Recognition of $M \times M$ by Its Complex Group Algebra Where $M$ Is a Simple $K_3$-Group

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Abstract: In this paper we prove that if $M$ is a simple $K_3$-group, then $M \times M$ is uniquely determined by its order and some information on irreducible character degrees and as a consequence of our results we show that $M \times M$ is uniquely determined by the structure of its complex group algebra.

Keywords: character degree; order; complex group algebra

1. Introduction

Let $G$ be a finite group, $\text{Irr}(G)$ be the set of irreducible characters of $G$, and denote by $\text{cd}(G)$, the set of irreducible character degrees of $G$. A finite group $G$ is called a $K_3$-group if $|G|$ has exactly three distinct prime divisors. By [1], simple $K_3$-groups are $A_5$, $A_6$, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$. Chen et al. in [2,3] proved that all simple $K_3$-groups and the Mathieu groups are uniquely determined by their orders and one or both of their largest and second largest irreducible character degrees. In [4], it is proved that $L_2(q)$ is uniquely determined by its group order and its largest irreducible character degree when $q$ is a prime or when $q = 2^a$ for an integer $a \geq 2$ such that $2^a - 1$ or $2^a + 1$ is a prime.

Let $p$ be an odd prime number. In [5–8], it is proved that the simple groups $L_2(q)$ and some extensions of them, where $q \mid p^3$ are uniquely determined by their orders and some information on irreducible character degrees.

In ([9], Problem 2')R. Brauer asked: Let $G$ and $H$ be two finite groups. If for all fields $F$, two group algebras $FG$ and $FH$ are isomorphic can we get that $G$ and $H$ are isomorphic? This is false in general. In fact, E. C. Dade [10] constructed two nonisomorphic metabelian groups $G$ and $H$ such that $FG \cong FH$ for all fields $F$. In [11], Tong-Viet posed the following question:

Question. Which groups can be uniquely determined by the structure of their complex group algebras?

In general, the complex group algebras do not uniquely determine the groups, for example, $CD_8 \cong CQ_8$. It is proved that nonabelian simple groups, quasi-simple groups and symmetric groups are uniquely determined up to isomorphism by the structure of their complex group algebras (see [12–18]). Khosravi et al. proved that $L_2(p) \times L_2(p)$ is uniquely determined by its complex group algebra, where $p \geq 5$ is a prime number (see [19]). In [20], Khosravi and Khademi proved that the characteristically simple group $A_5 \times A_5$ is uniquely determined by its order and its character degree graph (vertices are the prime divisors of the irreducible character degrees of $G$ and two vertices $p$ and $q$ are joined by an edge if $pq$ divides some irreducible character degree of $G$). In this paper, we prove that if $M$ is a simple $K_3$-group, then $M \times M$ is uniquely determined by its order and some information about its irreducible character degrees. In particular, this result is the generalization of ([19], Theorem 2.4) for $p = 5, 7$ and 17. Also as a consequence of our results we show that $M \times M$ is uniquely determined by the structure of its complex group algebra.
2. Preliminaries

If \( \chi = \sum_{i=1}^{k} e_{i} \chi_{i} \), where for each \( 1 \leq i \leq k \), \( \chi_{i} \in \text{Irr}(G) \) and \( e_{i} \) is a natural number, then each \( \chi_{i} \) is called an irreducible constituent of \( \chi \).

**Lemma 1.** (Itô's Theorem) ([21], Theorem 6.15) Let \( A \leq G \) be abelian. Then \( \chi(1) \) divides \( |G : A| \), for all \( \chi \in \text{Irr}(G) \).

**Lemma 2.** ([21], Corollary 11.29) Let \( N \trianglelefteq G \) and \( \chi \in \text{Irr}(G) \). If \( \theta \) is an irreducible constituent of \( \chi_{N} \), then \( \chi(1)/\theta(1) \mid |G : N| \).

