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Article

On p-Cyclic Orbital M-K Contractions in a Partial Metric Space

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Abstract: A cyclic map with a contractive type of condition called p-cyclic orbital M-Kcontraction is introduced in a partial metric space. Sufficient conditions for the existence and uniqueness of fixed points and the best proximity points for these maps in complete partial metric spaces are obtained. Furthermore, a necessary and sufficient condition for the completeness of partial metric spaces is given. The results are illustrated with an example.

Keywords: contraction; p-cyclic mappings; best proximity point; partial metric space

MSC: 47H10; 54H25

1. Introduction and Preliminaries

An important field of application of fixed point theory exist nowadays in the investigation of the stability of complex continuous-time and discrete-time dynamic systems [1–3]. Meir–Keeler self-mappings have received important attention in the context of fixed point theory perhaps due to the associated relaxing in the required conditions for the existence of fixed points compared with the usual contractive mappings [4–6]. A connection between p-cyclic contractive, p-cyclic Kannan and p-cyclic Meir–Keeler contractions was obtained in [7]. The notion of orbital contractions introduced in [8] weakens the contraction condition in a different way by assuming that the contractive condition is not satisfied for all pairs $(x, y) \in X \times X$. On the other hand, an extension of Banach's fixed point theorem is obtained in [9] by considering a cyclical contractive condition. Some generalizations of cyclic maps that involve Meir–Keeler maps were given in [5,10–12]. Another kind of generalization of the Banach contraction principle is by altering the underlying space. Such kinds of generalizations are also obtained in partial metric spaces. To understand partial metric spaces, one may refer to [13–15]. For generalizations of the Banach contraction theorem, in which the underlying space is a partial metric space, one may refer to [16–20]. In this article, we give the conditions for the existence of a unique fixed point and a best proximity point of p-cyclic orbital Meir–Keeler maps in partial metric spaces.

A partial metric space is a generalization of a metric space, as is well seen from the following definition:

Definition 1 ([13]). *A partial metric space is a pair (X, ρ), where $\rho : X \times X \to \mathbb{R}^+$ such that:*

(1) $0 \le \rho(x, x) \le \rho(x, y)$ *(non-negativity and small self-distances)*
(2) *If* $\rho(x, x) = \rho(x, y) = \rho(y, y)$ *then* $x = y$ *(indistancy implies equality)*
(3) $\rho(x, y) = \rho(y, x)$ *(symmetry)*

(4) $\rho(x,z) \leq \rho(x,y) + \rho(y,z) - \rho(y,y)$ *(triangularity).*

A metric space can be defined to be a partial metric space in which each self-distance is zero.

It is easy to see that if (X,ρ) is a partial metric space and if $\rho(x,y) = 0$, then $x = y$.

Since for any $x,y,z \in X$, $\rho(x,z) \leq \rho(x,y) + \rho(y,z) - \rho(y,y) \leq \rho(x,y) + \rho(y,z)$, we have $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. From the triangle inequality, it follows that $\rho(x,\cdot)$ and $\rho(\cdot,\cdot)$ are continuous functions, i.e., $\lim_{n\to\infty} \rho(x,z_n) = \rho(x,z)$, provided that $\lim_{n\to\infty} \rho(z,z_n) = 0$ and $\lim_{n\to\infty} \rho(x_n,z_n) = \rho(x,z)$, provided that $\lim_{n\to\infty} \rho(x,x_n) = 0$ and $\lim_{n\to\infty} \rho(z,z_n) = 0$.

Example 1. *For an example of a partial metric space, let us consider I to be the collection of all nonempty, closed and bounded intervals in $\mathbb{R}$. That is, $I = \{[a,b] : a \leq b\}$. For $[a,b],[c,d] \in I$, let:*

$$\rho([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\},$$

then it can be shown that ρ is a partial metric over I, and the self-distance of $[a,b]$ is the length b-a. This is related to the real line as follows:

$|a-b| = \rho([a,a],[b,b])$*, and so, by mapping each $a \in \mathbb{R}$ to $[a,a]$, we embed the usual metric structure of $\mathbb{R}$ into that of the partial metric structure of intervals.*

Following [15], we will recall some basic facts and definitions about partial metric spaces.

Each partial metric ρ on X induces a T_0 topology τ_ρ on X, which has the base of the family of open balls $\{B_\rho(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_\rho(x,\epsilon) = \{y \in X : \rho(x,y) < \epsilon + \rho(x,x)\}$ for all $x \in X$ and $\epsilon > 0$.

If ρ is a partial metric on X, then the function $\rho^s : X \times X \to [0,\infty)$ given by:

$$\rho^s(x,y) = 2\rho(x,y) - \rho(x,x) - \rho(y,y),$$

is a metric on X.

Definition 2. *Let (X,ρ) be a partial metric space.*

1. *A sequence $\{x_n\}$ in (X,ρ) converges to a point $x \in X$ if and only if* $\lim_{n,m\to\infty} \rho(x_n,x_m) = \lim_{n\to\infty} \rho(x_n,x) = \rho(x,x)$*;*
2. *A sequence $\{x_n\}$ in (X,ρ) is called a Cauchy sequence if there exists a finite limit* $\lim_{n,m\to\infty} \rho(x_n,x_m)$*;*
3. *(X,ρ) is called a complete partial metric space if every Cauchy sequence $\{x_n\}$ converges, with respect to τ_ρ, to a point $x \in X$, such that $\rho(x,x) = \lim_{n,m\to\infty} \rho(x_n,x_m)$.*

Lemma 1 ([18])**.** *A sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a partial metric space (X,ρ) if and only if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the metric space (X,ρ^s). A partial metric space (X,ρ) is complete if and only if the metric space (X,ρ^s) is complete. Moreover,*

$$\lim_{n\to\infty} \rho^s(x,x_n) = 0 \text{ if and only if } \rho(x,x) = \lim_{n\to\infty} \rho(x,x_n) = \lim_{n,m\to\infty} \rho(x_n,x_m).$$

Below, we obtain a new partial metric from the existing metric and partial metric.

