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Article

On Small Deviation Asymptotics In L_2 of Some Mixed Gaussian Processes

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Abstract: We study the exact small deviation asymptotics with respect to the Hilbert norm for some mixed Gaussian processes. The simplest example here is the linear combination of the Wiener process and the Brownian bridge. We get the precise final result in this case and in some examples of more complicated processes of similar structure. The proof is based on Karhunen–Loève expansion together with spectral asymptotics of differential operators and complex analysis methods.

Keywords: mixed Gaussian process; small deviations; exact asymptotics

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1. Introduction

The problem of small deviation asymptotics for Gaussian processes was intensively studied in last years. Such a development was stimulated by numerous links between the small deviation theory and such mathematical problems as the accuracy of discrete approximation for random processes, the calculation of the metric entropy for functional sets, and the law of the iterated logarithm in the Chung form. It is also known that the small deviation theory is related to the functional data analysis and nonparametric Bayesian estimation.

The history of the question is described in the surveys [1,2], see also [3] for recent results. The most explored is the case of L_2-norm. For an arbitrary square-integrable random process X on $[0,1]$ put

$$||X||_2 = \left(\int_0^1 X^2(t)dt \right)^{\frac{1}{2}}.$$

We are interested in the exact asymptotics as $\varepsilon \to 0$ of the probability $\mathbb{P}\{||X||_2 \le \varepsilon\}$.

Usually one studies the *logarithmic* asymptotics while the *exact* asymptotics was found only for several special processes. Most of them are so-called *Green Gaussian processes*. This means that the covariance function G_X is the Green function for the ordinary differential operator (ODO)

$$\mathcal{L}u \equiv (-1)^\ell \left(p_\ell(t)u^{(\ell)}\right)^{(\ell)} + \left(p_{\ell-1}(t)u^{(\ell-1)}\right)^{(\ell-1)} + \cdots + p_0(t)u, \tag{1}$$

($p_\ell(t) > 0$) subject to proper homogeneous boundary conditions. This class of processes contains, e.g., the integrated Brownian motion, the Slepian process and the Ornstein–Uhlenbeck process, see [4–10]. Notice that some strong and interesting results were obtained recently for non-Green processes by Kleptsyna et al., see [11] and references therein.

In the present paper, we are interested in small deviations of so-called *mixed* Gaussian processes which are the sum (or the linear combination) of two independent Gaussian processes, usually with zero mean values. Mixed random processes arise quite naturally in the mathematical theory of finances and engineering applications and are known long ago.

Cheredito [12] considered the linear combination of the standard Wiener process W and the fractional Brownian motion (fBm) W^H with the Hurst index H, namely the process

$$Y^H_{(\beta)}(t) = W(t) + \beta W^H(t),$$

where $\beta \neq 0$ is a real constant. It is assumed that the processes W and W^H are independent. The covariance function of this process is $\min(s,t) + \beta^2 G_{W^H}(s,t)$, where the covariance function of the fBm is given by the well-known formula

$$G_{W^H}(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}),$$

and $H \in (0,1)$ is the so-called Hurst index. For $H = 1/2$ the fBm process turns into the usual Wiener process.

This paper strongly stimulated the probabilistic study of such process and its generalizations concerning the regularity of its trajectories, its martingale properties, the innovation representations, etc. The papers [13–15] are the typical examples.

The small deviations of the process $Y^H_{(\beta)}$ were studied at the logarithmical level in [16], where the following result was obtained. We cite it in the simplified form (without the weight function).

Proposition 1. *As $\varepsilon \to 0$ the following asymptotics holds*

$$\ln \mathbb{P}\{||Y^H_{(\beta)}||_2 \le \varepsilon\} \sim \begin{cases} \ln \mathbb{P}\{||W||_2 \le \varepsilon\}, & \text{if} \quad H > 1/2; \\ \beta^{1/H} \ln \mathbb{P}\{||W^H||_2 \le \varepsilon\}, & \text{if} \quad H < 1/2. \end{cases}$$

From [17] we know that as $\varepsilon \to 0$

$$\ln \mathbb{P}\{||W^H|| \le \varepsilon\} \sim -\frac{H}{(2H+1)^{\frac{2H+1}{2H}}} \left(\frac{\Gamma(2H+1)\sin(\pi H)}{(\sin(\frac{\pi}{2H+1}))^{2H+1}} \right)^{\frac{1}{2H}} \varepsilon^{-1/H},$$

and the exact small deviation asymptotics of W is given below, see (3).