**Lemma 3.** ([2], Lemma 1) Let \( G \) be a non-solvable group. Then \( G \) has a normal series \( 1 \leq H \leq K \leq G \) such that \( K/H \) is a direct product of isomorphic nonabelian simple groups and \( |G/K| \mid |\text{Out}(K/H)| \).

**Lemma 4.** (Itô-Michler Theorem) [22] Let \( \rho(G) \) be the set of all prime divisors of the elements of \( \text{cd}(G) \). Then \( p \notin \rho(G) \) is \( \{ p : p \text{ is a prime number}, p \mid \chi(1), \chi \in \text{Irr}(G) \} \) if and only if \( G \) has a normal abelian Sylow \( p \)-subgroup.

**Lemma 5.** ([3], Lemma 2) Let \( G \) be a finite solvable group of order \( p_{1}^{a_{1}} p_{2}^{a_{2}} \cdots p_{n}^{a_{n}} \), where \( p_{1}, p_{2}, \ldots, p_{n} \) are distinct primes. If \( (k_{p_{n}} + 1) \mid p_{n}^{b_{n}} \), for each \( i \leq n - 1 \) and \( k > 0 \), then the Sylow \( p_{n} \)-subgroup is normal in \( G \).

**Lemma 6.** ([19], Theorem 2.4) Let \( p \geq 5 \) be a prime number. If \( G \) is a finite group such that (i) \( |G| = |L_{2}(p)|^{2} \), (ii) \( p^{2} \in \text{cd}(G) \), (iii) there does not exist any element \( a \in \text{cd}(G) \) such that \( 2p^{2} \mid a \), (iv) if \( p \) is a Mersenne prime or a Fermat prime, then \( (p \pm 1)^{2} \in \text{cd}(G) \), then \( G \cong L_{2}(p) \times L_{2}(p) \).

3. The Main Results

**Lemma 7.** Let \( S \) be a simple \( K_{3} \)-group and let \( G \) be an extension of \( S \) by \( S \). Then \( G \cong S \times S \).

**Proof.** There exists a normal subgroup of \( G \) which is isomorphic to \( S \) and we denote it by the same notation. By [23], we know that \( |\text{Out}(S)| \leq 4 \) and \( G/\text{C}_{G}(S) \rightarrow \text{Aut}(S) \), which implies that \( \text{C}_{G}(S) \neq 1 \). As \( S \) is a non-solvable simple group, \( 5 \cap \text{C}_{G}(S) = 1 \) and it follows that \( \text{SC}_{G}(S) \cong S \times C_{G}(S) \). Also \( C_{G}(S) \cong SC_{G}(S)/S \leq G/S \cong S \) which implies that \( G \) is isomorphic to \( S \times S \). \( \square \)

**Theorem 1.** Let \( G \) be a finite group. Then \( G \cong A_{5} \times A_{5} \) if and only if \( |G| = |A_{5}|^{2} \) and \( 5^{2} \in \text{cd}(G) \).

**Proof.** Obviously by Itô's theorem, we get that \( O_{5}(S) = 1 \). First we show that \( G \) is not a solvable group. If \( G \) is a solvable group, then let \( H \) be a Hall subgroup of \( G \) of order \( 2^{3} 5^{2} \). Since \( G/H \rightarrow S_{5} \), we get that \( 5 \mid |H_{G}| \). If \( 5^{2} \mid |H_{G}| \), then \( 25 \in \text{cd}(H_{G}) \). On the other hand, \( 25^{2} < |H_{G}| \leq 2^{4} 5^{2} \), a contradiction. If \( |H_{G}| = 2^{4} 5^{2} \), then \( |G/H_{G}| = 45 \). Let \( L/H_{G} \) be a Sylow 5-subgroup of \( G/H_{G} \). Then \( L/H_{G} \cong G/H_{G} \) and \( L \leq G \) and \( |L| = 5^{2} 2^{4} \). Then \( 25 \in \text{cd}(L) \), which is a contradiction. If \( |H_{G}| \mid 2^{3} 5^{2} \), then \( P \), a Sylow 5-subgroup of \( H_{G} \) is a normal subgroup of \( G \), which is a contradiction by Lemma 4. Therefore \( G \) is a non-solvable group.

