|^{p(x,t)-2}v) \quad \text{in } \Omega_T, \\ v = & 0 \quad \text{on } \partial\Omega\times(0,T), \\ v(\cdot,0) = & v_0 \quad \text{on } \Omega\times\{0\}, \end{cases}$$

where the initial values $u_0, v_0 \in L^2(\Omega)$ of both problems are different, i.e., $u_0 \neq v_0$. The existence is guaranteed by Theorem 1. Moreover, we know that $u - v \in W_0^{p(\cdot)}(\Omega_T)$. Therefore, we choose $\varphi = u - v \in W_0^{p(\cdot)}(\Omega_T)$ as an admissible test function in both weak formulations

$$\int_{\Omega_T} [u \cdot \varphi_t - a(z,\nabla u)\cdot\nabla\varphi]\,\mathrm{d}z = \lambda \int_{\Omega_T} |u|^{p(x,t)-2}u\cdot\varphi\mathrm{d}z$$

and

$$\int_{\Omega_T} [v \cdot \varphi_t - a(z,\nabla v)\cdot\nabla\varphi]\,\mathrm{d}z = \lambda \int_{\Omega_T} |v|^{p(x,t)-2}v\cdot\varphi\mathrm{d}z,$$

since $W_0^{p(\cdot)}(\Omega_T)'$ is the dual of $W_0^{p(\cdot)}(\Omega_T)$. Now, we subtract the second equation from the first one. This yields

$$\int_{\Omega_T} [(u-v)\cdot(u-v)_t - (a(z,\nabla u) - a(z,\nabla v))\cdot\nabla(u-v)]\,\mathrm{d}z$$
$$= \lambda \int_{\Omega_T} [|u|^{p(x,t)-2}u - |v|^{p(x,t)-2}v]\cdot(u-v)\mathrm{d}z.$$

Using the monotonicity condition (5) and Lemma 2, we derive at

$$0 \geq \int_{\Omega_T} (u-v)_t\cdot(u-v)\mathrm{d}z = \frac{1}{2}\int_0^T \partial_t\|(u-v)\|_{L^2(\Omega)}\mathrm{d}t,$$

which implies the stability estimate (3), i.e.,

$$\int_\Omega |u(x,t)-v(x,t)|^2\mathrm{d}x \leq \int_\Omega |u_0(x)-v_0(x)|^2\mathrm{d}x$$

for a.e. $t \in [0,T)$. This shows the conclusion of the Theorem. □

4. Conclusions

In this manuscript we proved the existence of a unique weak solution to the Dirichlet problem (2). Moreover, we mentioned that we can also use this approach to show the existence of a unique weak solution to more general problems, please see Remark 3. Furthermore, we studied the stability of the unique weak solution to the Dirichlet problem (2). To this aim, we established the stability estimate (3) for two unique weak solutions to (2) with different initial values.Therefore, it turns out that these weak solutions are controlled by their initial value completely.

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