However, the exact small deviations of mixed processes have not been explored. In general case it looks like a very complicated problem. First steps were made in a special case when a Gaussian process is mixed with some finite-dimensional "perturbation". The general theory was built in [18], later some refined results were obtained in the case of Durbin processes (limiting processes for empirical processes with estimated parameters), see [19] as a typical example.

We can give the solution in two cases. In Section 2 we consider the linear combination of two processes whose covariance functions are Green functions for two different boundary value problems to the same differential equation. The simplest example here is given by the standard Wiener process $W(t)$ and the Brownian bridge $B(t)$. Also we provide the exact small deviation asymptotics for more complicated mixtures containing the Ornstein–Uhlenbeck processes.

In Section 3 we deal with pairs of processes whose covariance functions are kernels of integral operators which are powers (or, more general, polynomials) of the same integral operator. The basic example here is the Brownian bridge and the integrated centered Wiener process

$$\overline{W}(t) = \int_0^t \left(W(s) - \int_0^1 W(u)du \right) ds. \tag{2}$$

Another series of examples is given by the Wiener process and the so-called Euler integrated Wiener process.

2. Mixed Green Processes Related to the Same Ordinary Differential Operator (ODO)

Let X_1 and X_2 be independent zero mean Gaussian processes on $[0,1]$. We assume that their covariance functions $G_1(s,t)$ and $G_2(s,t)$ are the Green functions for the same ODO (1) with different boundary conditions. This means they satisfy the equation

$$\mathcal{L}G_i(s,t) = \delta(s-t), \qquad i = 1,2.$$

in the sense of distributions and satisfy corresponding boundary conditions.

We consider the mixed process

$$Z^\beta(t) = X_1(t) + \beta X_2(t), \qquad t \in [0,1].$$

Since X_1 and X_2 are independent, it is easy to see that its covariance function equals

$$G_{Z^\beta}(s,t) = G_1(s,t) + \beta^2 G_2(s,t)$$

and satisfies the equation

$$\mathcal{L}G_{Z^\beta}(s,t) = (1+\beta^2)\delta(s-t)$$

in the sense of distributions. Therefore, it is the Green function for the ODO $\frac{1}{1+\beta^2}\mathcal{L}$ subject to some (in general, more complicated) boundary conditions. This allows us to apply general results of [6,8] on the small ball behavior of the Green Gaussian processes and to obtain the asymptotics of $\mathbb{P}\{||Z^\beta||_2 \le \varepsilon\}$ as $\varepsilon \to 0$ up to a constant. Then the sharp constant can be found by the complex variable method as shown in [7], see also [20].

To illustrate this algorithm we begin with the simplest mixed process

$$Z_1^\beta(t) = B(t) + \beta W(t), \qquad t \in [0,1].$$

The covariance function $G_{Z_1^\beta}$ is given by $(1+\beta^2)\min(s,t) - st$, and the integral equation for eigenvalues is equivalent to the boundary value problem

$$-f''(t) = \frac{1+\beta^2}{\lambda} f(t), \qquad f(0) = 0, \quad f(1) + \beta^2 f'(1) = 0.$$

It is easy to see that the process $\frac{1}{\sqrt{1+\beta^2}} Z_1^\beta(t)$ coincides in distribution with the process $W^{(\beta)}$, so-called *"elongated" Brownian bridge from zero to zero with length* $1+\beta^2$, see ([21], Section 4.4.20). Therefore, we obtain, as $\varepsilon \to 0$,

$$\mathbb{P}\{||Z_1^\beta||_2 \le \varepsilon\} = \mathbb{P}\Big\{||W^{(\beta)}||_2 \le \frac{\varepsilon}{\sqrt{1+\beta^2}}\Big\} \overset{(*)}{\sim} \frac{\sqrt{1+\beta^2}}{|\beta|} \cdot \mathbb{P}\Big\{||W||_2 \le \frac{\varepsilon}{\sqrt{1+\beta^2}}\Big\}$$

(the relation $(*)$ was derived in ([7], Proposition 1.9), see also ([18], Example 6)).