Since \( G \) is non-solvable, by Lemma 3, \( G \) has a normal series \( 1 \leq H \leq K \leq G \) such that \( K/H \) is a direct product of isomorphic nonabelian simple groups and \( |G/K| \mid |\text{Out}(K/H)| \). As \( |G| = 2^{4} 3^{2} 5^{2} \), we have \( K/H \cong A_{5}, A_{5} \) or \( A_{5} \times A_{5} \) by [23]. If \( K/H \cong A_{6} \), then \( |H| = 5 \). Using Lemma 2, \( 5 \in \text{cd}(H) \), a contradiction. If \( K/H \cong A_{5} \), then \( |H| = 60 \) or \( |H| = 30 \). By Lemma 2, \( 5 \in \text{cd}(H) \). If \( H \) is a solvable group, then by Lemma 5, \( P \leq H \), where \( P \in \text{Syl}_{5}(H) \), which is a contradiction. Therefore \( |H| = 60 \) and so \( H \cong A_{5} \). Hence \( G \) is an extension of \( A_{5} \) by \( A_{5} \) and by Lemma 7, \( G \cong A_{5} \times A_{5} \). If \( K/H \cong A_{5} \times A_{5} \), then \( |H| = 1 \) and \( G \cong A_{5} \times A_{5} \). \( \square \)

**Theorem 2.** Let \( G \) be a finite group. Then \( G \cong L_{2}(17) \times L_{2}(17) \) if and only if \( |G| = |L_{2}(17)|^{2} \) and \( 17^{2} \in \text{cd}(G) \).
Proof. Obviously $O_{17}(G) = 1$. On the contrary let $G$ be a solvable group. First we show that there exists no normal subgroup $N$ of $G$ such that

(a) $|N| = 2^k 3^i 17^j$, where $k \neq 0$ and $i < 8$; or (b) $|N| = 2^8 17^2$; or (c) $|N| = 2^8 17$. Let $N$ be a normal subgroup of $G$. If $|N| = 2^k 3^i 17^j$, where $k \neq 0$ and $i < 8$, then by Lemma 5, $P \trianglelefteq G$, where $P \in \text{Syl}_{17}(G)$. Hence $O_{17}(G) \neq 1$, which is a contradiction. If $|N| = 2^8 17^2$, then $17^2 \in \text{cd}(N)$, which is impossible. If $|N| = 2^8 17$, then $|G/N| = 3^i 17$. If $T/N \in \text{Syl}_{17}(G/N)$, then $T/N \trianglelefteq G/N$. Therefore $T \trianglelefteq G$, where $|T| = 17^2 2^8$ and this is a contradiction as we stated above.

Let $M$ be a minimal normal subgroup of $G$, which is an elementary abelian $p$-group. Obviously $p \neq 17$. Let $p = 2$. Then $|M| = 2^i$, where $0 < i \leq 8$ and so $|G/M| = 2^{8-i} 3^i 17^2$. Then $T/M \trianglelefteq G/M$, where $T/M \in \text{Syl}_{17}(G/M)$. Therefore $T \trianglelefteq G$ and $|T| = 17^2 2^8$, which is a contradiction. Hence $p = 3$ and $|M| = 3^i$, where $1 \leq i \leq 4$.

If $i = 4$, then $G/C_G(M) \rightarrow \text{Aut}(M) \cong \text{GL}(4,3)$ and $|\text{GL}(4,3)| = 2^9 \times 3^6 \times 5 \times 13$. Hence $17^2 \mid |C_G(M)|$. Since $M$ is an abelian subgroup of $G$, thus $3^4 \mid |C_G(M)|$. If $|C_G(M)| = 17^2 3^j 2^i$, where $j \neq 8$, then by the above discussion we get a contradiction. Otherwise, $C_G(M) = G$ and so by Burnside normal $p$-complement theorem, $G$ has a normal 3-complement of order $17^2 2^8$, which is a contradiction.