Example 2. *Let $X = [0,\infty)$, and let $\rho : X \times X \to [0,\infty)$ be given by $\rho(x,y) = \max\{x,y\}$ for all $x,y \in X$. Then, it is easy to see that (X,ρ) is a complete partial metric space.*

Example 3 ([19])**.** *Let (X,d) and (X,ρ) be a metric space and a partial metric space, respectively. Functions ρ_i:$X \times X \to [0,+\infty)$, for $i = 1,2,3$, are defined by:*

1. $\rho_1(x,y) = d(x,y) + \rho(x,y)$
2. $\rho_2(x,y) = d(x,y) + \max\{\omega(x),\omega(y)\}$

3. $\rho_3(x,y) = d(x,y) + a$

for all $x, y \in X$ introduce partial metrics on X, where $\omega : X \to [0, +\infty)$ is an arbitrary function and $a \geq 0$.

Now, we recall the notion of L-functions introduced by Lim in [21], which is useful to prove the main result.

Definition 3 ([21])**.** *A function $\phi : [0, \infty) \to [0, \infty)$ is called an L-function if $\phi(0) = 0$; $\phi(s) > 0$ for every $s > 0$ and for every $s > 0$, there exists $u > s$ such that $\phi(t) \leq s$ for $t \in [s, u]$.*

Note that every L-function satisfies the condition $\phi(s) \leq s$ for every $s \geq 0$. Suzuki generalized Lim's results in [22]. The following result of Lim is used in the proof of the main result.

Lemma 2 ([22])**.** *Let Y be a nonempty set, and let $f, g : Y \to [0, \infty)$. Then, the following are equivalent:*

1. *For each $\epsilon > 0$, there exists a $\delta > 0$, such that $f(x) < \epsilon + \delta \Rightarrow g(x) < \epsilon$.*
2. *There exists an L-function ϕ (which may be chosen to be a non-decreasing and continuous) such that $f(x) > 0 \Rightarrow g(x) < \phi(f(x))$, $x \in Y$ and $f(x) = 0 \Rightarrow g(x) = 0$, $x \in Y$.*

2. Main Result

For the investigation of the best proximity points, the notion of the uniform convexity of a Banach space plays a crucial role. In [23], Suzuki et al. introduced the notion of the metric space, which satisfies property UC. In a similar way, we generalize this notion in the partial metric space.

Definition 4. *Let (X, ρ) be a partial metric space and A and B be subsets. The pair (A, B) is said to satisfy the property UC if the following holds: If $\{x_n\}, \{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B, such that $\lim_{n\to\infty} \rho(x_n, y_n) = \mathrm{dist}(A, B)$ and for any $\varepsilon > 0$, there is $n \in \mathbb{N}$, so that $\rho(z_m, y_n) < \mathrm{dist}(A, B) + \varepsilon$ for all $n \geq N$, then for any $\varepsilon > 0$, there is $N_1 \in \mathbb{N}$, so that $\rho(x_n, z_m) < \varepsilon$ for $m, n \geq N_1$.*

Now, let us recall the notion of cyclic maps.

Definition 5. *Let X be a nonempty set and $A_1, A_2, ..., A_p$ be nonempty subsets of X. A map $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is called a p-cyclic map if $T(A_i) \subseteq A_{i+1}$, for all $i = 1, 2, ..., p$, where we use the convention $A_{p+1} = A_1$.*

Definition 6. *Let (X, ρ) be a partial metric space and $A_1, A_2, ..., A_p$ be nonempty subsets of X. A point $x \in A_i$ is said to be a best proximity point of T in A_i, if $\rho(x, Tx) = \mathrm{dist}(A_i, A_{i+1})$, $1 \leq i \leq p$.*

We introduce the notion of p-cyclic orbital M-Kcontraction as follows:

Definition 7. *Let (X, ρ) be a partial metric space, $A_1, A_2, \ldots,$
A_p be nonempty subsets of X and $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map. The map T is called a p-cyclic orbital M-K contraction if for some $x \in A_i$, for each $\epsilon > 0$, there exists $\delta > 0$, such that the following condition:*

$$\rho(T^{pn+k-1}x, T^k y) < D_k + \varepsilon + \delta \Rightarrow \rho(T^{pn+k}x, T^{k+1}y) < D_{k+1} + \varepsilon, \tag{1}$$

holds for all $n \in \mathbb{N} \cup \{0\}$ and for all $y \in A_i$, where $D_k \geq 0$, for $k = 1, 2, ..., p$.

Theorem 1. *Let (X, ρ) be a complete partial metric space. Let $A_1, A_2, ..., A_p$ be nonempty and closed subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital M-K contraction map with constants D_k equal to zero or $D_k = \mathrm{dist}(A_{i+k-1}, A_{i+k})$, $k = 1, 2, \ldots, p$.*

1. *If $D_k = 0$ for all $k = 1, 2, \ldots p$, then $\cap_{i=1}^{p} A_i$ is nonempty, and T has a unique fixed point $\xi \in \cap_{i=1}^{p} A_i$. For any $x \in \cup_{i=1}^{p} A_i$, satisfying (1) with $D_k = 0$, $\lim_{n\to\infty} T^n x = \xi$ holds.*

2. *If $D_k = \operatorname{dist}(A_{i+k-1}, A_{i+k})$ for all $k = 1, 2, \ldots, p$ and (X, ρ) is a partial metric space with property UC, then for every $x \in A_i$ satisfying (1), the sequence $\{T^{pn}x\}$ converges to a unique point $z \in A_i$, which is the best proximity point, as well as the unique periodic point of T in A_i. Furthermore, $T^k z$ is a best proximity point of T in A_{i+k}, which is also a unique periodic point of T in A_{i+k}, for each $k = 1, 2, ..., (p-1)$.*

3. Auxiliary Results

Without loss of generality, let us assume that $x \in A_1$. The proof of the following lemma follows the technique used in [11].