The last asymptotics was obtained long ago:

$$\mathbb{P}\{||W||_2 \le \varepsilon\} \sim \frac{4}{\sqrt{\pi}} \cdot \varepsilon \cdot \exp\Big(-\frac{1}{8}\varepsilon^{-2}\Big), \qquad \varepsilon \to 0, \tag{3}$$

and we arrive at the following Theorem:

Theorem 1. *The following asymptotic relation holds as* $\varepsilon \to 0$*:*

$$\mathbb{P}\{||B+\beta W||_2 \le \varepsilon\} \sim \frac{4}{\sqrt{\pi}} \cdot \frac{\varepsilon}{|\beta|} \cdot \exp\Big(-\frac{1+\beta^2}{8}\,\varepsilon^{-2}\Big).$$

The next process we consider is

$$Z_2^\beta(t) = \mathring{U}_{(\alpha)}(t) + \beta U_{(\alpha)}(t), \qquad t \in [0,1].$$

Here $\mathring{U}_{(\alpha)}(t)$ is the Ornstein–Uhlenbeck process starting at the origin and $U_{(\alpha)}(t)$ is the stationary Ornstein–Uhlenbeck process. Both them are Gaussian processes with zero mean-value. Their covariance functions are, respectively,

$$G_{\mathring{U}_{(\alpha)}}(s,t) = (e^{-\alpha|t-s|} - e^{-\alpha(t+s)})/(2\alpha); \qquad G_{U_{(\alpha)}}(s,t) = e^{-\alpha|t-s|}/(2\alpha).$$

Direct calculation shows that the integral equation for eigenvalues of Z_2^β is equivalent to the boundary value problem

$$-f''(t) + \alpha^2 f(t) = \frac{1+\beta^2}{\lambda}\, f(t); \qquad (f' - \alpha(1+2\beta^{-2})f)(0) = (f'+\alpha f)(1) = 0.$$

By standard method we derive that if $r_1 < r_2 < ...$ are the positive roots of transcendental equation

$$F_1(\zeta) := (\zeta^2 - \alpha^2(1+2\beta^{-2}))\,\frac{\sin(\zeta)}{\zeta} - 2\alpha(1+\beta^{-2})\cos(\zeta) = 0$$

then $\lambda_n(Z_2^\beta) = \frac{1+\beta^2}{r_n^2+\alpha^2}$, $n \ge 1$.

Recall that the eigenvalues of the stationary Ornstein–Uhlenbeck process were derived in [22]. By rescaling we obtain $\lambda_n(\sqrt{1+\beta^2}\,U_{(\alpha)}) = \frac{1+\beta^2}{\rho_n^2+\alpha^2}$, $n \ge 1$, where $\rho_1 < \rho_2 < ...$ are the positive roots of transcendental equation

$$F_2(\zeta) := (\zeta^2 - \alpha^2)\,\frac{\sin(\zeta)}{\zeta} - 2\alpha\cos(\zeta) = 0.$$

We claim that $\lambda_n(Z_2^\beta)$ and $\lambda_n(\sqrt{1+\beta^2}\,U_{(\alpha)})$ are asymptotically close, and therefore, using the Wenbo Li comparison theorem, see [22], we can write

$$\mathbb{P}\{||Z_2^\beta||_2 \le \varepsilon\} \sim C_{\text{dist}} \cdot \mathbb{P}\Big\{||U_{(\alpha)}||_2 \le \frac{\varepsilon}{\sqrt{1+\beta^2}}\Big\}, \qquad \varepsilon \to 0, \tag{4}$$

where the distortion constant is given by

$$C_{\text{dist}} = \left(\prod_{n=1}^{\infty} \frac{\lambda_n(\sqrt{1+\beta^2}\,U_{(\alpha)})}{\lambda_n(Z_2^\beta)}\right)^{\frac{1}{2}} = \left(\prod_{n=1}^{\infty} \frac{r_n^2+\alpha^2}{\rho_n^2+\alpha^2}\right)^{\frac{1}{2}}.$$

To justify (4) we should prove the convergence of the last infinite product. As in [7], we use the complex variable method.