If $i = 3$, then $|G/M| = 2^5 17^2$. Let $H/M$ be a Hall subgroup of $G/M$ of order $2^5 17^2$. Then $|H| = 2^3 3^i 17^2$. Since $G/H_G \rightarrow S_3$, thus $3^i 17^2 \mid |H_G|$. If $2^3 \nmid |H_G|$, then by the above discussion we get a contradiction. Therefore $|H_G| = 2^5 3^3 17^2$, i.e., $H \trianglelefteq G$. Let $B$ be a Hall subgroup of $H$ of order $|B| = 2^5 17^2$. Then similarly to the above discussion we get that $H_G$ has a normal subgroup of order $17^2 2^8$, which is a contradiction. If $|H_G| = 2^5 \times 17^3$, then $|G/H_G| = 3^i 17$. Therefore $T/H \trianglelefteq G/H_G$, where $T/H \in \text{Syl}_{17}(G/H_G)$. Hence $T \trianglelefteq G$ and $|T| = 2^5 17^2$, which is a contradiction.

Therefore $G$ is nonsolvable and by Lemma 3, $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_2(17)$ or $L_2(17) \times L_2(17)$ and $|G/K| = |\text{Out}(K/H)|$.

If $K/H \cong L_2(17)$, then $|H| = 2^3 3^2 17$ or $2^4 3^2 17$ and so $17 \in \text{cd}(H)$. If $H$ is a solvable group, then by Lemma 5, $P \trianglelefteq H$, where $P \in \text{Syl}_{17}(H)$, which is a contradiction by Lemma 4. Otherwise by Lemma 3 and [23] we get that $H \cong L_2(17)$. Therefore $G$ is an extension of $L_2(17)$ by $L_2(17)$ and by Lemma 7, $G \cong L_2(17) \times L_2(17)$.

Obviously if $K/H \cong L_2(17) \times L_2(17)$, then $G \cong L_2(17) \times L_2(17)$.

In the sequel, we show that if $G$ is a finite group of order $|L_2(7) \times L_2(7)|$, such that $G$ has an irreducible character of order $7^2$ or $2^6$, then we can not conclude that $G \cong L_2(7) \times L_2(7)$. So we need more assumptions to characterize $L_2(7) \times L_2(7)$.

**Remark 1.** Using the notations of GAP [24], if $A = \text{SmallGroup}(56,11)$ and $H = A \times A \times \mathbb{Z}_9$, then $|H| = |L_2(7) \times L_2(7)|$ and $H$ has an irreducible character of degree $7^2$.

Similarly if $B = \text{SmallGroup}(784,160)$ and $K = B \times S_3 \times S_3$, then $|H| = |L_2(7) \times L_2(7)|$ and $H$ has an irreducible character of degree $2^6$.

**Theorem 3.** Let $G$ be a finite group. Then $G \cong L_2(7) \times L_2(7)$ if and only if $|G| = 2^9 3^2 7^2$ and $2^6, 7^2 \in \text{cd}(G)$.

**Proof.** If $G$ is a solvable group, then let $H$ be a Hall subgroup of $G$ of order $2^9 3^2 7^2$. Since $G/H_G \rightarrow S_9$, we have $|H_G| = 2^9 7^3$, where $0 \leq i \leq 6$ and $1 \leq j \leq 2$. Using Lemma 2, $2^i 7^j \in \text{cd}(H_G)$. If $O_2(H_G) \neq 1$,
then by Lemma 2, $|O_2(H_G)| \in \text{cd}(O_2(H_G))$, which is a contradiction. Similarly $O_7(H_G) = 1$, which shows that $G$ is a nonsolvable group.

Therefore $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_2(8), L_2(7)$ or $L_2(7) \times L_2(7)$ and $|G/K| = |\text{Out}(K/H)|$.

If $K/H \cong L_2(8)$, then $|H| = 56$. Using Lemma 2, $8 \in \text{cd}(H)$ and since $64 > 56$, we get a contradiction.