Lemma 3. *Let (X, ρ) be a partial metric space. Let $A_1, A_2, ..., A_p$ be nonempty subsets of X. Let $T{:}\cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital M-K contraction map with constants D_k equal to zero or $D_k = \operatorname{dist}(A_k, A_{1+k})$. Then, there exists an L-function ϕ such that for an $x \in A_1$ satisfying (1), the following holds: if $\rho(T^{pn+k-1}x, T^k y) > D_k$, then:*

$$\rho(T^{pn+k}x, T^{k+1}y) - D_{k+1} < \phi(\rho(T^{pn+k-1}x, T^k y) - D_k) \tag{2}$$

$$\text{and if } \rho(T^{pn+k-1}x, T^k y) = D_k, \text{then } \rho(T^{pn+k}x, T^{k+1}y) = D_{k+1}, \tag{3}$$

for each $k = 1, 2, ..., p$, for all $n \in \mathbb{N}$ and for all $y \in A_1$.

Proof. Let $x \in A_1$ satisfy (1). For each $k = 0, 1, 2, ..., p-1$, define the following sets: $C_k = \{T^{pn+k-1}x : n \in \mathbb{N}\}$ and $B_k = \{T^k y : y \in A_1\}$. Let $f_k, g_k : C_k \times B_k \to [0, \infty)$ be defined as follows: $f_k(a_k, b_k) = \rho(T^{pn+k-1}x, T^k y) - D_k$ and $g_k(a_k, b_k) = \rho(T^{pn+k}x, T^{k+1}y) - D_{k+1}$. Since T is a p-cyclic orbital M-K contraction map, each f_k and g_k satisfies the condition (1) of Lemma 2, and hence, (2) and (3) hold. □

Remark 1. *From Lemma 3, it follows that for a p-cyclic orbital M-K contraction map T, the sequence $\{\rho(T^{pn+k-1}x, T^k y) - D_k\}_{n=1}^{\infty}$, $k = 1, 2, ..., p$ is non-increasing.*

Lemma 4. *Let (X, ρ) be a partial metric space. Let $A_1, A_2, ..., A_p$ be nonempty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital M-K contraction map with constants D_k equal to zero or $D_k = \operatorname{dist}(A_k, A_{1+k})$. Then, for any $x \in A_1$ satisfying (1), for all $y \in A_1$ and for each $k \in \{0, 1, 2, ..., p-1\}$, the sequence $\{\rho(T^{pn+k}x, T^{pn+k+1}y)\}_{n=1}^{\infty}$ converges to D_{k+1}.*

Proof. Let $s_n = \rho(T^{pn+k}x, T^{pn+k+1}y) - D_{k+1}$. Then, $s_n \geq 0$ for all $n \in \mathbb{N}$. By Remark 1, $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$. If $s_n = 0$ for some n, then the lemma follows. Suppose $s_n > 0$ for every $n \in \mathbb{N}$. Then, by Lemma 3, there exists an L-function ϕ satisfying (2) and (3). Since $s_{n+1} \leq s_n$, $\{s_n\}$ converges to an $r \geq 0$. Suppose $r > 0$. Then, for this $r > 0$, by (1), there exists a $\delta > 0$, such that $r \leq s_n < r + \delta$ and such that $s_{n+1} < \phi(s_n) \leq r$. That is, $s_{n+1} < r$, which is a contradiction. Hence, $r = 0$. Thus, the following holds $\rho(T^{pn+k}x, T^{pn+k+1}y) \to D_{k+1}$, when $n \to \infty$. □

Corollary 1. *Let (X, ρ) be a partial metric space. Let $A_1, A_2, ..., A_p$ be nonempty subsets of X. Let $T{:}\cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital M-K contraction map with constants D_k equal to zero. Then, for any $x \in A_1$, satisfying (1), the following holds:*

$$\rho(T^{pn+k}x, T^{k+1}y) \leq \rho(T^{pn+k-1}x, T^k y) \tag{4}$$

for all $n \in \mathbb{N}$, for all $y \in A_1$ and for each $k = 0, 1, 2, \ldots, p-1$.

Corollary 2. *Let (X, ρ) be a partial metric space. Let $A_1, A_2, ..., A_p$ be nonempty subsets of X. Let $T{:}\cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital Meir–Keeler contraction map with constants D_k, equal to zero for $k = 0, 1, 2, \ldots, p-1$; the following holds:*

$$\lim_{n \to \infty} \rho(T^{pn+k}x, T^{pn+k+1}y) = 0. \tag{5}$$

Lemma 5. *Let (X, ρ) be a complete partial metric space with property UC and $A_1, A_2, ..., A_p$ be non-empty and closed subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital M-K contraction map with constants $D_k = \operatorname{dist}(A_k, A_{k+1})$. Let $x \in A_1$ satisfy (1). Suppose that for each $k = 0, 1, 2, ..., p-1$, the sequence $\{T^{pn+k}x\}$ converges to $z_k \in A_{1+k}$. Then:*

(a) $\operatorname{dist}(A_1, A_2) = \operatorname{dist}(A_2, A_3) = \cdots = \operatorname{dist}(A_{p-1}, A_p) = \operatorname{dist}(A_p, A_1)$;
(b) *z_k is a best proximity point of T in A_{1+k} and $z_k = T^k z_0$, for $i = 1, 2, ..., p$;*
(c) *z_k is a periodic point of T with period p in A_{1+k}.*

Proof. From Lemma 4, it follows that $\lim_{n\to\infty} d(T^{pn+k-1}x, T^{pn+k}y) = D_k = \operatorname{dist}(A_k, A_{1+k})$.