For large N in the disk $|\zeta| < \pi(N-\frac{1}{2})$ there are exactly $2N$ zeros $\pm r_j$, $j = 1,...,N$, of $F_1(\zeta)$, and exactly $2N$ zeros $\pm\rho_j$, $j = 1,...,N$, of $F_2(\zeta)$. By the Jensen theorem, see ([23], Section 3.6.1), we have

$$\ln\left(\frac{|F_1(0)|}{|F_2(0)|} \cdot \prod_{n=1}^{N} \frac{\rho_n^2}{r_n^2}\right) = \frac{1}{2\pi}\int_0^{2\pi} \ln\frac{|F_1(\pi(N-\frac{1}{2})\exp(i\varphi))|}{|F_2(\pi(N-\frac{1}{2})\exp(i\varphi))|}\,d\varphi.$$

It is easy to see that if we take $|\zeta| = \pi(N-\frac{1}{2})$ then

$$\frac{|F_1(\zeta)|}{|F_2(\zeta)|} \rightrightarrows 1, \qquad N \to \infty.$$

Therefore,

$$\prod_{n=1}^{\infty} \frac{r_n^2}{\rho_n^2} = \frac{|F_1(0)|}{|F_2(0)|}. \tag{5}$$

Now we use Hadamard's theorem on canonical product, see ([23], Section 8.24):

$$F_1(\zeta) \equiv F_1(0) \cdot \prod_{n=1}^{\infty} \left(1 - \frac{\zeta^2}{r_n^2}\right); \qquad F_2(\zeta) \equiv F_2(0) \cdot \prod_{n=1}^{\infty} \left(1 - \frac{\zeta^2}{\rho_n^2}\right).$$

In view of (5) this gives

$$C_{\text{dist}}^2 = \prod_{n=1}^{\infty} \frac{r_n^2 + \alpha^2}{\rho_n^2 + \alpha^2} = \frac{|F_1(0)|}{|F_2(0)|} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{\alpha^2}{r_n^2}\right) \Big/ \prod_{n=1}^{\infty} \left(1 + \frac{\alpha^2}{\rho_n^2}\right) = \frac{|F_1(i\alpha)|}{|F_2(i\alpha)|} = 1 + \beta^{-2}.$$

Thus, (4) is proved. Since the small deviation asymptotics of $U_{(\alpha)}$ is known, see ([7], Proposition 2.1) and ([20], Corollary 3), we obtain the following Theorem:

Theorem 2. *The following asymptotic relation holds as $\varepsilon \to 0$:*

$$\mathbb{P}\{||\mathring{U}_{(\alpha)} + \beta U_{(\alpha)}||_2 \le \varepsilon\} \sim \sqrt{\frac{\alpha e^{\alpha}}{\pi}} \cdot \frac{8\varepsilon^2}{|\beta|\sqrt{1+\beta^2}} \cdot \exp\left(-\frac{1+\beta^2}{8}\,\varepsilon^{-2}\right).$$

Finally, we consider the stationary process

$$Z_3^{\beta}(t) = \mathcal{B}_{(\alpha)}(t) + \beta U_{(\alpha)}(t), \qquad t \in [0,1],$$

where $\mathcal{B}_{(\alpha)}$ is the Bogoliubov periodic process ([24–26]) with zero mean and covariance function

$$G_{\mathcal{B}_{(\alpha)}}(s,t) = \frac{1}{2\alpha \sinh(\alpha/2)} \cosh\left(\alpha|t-s| - \frac{\alpha}{2}\right).$$

A portion of tedious calculations gives the boundary value problem for eigenvalues of Z_3^{β}:

$$-f''(t) + \alpha^2 f(t) = \frac{1+\beta^2}{\lambda} f(t); \qquad f'(0) - \mathcal{A}_1 f(0) + \mathcal{A}_2 f(1) = f'(1) + \mathcal{A}_1 f(1) - \mathcal{A}_2 f(0) = 0.$$