If $K/H \cong L_2(7)$, then $|H| = 2^3 \times 3 \times 7$ or $2^3 \times 3 \times 7$. If $|H| = 2^3 \times 3 \times 7$, then by Lemma 2, $7 \in \text{cd}(H)$. Since there exists no nonabelian simple group $S$ such that $|S| = |H|$, we get that $H$ is a solvable group. Then by Lemma 5, $P \trianglelefteq H$ where $P \in \text{Syl}_7(H)$, which is a contradiction by Lemma 4. So $|H| = 2^3 \times 3 \times 7$, by the same argument for the proof of Theorem A in [2], we get that $H \cong L_2(7)$.

Therefore $G$ is an extension of $L_2(7)$ by $L_2(7)$ and by Lemma 7, $G \cong L_2(7) \times L_2(7)$. $\square$

Remark 2. We note that Theorems 1, 2 and 3 are generalizations of Lemma 6 for special cases $p = 5, 7, 17$.

Lemma 8. Let $G$ be a finite group. If $|G| = 2^i3^j5^k$, where $i \geq 3$ or $j \geq 1$, and $2^i, 3^j \in \text{cd}(G)$, then $G$ is not solvable. If $|G| = 2^i3^j5^k$, where $i \geq 6$ or $j \geq 2$, and $2^i, 3^j \in \text{cd}(G)$, then $G$ is not solvable.

Proof. On the contrary let $G$ be a solvable group.

Let $O_2(G) \neq 1$ and $|O_2(G)| = 2^i$, where $1 \leq t \leq i$. By the assumption, there exists $\chi \in \text{Irr}(G)$ such that $\chi(1) = 2^i$. If $\sigma \in \text{Irr}(O_2(G))$ such that $[\chi_{O_2(G)}(\sigma)] \neq 0$, then by Lemma 2, $2^i/\sigma(1)$ is a divisor of $|G : O_2(G)| = 2^{i-t}$. Since $\sigma(1) \mid |O_2(G)|$, we get that $\sigma(1) = 2^i$, which is a contradiction. Similarly $O_5(G) = 1$.

Therefore $\text{Fit}(G) = O_5(G) \neq 1$. We know that $G/C_G(\text{Fit}(G)) \hookrightarrow \text{Aut}(\text{Fit}(G))$ and since $G$ is a solvable group, $C_G(\text{Fit}(G)) \leq \text{Fit}(G)$. Therefore $|G|$ is a divisor of $|\text{Fit}(G)| \cdot |\text{Aut}(\text{Fit}(G))|$ and easily we can see that in each case we get a contradiction. $\square$

Similarly to the above we have the following result:

Lemma 9. Let $G$ be a finite group.

(a) If $|G| = 2^i3^j7^k$, where $i \geq 2$ or $j \geq 2$, and $2^i, 3^j \in \text{cd}(G)$, then $G$ is not solvable.

(b) If $|G| = 2^i3^j7^k$, where $i \geq 6$ or $j \geq 2$, and $2^i, 3^j \in \text{cd}(G)$, then $G$ is not solvable.

Theorem 4. Let $G$ be a finite group.

(a) If $|G| = 2^63^45^2$ and $2^6, 3^4 \in \text{cd}(G)$, then $G \cong A_6 \times A_6$ or $G \cong \mathbb{Z}_5 \times U_4(2)$;

(b) If $|G| = 2^12^33^45^2$ and $2^6, 3^4 \in \text{cd}(G)$, then $G \cong U_4(2) \times U_4(2)$.

Proof. Lemma 8 gives us that $G$ is not solvable and so $G$ has a normal series $1 \leq H \leq K \leq G$ such that $K/H$ is a direct product of isomorphic nonabelian simple groups and $|G/K| = |\text{Out}(K/H)|$.

(a) By assumptions $K/H$ is isomorphic to $A_5, A_6, U_4(2), A_5 \times A_5$ or $A_6 \times A_6$.