(a) For any $k \in \{0, 1, 2, \ldots, p-1\}$, using the continuity of the function $\rho(x, \cdot)$ and Corollary 1, we get:

$$\begin{aligned} \operatorname{dist}(A_{k+1}, A_{k+2}) &\leq \rho(z_k, Tz_k) = \lim_{n\to\infty} \rho(T^{pn+k}x, Tz_k) \leq \lim_{n\to\infty} \rho(T^{pn+k-1}x, z_k) \\ &= \lim_{n\to\infty} \rho(T^{pn+k-1}x, T^{pn+k}x) = \operatorname{dist}(A_k, A_{k+1}). \end{aligned}$$

Thus:

$$\operatorname{dist}(A_1, A_2) \leq \operatorname{dist}(A_2, A_3) \leq \cdots \leq \operatorname{dist}(A_{p+1}, A_2) = \operatorname{dist}(A_1, A_2).$$

(b) For each $k = 0, 1, 2, ..., p-1$, we get:

$$\begin{aligned} \operatorname{dist}(A_{1+k}, A_{2+k}) &\leq \rho(z_k, Tz_k) = \lim_{n\to\infty} \rho(T^{pn+k}x, Tz_k) \leq \lim_{n\to\infty} \rho(T^{pn+k-1}x, z_k) \\ &= \lim_{n\to\infty} \rho(T^{pn+k-1}x, T^{pn+k}x) = \operatorname{dist}(A_k, A_{1+k}) = \operatorname{dist}(A_{1+k}, A_{k+2}). \end{aligned}$$

Hence, $\rho(z_k, Tz_k) = \operatorname{dist}(A_{1+k}, A_{k+2})$. Consider:

$$\begin{aligned} \rho(z_1, T^2 z_0) &= \lim_{n\to\infty} \rho(T^{pn+1}x, T^2 z_0) \leq \lim_{n\to\infty} \rho(T^{pn}x, Tz_0) \leq \lim_{n\to\infty} \rho(T^{pn-1}x, z_0) \\ &= \lim \rho(T^{pn-1}x, T^{pn}x) = \operatorname{dist}(A_p, A_1) = \operatorname{dist}(A_2, A_3). \end{aligned}$$

It is obvious that $\rho(Tz_0, T^2 z_0) = \operatorname{dist}(A_2, A_3)$, and using the property UC of the underlying space, it follows that $z_1 = Tz_0$. Hence, $z_1 = Tz_0$. Similarly, we can prove that if $x \in A_1$ and $T^{pn}x \to z_0$, then $T^k z_0 = z_k$, for $k = 1, 2, ..., p-1$.

(c) Since $T^{p(n-1)+k} \longrightarrow z_k$,

$$\rho(z_k, T^{p+1}z_k) = \lim_{n\to\infty} \rho(T^{pn+k}x, T^{p+1}z_k) \leq \lim_{n\to\infty} \rho(T^{p(n-1)+k}x, T_k^z) = \rho(z_k, Tz_k) = \operatorname{dist}(A_k, A_{k+1}).$$

Now, $\rho(z_k, Tz_k) = \operatorname{dist}(A_k, A_{k+1})$ and $\rho(z_k, T^{p+1}z_k) = \operatorname{dist}(A_k, A_{k+1})$. Since the underlying space satisfies property UC, $Tz_k = T^{p+1}z_k$. Now:

$$\rho(T^p z_k, Tz_k) = \rho(T^p z_k, T^{p+1}z_k) \leq \rho(z_k, Tz_k) = \operatorname{dist}(A_k, A_{k+1}).$$

By a similar argument as above, $z_k = T^p z_k$. □

4. Proof of the Main Result

Proof. (i) Let $x \in A_1$ satisfy (1). Let us prove that $\lim_{m,n\to\infty} \rho(T^{pm}x, T^{pn}x) = 0$. Let $\varepsilon > 0$ be given. Then, there exists a $\delta > 0$ satisfying (1). Without loss of generality, let $\delta < \varepsilon/2$. Since by Corollary 2 $\lim_{n\to\infty} \rho(T^{pn+k}x, T^{pn+k+1}x) = 0$ there is an $n_0 \in \mathbb{N}$, such that for each $k \in \{0, 1, 2, \ldots, p-1\}$, it holds:

$$\rho(T^{pn+k}x, T^{pn+k+1}x) < \frac{\delta}{p}, \text{ for all } n \geq n_0. \tag{6}$$

Fix $n \geq n_0$. Let $m \in \mathbb{N}$ be such that $m \geq n$. We will prove that the inequality $\rho(T^{pn}x, T^{pm}x) < \varepsilon$ holds for every $m \geq n$.

Since for all $m \geq n$, we have $\rho(T^{pm}x, T^{pn}x) \leq \rho(T^{pm}x, T^{pm+1}x) + \rho(T^{pm+1}x, T^{pn}x)$, from the inequality $\rho(T^{pm}x, T^{pm+1}x) < \frac{\delta}{p} < \frac{\varepsilon}{4}$, it is enough if we prove that for all $m \geq n$:

$$\rho(T^{pn}x, T^{pm+1}x) < \frac{3\varepsilon}{4}. \tag{7}$$

Let us prove (7) by induction. From (6), it follows that (7) holds for $m = n$. Assume that Inequality (7) holds for some $m > n$. We prove that (7) holds for $m+1$. Let $\varepsilon_0 = \rho(T^{pn}x, T^{pm+1}x)$. Then, by our assumption $\varepsilon_0 = \rho(T^{pn}x, T^{pm+1}x) < \frac{3\varepsilon}{4}$. If $\epsilon_0 = 0$, then $T^{pn}x = T^{pm+1}x$. Therefore, $T^{p(m+1)+1}x = T^{pm+p+1}x = T^{pn+p}x$. Now:

$$\begin{aligned}
\rho(T^{pn}x, T^{p(m+1)+1}x) &= \rho(T^{pn}x, T^{pn+p}x) \\
&\leq \rho(T^{pn}x, T^{pn+1}x) + \rho(T^{pn+1}x, T^{pn+2}x) + ... + \rho(T^{pn+p-1}x, T^{pn+p}x) \\
&< \delta < \varepsilon/2.
\end{aligned}$$