Here

$$\mathcal{A}_1 = \alpha \frac{(1+\beta^2\gamma)^2 + 1}{(1+\beta^2\gamma)^2 - 1}; \qquad \mathcal{A}_2 = 2\alpha \frac{1+\beta^2\gamma}{(1+\beta^2\gamma)^2 - 1}; \qquad \gamma = 1 - e^{-\alpha}.$$

Just as in the previous example we obtain $\lambda_n(Z_3^{\beta}) = \frac{1+\beta^2}{\mathfrak{r}_n^2+\alpha^2}$, $n \ge 1$, where $\mathfrak{r}_1 < \mathfrak{r}_2 < \ldots$ are the positive roots of transcendental equation

$$F_3(\zeta) := (\zeta^2 - \alpha^2)\,\frac{\sin(\zeta)}{\zeta} - 2\mathcal{A}_1 \cos(\zeta) + 2\mathcal{A}_2 = 0.$$

Arguing as before, we derive

$$\mathbb{P}\{||Z_3^{\beta}||_2 \le \varepsilon\} \sim \widetilde{C}_{\text{dist}} \cdot \mathbb{P}\Big\{||U_{(\alpha)}||_2 \le \frac{\varepsilon}{\sqrt{1+\beta^2}}\Big\}, \qquad \varepsilon \to 0,$$

where

$$\widetilde{C}^2_{\text{dist}} = \prod_{n=1}^{\infty} \frac{\mathfrak{r}_n^2 + \alpha^2}{\rho_n^2 + \alpha^2} = \frac{|F_3(i\alpha)|}{|F_2(i\alpha)|} = \frac{\gamma(1+\beta^2)^2}{\beta^2(2+\beta^2\gamma)},$$

and thus we obtain the following Theorem:

Theorem 3. *The following asymptotic relation holds as* $\varepsilon \to 0$*:*

$$\mathbb{P}\{||\mathcal{B}_{(\alpha)} + \beta U_{(\alpha)}||_2 \le \varepsilon\} \sim \sqrt{\frac{\alpha(e^\alpha - 1)}{\pi}} \cdot \frac{8\varepsilon^2}{|\beta|\sqrt{2+\beta^2(1-e^{-\alpha})}} \cdot \exp\left(-\frac{1+\beta^2}{8}\,\varepsilon^{-2}\right).$$

3. Mixed Processes Related to Polynomials of Covariance Operator

Recall that the covariance operator $\mathbb{G}_X$ related to the Gaussian process X is the integral operator with kernel G_X.

Lemma 1. *Let covariance operators* $\mathbb{G}_X$ *and* $\mathbb{G}_Z$ *are linked by relation* $\mathbb{G}_Z = \mathcal{P}(\mathbb{G}_X)$*, where* $\mathcal{P}$ *is a polynomial*

$$\mathcal{P}(x) := x + a_2x^2 + \cdots + a_{k-1}x^{k-1} + a_kx^k.$$

Then the following asymptotic relation holds as $\varepsilon \to 0$*:*

$$\mathbb{P}\{||Z||_2 \le \varepsilon\} \sim \widehat{C}_{\text{dist}} \cdot \mathbb{P}\{||X||_2 \le \varepsilon\}.$$

Proof. By the Wenbo Li comparison theorem, we should prove that the following infinite product converges:

$$\widehat{C}^2_{\text{dist}} = \prod_{n=1}^{\infty} \frac{\lambda_n(X)}{\lambda_n(Z)}.$$

It is well known that the set of eigenvalues of $\mathcal{P}(\mathbb{G}_X)$ coincides with the set $\mathcal{P}(\{\lambda_n(X)\}_{n\in\mathbb{N}})$. Moreover, since $\mathcal{P}$ increases in a neighborhood of the origin, for sufficiently large n we have just $\lambda_n(Z) = \mathcal{P}(\lambda_n(X))$. Thus,

$$\widehat{C}^2_{\text{dist}} = \prod_{n=1}^{\infty} \frac{\lambda_n(X)}{\mathcal{P}(\lambda_n(X))} = \prod_{n=1}^{\infty}(1 + O(\lambda_n(X))).$$

Since X is square integrable, the series $\sum_n \lambda_n(X)$ converges. Therefore, the infinite product also converges, and the lemma follows. □

The first example is the mixed process

$$Z_4^\beta(t) = B(t) + \beta\overline{W}(t), \qquad t \in [0,1],$$

where the integrated centered Wiener process $\overline{W}$ is defined in (2).