If $K/H \cong A_5$, then $|H| = 2^33^55$ or $|H| = 2^33^55^2$. By Lemma 8, $H$ is not soluble and $H$ has a normal series $1 \leq A \leq B \leq H$ such that $B/A$ is a direct product of $m$ copies of a nonabelian simple group $S$ and $|H/B| = |\text{Out}(B/A)|$. If $|H| = 2^33^55^2$, we have $B/A \cong A_5$ or $A_6$. Then $|A| = 36, 18, 6$ or $3$, which is a contradiction. If $|H| = 2^33^55^3$, then similarly we get a contradiction.

If $K/H \cong A_6$, then $|H| = 2^33^55^3$, where $1 \leq i \leq 3$. By Lemma 2, $2^3, 3^2 \in \text{cd}(H)$. Using Lemma 8, $H$ is not a solvable group and so $i \neq 1$. Also $H$ has a normal series $1 \leq A \leq B \leq H$ such that $B/A$ is a direct product of $m$ copies of a nonabelian simple group $S$ and $|H/B| = |\text{Out}(B/A)|$. If $|H| = 2^33^55^3$, by Theorem B in [2], we get that $H \cong A_6$, and so by Lemma 7, $G \cong A_6 \times A_6$. If $|H| = 2^33^55^4$, then $|A| = 3$, which is a contradiction.
If $K/H \cong U_4(2)$, then $|H| = 5$ and $G = K$. Therefore $G$ is an extension of $\mathbb{Z}_5$ by $U_4(2)$. We know that $G/C_G(H) \cong \text{Aut}(H)$ and $(G/H)/(C_G(H)/H) \cong G/C_G(H)$. So $G$ is a central extension of $H$ by $U_4(2)$. Since the Schur multiplier of $U_4(2)$ is 2, we get that $G \cong \mathbb{Z}_5 \times U_4(2)$.

Let $K/H \cong A_5 \times A_5$. We know that $\text{Out}(K/H) \cong \text{Out}(A_5) \times S_2$, and so $|G/K| \mid 8$. Thus $|H| = 2^i 3^j$, where $0 \leq i \leq 2$, which is a contradiction.

Finally, if $K/H \cong A_6 \times A_6$, then $G \cong A_6 \times A_6$.

(b) In this case, we have $K/H \cong A_5, A_6, U_4(2), A_5 \times A_5, A_6 \times A_6$ or $U_4(2) \times U_4(2)$.

If $K/H \cong A_5$, then $|H| = 2^{10} 3^{17}$ or $2^9 3^{15}$. By Lemma 8, $H$ is not a solvable group and $H$ has a normal series $1 \leq A \leq B \leq H$ such that $B/A$ is a nonabelian simple group. Therefore $A$ is a $\{2,3\}$-group such that $O_2(A) = O_3(A) = 1$ and this is a contradiction.

If $K/H \cong A_6$, then similarly to the above we get a contradiction.

If $K/H \cong U_4(2)$, then $|H| = 2^i 3^j$, where $5 \leq i \leq 6$. By Lemma 2, $2^i 3^j \in \text{cd}(H)$. Therefore $H$ is not a solvable group and $H$ has a normal series $1 \leq A \leq B \leq H$ such that $B/A$ is a nonabelian simple group. If $|H| = 2^5 3^4$, then $A$ is a $\{2,3\}$-group such that $O_2(A) = O_3(A) = 1$ and this is a contradiction. If $|H| = 2^5 3^4$, by Theorem A in [2], we get that $H \cong U_4(2)$ and by Lemma 7, $G \cong U_4(2) \times U_4(2)$.

Let $K/H \cong A_6 \times A_5$. We know that $\text{Out}(K/H) \cong \text{Out}(A_5) \times S_2$. Therefore $|G/K| \mid 8$ and thus $|H| = 2^i 3^j$, where $1 \leq i \leq 6$, which is a contradiction.