Now, if $\epsilon_0 > 0$, since $\epsilon_0 = \rho(T^{pn}x, T^{pm+1}x) < \frac{3\varepsilon}{4} = \varepsilon/4 + \varepsilon/2$, we have by (1) that the inequality holds $\rho(T^{pn+1}x, T^{pm+2}x) < \epsilon/4$. Using this and (6), we get:

$$\begin{aligned}
\rho(T^{pn}x, T^{p(m+1)+1}x) \leq\; & \rho(T^{pn}x, T^{pn+1}x) + \rho(T^{pn+1}x, T^{pm+2}x) + \rho(T^{pm+2}x, T^{pm+3}x) + \\
& ... + \rho(T^{pm+p}x, T^{pm+p+1}x) < \frac{\delta}{p} + \frac{\varepsilon}{4} + (p-1)\frac{\delta}{p} = \frac{3\varepsilon}{4}.
\end{aligned}$$

Thus, (7) holds for $m+1$ in this case, as well. Hence, $\{T^{pn}x\}$ is a Cauchy sequence in X, and since X is a complete partial metric space, there exists a $z \in A_1$ such that:

$$0 = \lim_{m,n\to\infty} \rho(T^{pn}x, T^{pm}x) = \lim_{n\to\infty} \rho(z, T^{pn}x) = \rho(z,z).$$

This implies $\rho(z,z) = 0$. Now, from the inequalities:

$$0 \leq \lim_{n\to\infty} \rho(T^{pn+p-1}x, z) \leq \lim_{n\to\infty} \rho(T^{pn+p-1}x, T^{pn+p}x) + \rho(T^{pn+p}x, z) = 0$$

and the assumption that $\{T^{pn+p-1}x\} \subseteq A_p$, it follows that $\{T^{pn+p-1}x\}_{n=1}^{\infty}$ also converges to z. From the continuity of the function $\rho(x,\cdot)$ and Corollary 1, we get:

$$\rho(z, Tz) = \lim_{n\to\infty} \rho(T^{pn+p}x, Tz) \leq \lim_{n\to\infty} \rho(T^{pn+p-1}x, z) = 0.$$

Hence, $\rho(z, Tz) = 0$, i.e., $z = Tz$. To prove the uniqueness of z, let us suppose that there is $\xi \in A_i$, $\xi \neq z$, such that $\rho(\xi, T\xi) = 0$. From $\xi = T\xi$, it follows that $\xi = T^n\xi$ for all $n \in \mathbb{N}$, and therefore, $\xi \in \cap_{i=1}^{p} A_i$. Using Lemma 2, we get $\rho(z,\xi) = \lim_{n\to\infty} \rho(T^{pn}x, T^{pn+1}\xi) = 0$.

(ii) Let $\varepsilon > 0$ be given. Let ϕ be the L-function as given in Lemma 3. Then, for this $\varepsilon > 0$, there is $\delta_1 > 0$, such that:

$$\phi(\varepsilon + \delta_1) \leq \varepsilon \tag{8}$$

Since T is a p-cyclic orbital M-K contraction map, there exists an $x \in A_1$ and a $\delta_2 > 0$ satisfying (1). Let us put $\delta = \min\{\delta_1, \delta_2\}$. Without loss of generality, let $\delta < \epsilon$. By Lemma 4, we have that $\lim_{n\to\infty} \rho(T^{pn+1}x, T^{pn+2}x) = \text{dist}(A_2, A_3)$ and $\lim_{n\to\infty} \rho(T^{p(n+1)+1}x, T^{pn+2}x) = \text{dist}(A_2, A_3)$. From the assumption that X is a partial metric space with property UC, we get $\lim_{n\to\infty} \rho(T^{p(n+1)+1}x, T^{pn+1}x) = 0$. Therefore, it is possible to choose an $n_1 \in \mathbb{N}$ such that:

$$\rho(T^{pn+1}x, T^{p(n+1)+1}x) < \delta/2, \text{ for all } n \geq n_1 \tag{9}$$

and by Lemma 4:

$$\rho(T^{pn}x, T^{pn+1}x) < \mathrm{dist}(A_1, A_2) + 2\varepsilon \text{ for all } n \geq n_1. \tag{10}$$

Fix $n \geq n_1$. We show that:

$$\rho(T^{pm}x, T^{pn+1}x) < \mathrm{dist}(A_1, A_2) + \epsilon + \delta, \ \ \forall \ \ m, n \geq n_1 \tag{11}$$

by the method of induction. It is obvious that Condition (11) is true for $m = n$. Assume that the condition (11) is true for an $m > n$. To prove this condition for $m + 1$, consider:

$$\rho(T^{p(m+1)}x, T^{pn+1}x) \leq \rho(T^{pn+1}x, T^{p(n+1)+1}x) + \rho(T^{p(n+1)+1}x, T^{p(m+1)}x). \tag{12}$$

Now, from Lemma 3, we get:

$$\begin{aligned} \rho(T^{p(n+1)+1}x, T^{p(m+1)}x) - D_1 &< \phi(\rho(T^{p(n+1)}x, T^{p(m+1)-1}x) - D_p) \\ &\leq \rho(T^{p(n+1)}x, T^{p(m+1)-1}x) - D_p) \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &< \phi(\rho(T^{pn+1}x, T^{pm}x) - D_1) \leq \phi(\varepsilon + \delta) < \varepsilon. \end{aligned}$$