The integral equation for the eigenvalues of $\overline{W}$ is equivalent to the boundary value problem [4]

$$y^{(IV)} = \frac{1}{\lambda}y, \qquad y(0) = y(1) = y''(0) = y''(1) = 0. \tag{6}$$

It is easy to see that the operator of the problem (6) is just the square of the operator of the boundary value problem

$$-y'' = \frac{1}{\lambda}y, \qquad y(0) = y(1) = 0, \tag{7}$$

which corresponds to the Brownian bridge. Therefore, we have the relation $\mathbb{G}_{\overline{W}} = \mathbb{G}_B^2$ (surely, this can be checked directly). Thus,

$$\mathbb{G}_{Z_4^\beta} = \mathbb{G}_B + \beta^2\mathbb{G}_B^2.$$

Therefore, we can apply Lemma 1. Since the small ball asymptotics for the Brownian bridge was obtained long ago

$$\mathbb{P}\{||B||_2 \le \varepsilon\} \sim \sqrt{\frac{8}{\pi}} \cdot \exp\Big(-\frac{1}{8}\,\varepsilon^{-2}\Big),$$

it remains to calculate

$$\widehat{C}^2_{\text{dist}} = \prod_{n=1}^{\infty} \frac{\lambda_n(B)}{\lambda_n(B) + \beta^2\lambda_n^2(B)} = \prod_{n=1}^{\infty} \frac{(\pi n)^2}{(\pi n)^2 + \beta^2}.$$

The application of Hadamard's theorem to the function $F_4(\zeta) = \frac{\sin(\zeta)}{\zeta}$ gives

$$\prod_{n=1}^{\infty} \Big(1 + \frac{\beta^2}{(\pi n)^2}\Big) = \frac{F_4(i\beta)}{F_4(0)} = \frac{\sinh(\beta)}{\beta},$$

and we arrive at the following Theorem:

Theorem 4. *The following asymptotic relation holds as* $\varepsilon \to 0$*:*

$$\mathbb{P}\{||B + \beta\overline{W}||_2 \le \varepsilon\} \sim \sqrt{\frac{8\beta}{\pi\sinh(\beta)}} \cdot \exp\Big(-\frac{1}{8}\,\varepsilon^{-2}\Big).$$

Now we consider a family of mixed processes ($m \in \mathbb{N}$)

$$\widehat{Z}^{\beta}_{2m}(t) = W(t) + \beta W^{\mathcal{E}}_{2m}(t); \qquad \widehat{Z}^{\beta}_{2m-1}(t) = W(1-t) + \beta W^{\mathcal{E}}_{2m-1}(t), \qquad t \in [0,1],$$

where $W^{\mathcal{E}}_m$ is so-called *Euler integrated* Brownian motion, see [27,28]:

$$W^{\mathcal{E}}_0(t) = W(t); \qquad W^{\mathcal{E}}_{2m-1}(t) = \int_t^1 W^{\mathcal{E}}_{2m-1}(s)\,ds; \qquad W^{\mathcal{E}}_{2m}(t) = \int_0^t W^{\mathcal{E}}_{2m-1}(s)\,ds.$$

It was shown in [28], see also ([6], Proposition 5.1), that the covariance operator of $W^{\mathcal{E}}_m$ can be expressed as

$$\mathbb{G}_{W^{\mathcal{E}}_{2m}} = \mathbb{G}^{2m+1}_{W}; \qquad \mathbb{G}_{W^{\mathcal{E}}_{2m-1}} = \mathbb{G}^{2m}_{\widetilde{W}},$$

where $\widetilde{W}(t) = W(1-t)$. Obviously, the small ball asymptotics for $\widetilde{W}$ and for W coincide.

Thus, we can apply Lemma 1, and it remains to calculate

$$\widehat{C}^2_{\text{dist}} = \prod_{n=1}^{\infty} \frac{\lambda_n(W)}{\lambda_n(W) + \beta^2\lambda_n^{k+1}(W)} = \prod_{n=1}^{\infty} \frac{(\pi(n-\frac{1}{2}))^{2k}}{(\pi(n-\frac{1}{2}))^{2k} + \beta^2}$$

(here $k = 2m$ or $k = 2m-1$).