Therefore $K/H \cong U_4(2) \times U_4(2)$, and so $G \cong U_4(2) \times U_4(2)$. □

**Corollary 1.** If $|G| = 2^6 3^{14} 5^2$ and $2^6, 3^4 \in \text{cd}(G)$ and $6 \notin \text{cd}(G)$, then $G \cong A_6 \times A_6$.

**Theorem 5.** If $|G| = 2^{10} 3^{6} 5^2$ and $2^6, 3^6 \in \text{cd}(G)$, then $G \cong U_3(3) \times U_3(3)$.

**Proof.** By Lemma 9 it follows that $G$ is not solvable and $G$ has a normal series $1 \leq H \leq K \leq G$ such that $K/H \cong L_2(7), L_2(8), U_3(3), L_2(7) \times L_2(7), L_2(8) \times L_2(8)$ or $U_3(3) \times U_3(3)$ and $|G/K| \mid |\text{Out}(K/H)|$.

If $K/H \cong L_2(7)$, then $|H| = 2^i 3^j$ or $2^6 3^5 7$. By Lemma 9, $H$ is not solvable and $H$ has a normal series $1 \leq A \leq B \leq H$ such that $B/A$ is a nonabelian simple group. If $|H| = 2^4 3^{5} 7$, then $A$ is a $\{2,3\}$-group such that $O_2(A) = O_3(A) = 1$, which is a contradiction. If $|H| = 2^5 3^{3} 7$, by Theorem C in [2], we get that $H \cong U_3(3)$ and by Lemma 7, $G \cong U_3(3) \times U_3(3)$.

Finally, if $K/H \cong U_3(3) \times U_3(3)$, then obviously $G \cong U_3(3) \times U_3(3)$. □

**Theorem 6.** If $G$ is a finite group such that

(i) $|G| = 2^6 3^{14} 7^2$,

(ii) $2^6, 3^4 \in \text{cd}(G)$,

(iii) $6, 12, 18 \notin \text{cd}(G)$,

then $G \cong L_2(8) \times L_2(8)$.

**Proof.** By Lemmas 3 and 9, we get that $G$ has a normal series $1 \leq H \leq K \leq G$ such that $K/H \cong L_2(7), L_2(8), U_3(3), L_2(7) \times L_2(7)$ or $L_2(8) \times L_2(8)$, and $|G/K| \mid |\text{Out}(K/H)|$.

If $K/H \cong L_2(7)$, then $|H| = 2^i 3^j$ or $2^6 3^{5} 7$. By Lemma 9, $H$ is not a solvable group and $H$ has a normal series $1 \leq A \leq B \leq H$ such that $B/A$ is a nonabelian simple group and $|H/B| \mid |\text{Out}(B/A)|$. 


If $|H| = 2^33^27$, we have $B/A \cong L_2(7)$ or $L_2(8)$. If $B/A \cong L_2(7)$, then $|A| = 3^2$, a contradiction. If $B/A \cong L_2(8)$, then by Ito’s theorem, $|A| = 1$ and $1 \leq B \cong L_2(8) \leq H$, where $|H : B| = 3$. By the proof of Lemma 1 in [2] (Lemma 3 in the present paper), $H/B$ is isomorphic to a subgroup of Out($B/A$) and by [23] we have $H \cong L_2(8).3$. Using GAP cd($H$) = {1,7,8,21,27}, Z($H$) = 1 and Aut($H$) = H. Now similarly to the proof of Lemma 7, $G \cong (L_2(8).3) \times L_2(7)$. Then $6 \in$ cd($G$), which is a contradiction by (iii). If $|H| = 2^33^27$, then by Lemma 9, $H$ is not a solvable group, and this is a contradiction by [23].

If $K/H \cong L_2(8)$, then $|H| = 2^3 \cdot 3^2 \cdot 7$ or $2^3 \cdot 3 \cdot 7$. Using Lemma 9, $H$ is not a solvable group. If $|H| = 2^3 \cdot 3^2 \cdot 7$, by the same argument as Theorem C in [2], we get that $H \cong L_2(8)$ and by Lemma 7, $G \cong L_2(8) \times L_2(8)$. If $|H| = 2^3 \cdot 3 \cdot 7$, then by Theorem A in [2], $H \cong L_2(7)$. Since $K/H \cong L_2(8)$, similarly to the proof of Lemma 7, we get that $K \cong L_2(7) \times L_2(8)$. So $G$ is an extension of $Z_3$ by $L_2(7) \times L_2(8)$. Since $6 \in$ cd($G$) or $18 \in$ cd($G$), we get a contradiction by (iii).