Using (9) in (12), we obtain $\rho(T^{p(m+1)}x, T^{pn+1}x) < \frac{\delta}{2} + \mathrm{dist}(A_1, A_2) + \varepsilon$. Hence, (11) holds for $(m+1)$ in place of m. From (11) and (10) and the assumption that the pair (A_1, A_2) satisfies the property UC, it follows that there exists $n_2 \in \mathbb{N}$, so that $\rho(T^{pn}x, T^{pm}x) < \varepsilon$ holds for all $m > n \geq n_2$. Hence, $\{T^{pn}x\}$ is a Cauchy sequence and converges to a $z \in A_1$. By Lemma 5, z is a best proximity point of T in A_1, and z is a periodic point of T in A_1. To prove that $z \in A_1$ is the unique periodic point of T in A_1, we proceed as follows. Let $y \in A_1$ satisfy (1) such that $y \neq x$. Then, by what we have proven now, there exists a $\eta \in A_1$, such that by Lemma 5, η is a best proximity point of T in A_1; η is a periodic point of T in A_1; and by a similar argument as in Lemma 5, $T^{p+1}\eta = T\eta$. If $\rho(z, T\eta) = \mathrm{dist}(A_1, A_2)$, then since $\rho(\eta, T\eta) = \mathrm{dist}(A_1, A_2)$, since the underlying space satisfies property UC, we have $z = \eta$. Hence, suppose $\rho(z, T\eta) - \mathrm{dist}(A_1, A_2) > 0$. Then:

$$\begin{aligned} \rho(Tz, T^2\eta) - \mathrm{dist}(A_1, A_2) &< \phi(\rho(z, T\eta) - \mathrm{dist}(A_1, A_2)) \leq \rho(z, T\eta) - \mathrm{dist}(A_1, A_2) \\ &= \rho(T^p z, T^{p+1}\eta) - \mathrm{dist}(A_1, A_2) \leq \rho(Tz, T^2\eta) - \mathrm{dist}(A_1, A_2), \end{aligned}$$

which is a contradiction. □

5. Examples

We start with a lemma, which is useful in checking whether a partial metric space is complete.

Lemma 6. *Let (X, d) be a complete metric space and (X, ρ) be a partial metric space. Let $\omega : X \to [0, +\infty)$ and $\rho(x, y) = d(x, y) + \max\{\omega(x), \omega(y)\}$. The partial metric space (X, ρ) is complete if and only if ω satisfies the condition: if $\limsup_{d(x_n, x) \to 0} \omega(x_n) < \infty$, then $\lim_{n \to \infty} \omega(x_n) = \omega(x)$.*

Proof. (X, ρ) is complete if and only if (X, ρ^s) is complete ([19]). By the definition of the partial metric ρ, we get that:

$$\begin{aligned} \rho^s(x, y) &= 2\rho(x, y) - \rho(x, x) - \rho(y, y) \\ &= 2d(x, y) + 2\max\{\omega(x), \omega(y)\} - \omega(x) - \omega(y) \\ &= 2d(x, y) + |\omega(x) - \omega(y)|. \end{aligned} \tag{13}$$

Sufficiency: Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in (X, ρ), and ω satisfies the condition in the lemma. From [19], it follows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, ρ^s). That is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that the inequality $\rho^s(x_n, x_m) < \varepsilon$ holds for every $n, m \geq N$. Thus, from (13), we get $2d(x_n, x_m) + |\omega(x_n) - \omega(x_m)| < \varepsilon$, and therefore, the inequality $d(x_n, x_m) < \varepsilon$ holds for every

$n, m \geq N$. Consequently, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d), hence converging to x. From the assumption that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, ρ), it follows that the limit:

$$\lim_{n,m\to\infty} \rho(x_n, x_m) = \lim_{n,m\to\infty} (d(x_n, x_m) + \max\{\omega(x_n), \omega(x_m)\}) = \lim_{n,m\to\infty} \max\{\omega(x_n), \omega(x_m)\}$$

exists and is finite.

There are two cases: (I) $\limsup\limits_{n\to\infty} \omega(x_n) \leq \omega(x)$ and (II) $\limsup\limits_{n\to\infty} \omega(x_n) \geq \omega(x)$.

(I) Let us assume that $\limsup_{n\to\infty} \omega(x_n) \leq \omega(x)$. Then, $\lim_{n\to\infty} \max\{\omega(x), \omega(x_n)\} = \omega(x)$. Consequently, we get:

$$\rho(x, x) = \omega(x) = \lim_{n\to\infty} \rho(x_n, x) = \lim_{n\to\infty} (d(x_n, x) + \max\{\omega(x), \omega(x_n)\}) = \omega(x).$$

(II) Let us assume that $\limsup_{n\to\infty} \omega(x_n) \geq \omega(x)$. From the assumption of the Lemma, it follows that $\lim\limits_{n\to\infty} \max\{\omega(x), \omega(x_n)\} = \omega(x)$. Consequently, we get:

$$\rho(x, x) = \omega(x) = \lim_{n\to\infty} \rho(x_n, x) = \lim_{n\to\infty} (d(x_n, x) + \max\{\omega(x), \omega(x_n)\}) = \omega(x).$$

Thus, $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in (X, ρ) in both cases. To prove the necessity condition, let (X, ρ) be a complete partial metric space. We will prove that ω satisfies the conditions of the Lemma. Let us suppose the contrary, i.e., there exists a sequence $\{x_n\}_{n=1}^{\infty}$ that is convergent to some point $x \in X$ with respect to the metric d, $\limsup_{n\to\infty} \omega(x_n) \leq M_1 < \infty$, and there is $\varepsilon_0 > 0$, so that the inequality $|\omega(x_n) - \omega(x)| \geq \varepsilon$ for every $n \geq N_0$, for some $N_0 \in \mathbb{N}$. From the convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ in (X, d), it follows that there is $M > 0$, such that $d(x_n, x_m) \leq M$. By the inequality $\rho(x_n, x_m) = d(x_n, x_m) + \max\{\omega(x_n), \omega(x_m)\} \leq M + M_1$, it follows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, ρ). Now, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, ρ^s) ([19]), and therefore, it is convergent. Let us denote its limit by z. From (13), it follows that $\lim_{n\to\infty} \rho^s(x_n, z) = \lim_{n\to\infty} (d(x_n, z) + |\omega(x_n) - \omega(z)|) = 0$. Consequently, $\lim_{n\to\infty} d(x_n, z) = 0$, and thus, $z = x$. Therefore, $\lim_{n\to\infty} |\omega(x_n) - \omega(z)| = 0$, which is a contradiction. □