Application of Hadamard's theorem to the function $\cos(\zeta)$ gives

$$\prod_{n=1}^{\infty} \Big(1 - \frac{\zeta^2}{(\pi(n-\frac{1}{2}))^2}\Big) = \cos(\zeta). \tag{8}$$

Put $z = \exp\big(\frac{i\pi}{2k}\big)$ and multiply relations (8) for $\zeta = \beta^{\frac{1}{k}}z, \zeta = \beta^{\frac{1}{k}}z^3, \ldots, \zeta = \beta^{\frac{1}{k}}z^{2k-1}$. This gives

$$\widehat{C}^2_{\text{dist}} = \bigg(\prod_{j=1}^{k} \cos\big(\beta^{\frac{1}{k}}z^{2j-1}\big)\bigg)^{-1}.$$

We take into account that

$$\begin{aligned}\cos\big(\beta^{\frac{1}{k}}z^{2j-1}\big)\cdot\cos\big(\beta^{\frac{1}{k}}z^{2k-2j+1}\big) &= |\cos\big(\beta^{\frac{1}{k}}z^{2j-1}\big)|^2\\ &= \sinh^2\big(\beta^{\frac{1}{k}}\sin(\tfrac{\pi(2j-1)}{2k})\big)+\cos^2\big(\beta^{\frac{1}{k}}\cos(\tfrac{\pi(2j-1)}{2k})\big)\end{aligned}$$

and obtain the following Theorem:

Theorem 5. *For $m\in\mathbb{N}$, the following asymptotic relations hold as $\varepsilon\to 0$:*

$$\begin{aligned}&\mathbb{P}\{||W+\beta W_{2m}^{\mathcal{E}}||_2\le\varepsilon\}\sim\frac{4}{\sqrt{\pi}}\\ &\times\frac{1}{\sqrt{\prod\limits_{j=1}^{m}\big(\sinh^2\big(\beta^{\frac{1}{2m}}\sin(\frac{\pi(2j-1)}{4m})\big)+\cos^2\big(\beta^{\frac{1}{2m}}\cos(\frac{\pi(2j-1)}{4m})\big)\big)}}\cdot\varepsilon\cdot\exp\Big(-\frac{1}{8}\varepsilon^{-2}\Big);\end{aligned}$$

$$\begin{aligned}&\mathbb{P}\{||\widetilde{W}+\beta W_{2m-1}^{\mathcal{E}}||_2\le\varepsilon\}\sim\frac{4}{\sqrt{\pi\cosh\big(\beta^{\frac{1}{2m-1}}\big)}}\\ &\times\frac{1}{\sqrt{\prod\limits_{j=1}^{m-1}\big(\sinh^2\big(\beta^{\frac{1}{2m-1}}\sin(\frac{\pi(2j-1)}{4m-2})\big)+\cos^2\big(\beta^{\frac{1}{2m-1}}\cos(\frac{\pi(2j-1)}{4m-2})\big)\big)}}\cdot\varepsilon\cdot\exp\Big(-\frac{1}{8}\varepsilon^{-2}\Big).\end{aligned}$$

4. Discussion

We have initiated the study of the complicated problem of exact small deviations asymptotics in L_2 for mixed Gaussian processes with independent components. After the survey of the problem, we consider the linear combination of two processes whose covariance functions are Green functions for two different boundary value problems to the same differential equation. The simplest example here is given by the standard Wiener process $W(t)$ and the Brownian bridge $B(t)$. Also we provide the exact small deviation asymptotics for more complicated mixtures containing the Ornstein–Uhlenbeck processes.

Next, we deal with pairs of processes whose covariance functions are kernels of integral operators which are powers (or, more general, polynomials) of of the same integral operator. The basic example here is the Brownian bridge and the integrated centered Wiener process

$$\overline{W}(t)=\int_0^t\left(W(s)-\int_0^1 W(u)du\right)ds.$$

Another series of examples is given by the Wiener process and the so-called Euler integrated Wiener process.

It would be interesting to understand the genesis of boundary conditions and integral operators in the more general cases of mixed processes.

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