If $K/H \cong U_3(3)$, then $|H| = 42$ or $|H| = 21$.

If $|H| = 42$, then $H$ is solvable and $H'$ is a cyclic group, since $|H'| = 7$ and $|H/H'| = 6$. Now easily we see that the equation $\sum_{\varphi \in \text{Irr}(H)} \varphi^2(1) = |H|$, where $\varphi(1) \mid |H|$, has no solution and so we get a contradiction.

If $|H| = 21$, then by Lemma 2, we get that $3 \in$ cd($H$) and so $H$ is a Frobenius group of order 21, which is denoted by 7 : 3. Also $Z(H) = 1$ and Aut($H$) $\cong$ H.2. Now similarly to the proof of Lemma 7, we get that $K \cong (7 : 3) \times U_3(3)$. Since $|G : K| = 2$, we have $G \cong (7 : 3) \times U_3(3).2$ and so $6 \in$ cd($G$) or 12 $\in$ cd($G$), which is a contradiction by (iii).

If $K/H \cong L_2(7) \times L_2(7)$. We know that Out($K/H$) $\cong$ Out($L_2(7)$) : $S_2$. Then $|G/K| \mid 8$ and thus $|H| = 2^3$, which is a contradiction.

Finally $K/H \cong L_2(8) \times L_2(8)$, and so $G \cong L_2(8) \times L_2(8)$. \hfill $\square$

**Theorem 7.** If $|G| = |L_3(3)|^2$ and $2^33^6 \in$ cd($G$), then $G \cong L_3(3) \times L_3(3)$.

**Proof.** First we show that $G$ is not a solvable group. If $G$ is a solvable group, then $O_2(G) = O_3(G) = 1$ and so $\text{Fit}(G) = O_3(G) \neq 1$. Since $|\text{Aut}(Z_{13})| = 2^33$, $|\text{Aut}(Z_{169})| = 2^3 \cdot 3 \cdot 13$ and $|\text{Aut}(Z_{13} \times Z_{13})| = 2^3 \cdot 3^2 \cdot 7 \cdot 13$, therefore $|G| \mid |\text{Fit}(G)| \cdot |\text{Aut}(\text{Fit}(G))|$, which is a contradiction. Therefore $G$ is nonsolvable and $G$ has a normal series $1 \leq H \leq K \leq G$ such that $K/H \cong L_3(3)$ or $L_3(3) \times L_3(3)$, where $|G/K| \mid \text{Out}(K/H)$. If $K/H \cong L_3(3) \times L_3(3)$, then $G = L_3(3) \times L_3(3)$. If $K/H \cong L_3(3)$, then $|G/K| = 1$ or 2, and thus $|H| = 2^33^313$ or $|H| = 2^33^313$. If $H$ is a solvable group, then $\text{Fit}(H) \cong Z_{13}$ and $|H| \mid |\text{Fit}(H)| \cdot |\text{Aut}(\text{Fit}(H))|$, which is a contradiction. Hence $H$ is not a solvable group and so $H \cong L_3(3)$ and by Lemma 7, $G \cong L_3(3) \times L_3(3)$. \hfill $\square$

As a consequence of the above theorem, by ([25], Theorem 2.13), we have the following result which is a partial answer to the question arose in [11].

**Corollary 2.** Let $M$ be a simple $K_3$-group and $H = M \times M$. If $G$ is a group such that $\mathbb{C}G \cong \mathbb{C}H$, then $G \cong H$. Thus $M \times M$, where $M$ is a simple $K_3$-group, is uniquely determined by the structure of its complex group algebra.

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