Example 4. *Let us consider the metric space $([0, +\infty), d)$, endowed with the metric $d(x, y) = |x - y|$. Let us consider the function $\omega(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 1, & x = 0. \end{cases}$ Then, $([0, +\infty), \rho)$ is a complete partial metric space, where $\rho(x, y) = |x - y| + \max\{\omega(x), \omega(y)\}$.*

Let $\lim_{n\to\infty} |x_n - x| = 0$. Then, $\limsup_{n\to\infty} \omega(x_n) < \infty$ if and only if $x \neq 0$. By the continuity of ω at any different point from zero, it follows that $\lim_{n\to\infty} \omega(x_n) = \omega(x)$, provided that $\lim_{n\to\infty} x_n = x \neq 0$. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $([0, +\infty), \rho)$. Thus, the limit $\lim_{n,m\to\infty} (|x_n - x_m| + \max\{\omega(x_n), \omega(x_m)\})$ exists and is finite. This limit is finite if and only if $\lim_{n\to\infty} x_n = x \neq 0$. Consequently, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $([0, +\infty), \rho)$ if and only if $\lim_{n\to\infty} |x_n - x| = 0$ for some $x \neq 0$ and $\lim_{n\to\infty} \rho(x_n, x) = \omega(x) = \rho(x, x)$. Consequently, $([0, +\infty), \rho)$ is a complete partial metric space.

Example 5. *Let us consider the metric space $([0, +\infty), d)$, endowed with the metric $d(x, y) = |x - y|$. Let us consider the function $\omega(x) = \begin{cases} 1, & x \neq 1 \\ a, & x = 1. \end{cases}$ Then, $([0, +\infty), \rho)$ is a complete partial metric space with $\rho(x, y) = |x - y| + \max\{\omega(x), \omega(y)\}$ if and only if $a = 1$.*

Let $\lim_{n\to\infty} |x_n - x| = 0$. Then, $\limsup_{n\to\infty} \omega(x_n) < \infty$ and $\lim_{n\to\infty} \omega(x_n) = \omega(x)$, provided that $x \neq 1$ or $a = 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence, such that $x_n \neq 1$, which is convergent to one with respect to the metric d. Then, it is a Cauchy sequence in $([0, +\infty), \rho)$, because the limit $\lim_{n,m\to\infty}(|x_n - x_m| + \max\{\omega(x_n), \omega(x_m)\}) = a$ exists and is finite. From $a = \rho(1,1)$ and $\rho(x_n, 1) = (|x_n - 1| + \max\{\omega(x_n), \omega(1)\} = \max\{1, a\}$, it follows that $\{x_n\}_{n=1}^{\infty}$ is convergent in $([0, +\infty), \rho)$ if and only if $a = 1$.

Corollary 3. *Let (X, d) be a complete metric space and $\omega : X \to [0, +\infty)$ be a continuous function with respect to metric d, and let us consider the partial metric space (X, ρ), where $\rho(x, y) = d(x, y) + \max\{\omega(x), \omega(y)\}$. Then, if $A \subset X$ is closed in (X, d), then it is closed in (X, ρ).*

Corollary 4. *Let (X, d) be a complete metric space, $a > 0$, and let us consider the partial metric space (X, ρ), where $\rho(x, y) = d(x, y) + a$. The partial metric space (X, ρ) is complete.*

Lemma 7. *Let (X, d) be a complete metric space and $\omega : X \to [0, +\infty)$ be a continuous function with respect to the metric d. Let us consider the partial metric space (X, ρ), where $\rho(x, y) = d(x, y) + \max\{\omega(x), \omega(y)\}$. The partial metric space (X, ρ) is:*

(a) a Hausdorff space with respect to the topology; τ_ρ
(b) a normal topological space with respect to the topology.

Proof. (a) Let $x, y \in (X, \rho)$. Let us put $a = \rho(x, y) = d(x, y) + \max\{\omega(x), \omega(y)\}$. Let us consider the open balls $B_{\frac{a}{2}}(x) = \{u \in (X, d) : \rho(x, u) < \frac{a}{2}\}$ and $B_{\frac{a}{2}}(y) = \{u \in (X, d) : \rho(y, u) < \frac{a}{2}\}$ in (X, d). Then, $B_{a/2}(x) \cap B_{a/2}(y) = \emptyset$. Indeed, let us suppose the contrary, i.e., there exists $u \in B_{a/2}(x) \cap B_{a/2}(y)$. Then, from the inequality:

$$\begin{aligned} \rho(x, y) &= d(x, y) + \max\{\omega(x), \omega(y)\} \\ &\le d(x, u) + d(u, y) + \max\{\omega(x), \omega(u)\} + \max\{\omega(u), \omega(y)\} \\ &= \rho(x, u) + \rho(u, y) < a/2 + a/2 = a \end{aligned}$$

we get a contradiction.

(b) From (a), it follows that (X, ρ) is a T_1 space. Let $U, V \subset (X, \rho)$ be closed sets. Let $u \in U$ be arbitrarily chosen. Let us put:

$$a_u = \inf\{\rho(u, v) : v \in V\} = \inf\{d(u, v) + \max\{\omega(u), \omega(v)\} : v \in V\}.$$

The function $f_u(v) = \inf\{d(u, v) + \max\{\omega(u), \omega(v)\} : v \in V\} : V \to [0, +\infty)$ is a continuous function with respect to the metric d, and thus, $a_u = \inf\{f_u(v) : v \in V\}$ exists. Let us denote $B_{a_u/2}(u) = \{x \in (X, d) : \rho(x, u) < a/2\}$. In a similar fashion, let us denote $B_{a_v/2}(v) = \{x \in (X, d) : \rho(x, v) < a_v/2\}$ for every $v \in V$, where $a_v = \inf\{\rho(u, v) : u \in U\}$. The sets $\bigcup\limits_{u\in U} B_{a_u/2}(u)$ and $\bigcup\limits_{v\in V} B_{a_v/2}(v)$ are open sets and $U \subset \bigcup\limits_{u\in U} B_{a_u/2}(u)$, $V \subset \bigcup\limits_{v\in V} B_{a_v/2}(v)$. Then, $\left(\bigcup\limits_{u\in U} B_{a_u/2}(u)\right) \bigcap \left(\bigcup\limits_{v\in V} B_{a_v/2}(v)\right) = \emptyset$. Indeed, let us suppose the contrary. That is, there exists $x \in \left(\bigcup\limits_{u\in U} B_{a_u/2}(u)\right) \bigcap \left(\bigcup\limits_{v\in V} B_{a_v/2}(v)\right)$. Then, there are $u \in U$ and $v \in V$, such that $x \in B_{a_u/2}(u)$ and $x \in B_{a_v/2}(v)$. Then, from the inequality:

$$\begin{aligned} \rho(u, v) &= d(u, v) + \max\{\omega(u), \omega(v)\} \\ &\le d(u, x) + d(x, v) + \max\{\omega(u), \omega(x)\} + \max\{\omega(x), \omega(v)\} \\ &= \rho(u, x) + \rho(x, v) < a_u/2 + a_v/2 \\ &\le \max\{a_u/2 + a_u/2, a_v/2 + a_v/2\} = \max\{a_u, a_v\} \end{aligned}$$

we get a contradiction. □

Example 6. *Let us consider the Banach space $(\mathbb{R}^2, \|\cdot\|_2)$, where $\mathbb{R}^2 = \{x = (u, v) : u, v \in \mathbb{R}\}$ and $\|x\|_2 = \|(u, v)\|_2 = \sqrt{u^2 + v^2}$. Let us endow $\mathbb{R}^2$ with the partial metric $\rho(x, y) = \|x - y\|_2 + \max\{\|x\|_2^2, \|y\|_2^2\}$. From Example 3, $\rho(x, y)$ is a partial metric. From Lemma 6, it follows that $(\mathbb{R}^2, \rho)$ is a complete partial metric* *space. We consider the sets A_1, A_2, A_3, A_4 defined by $A_1 = \{(u,v) \in \mathbb{R} : u \geq 0, v \geq 0\}$, $A_2 = \{(u,v) \in \mathbb{R} : u \leq 0, v \geq 0\}$, $A_3 = \{(u,v) \in \mathbb{R} : u \leq 0, v \leq 0\}$, $A_4 = \{(u,v) \in \mathbb{R} : u \geq 0, v \leq 0\}$. From Corollary 3, it follows that A_1, A_2, A_3, A_4 are closed sets in $(\mathbb{R}^2, \rho)$. Let us define a cyclic map $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_{i+1}$ by:*

$$T(u,v) = \left(\frac{-|v|}{1+2\|(u,v)\|_2}, \frac{|u|}{1+2\|(u,v)\|_2}\right), \text{for } (u,v) \in A_1;$$
$$T(u,v) = \left(\frac{-|v|}{1+2\|(u,v)\|_2}, \frac{-|u|}{1+2\|(u,v)\|_2}\right), \text{for } (u,v) \in A_2;$$
$$T(u,v) = \left(\frac{|v|}{1+2\|(u,v)\|_2}, \frac{-|u|}{1+2\|(u,v)\|_2}\right), \text{for } (u,v) \in A_3; \text{ and}$$
$$T(u,v) = \left(\frac{|v|}{1+2\|(u,v)\|_2}, \frac{|u|}{1+2\|(u,v)\|_2}\right), \text{for } (u,v) \in A_4.$$

Let $x = (0,0)$. Let us choose an arbitrary $y_0 = (u_0, v_0) \in A_i$. Let us denote $T^k y_0 = y_k = (u_k, v_k)$. Then, $\rho(T^k(u_0,v_0), T^{4n+k-1}x) = \rho((u_k,v_k),(0,0)) = \|(u_k,v_k)\|_2 + \|(u_k,v_k)\|_2^2 = \sqrt{u_k^2+v_k^2} + u_k^2 + v_k^2$. Now:

$$\begin{aligned} R_4 &= \rho(T^{k+1}y_0, T^{4n+k}x) = \rho(T^{k+1}(u_0,v_0), T^{4n+k}x) \\ &= \rho(T(u_k,v_k),(0,0)) \\ &= \frac{\sqrt{u_k^2+v_k^2}}{(1+2\|(u_k,v_k)\|_2)} + \frac{u_k^2+v_k^2}{(1+2\|(u_k,v_k)\|_2)^2} \\ &\leq \frac{\sqrt{u_k^2+v_k^2+u_k^2+v_k^2}}{(1+2\|(u_k,v_k)\|_2)} = \frac{\|(u_k,v_k)\|_2+\|(u_k,v_k)\|_2^2}{(1+2\|(u_k,v_k)\|_2)} = \frac{\|y_k\|_2+\|y_k\|_2^2}{(1+2\|y_k\|_2)}. \end{aligned} \tag{14}$$

Now, $\rho(x,y) = \|y_k\|_2 + \|y_k\|_2^2$.
By solving the equation $\|y_k\|_2^2 + \|y_k\|_2 + \rho(x,y) = 0$, we get

$$\|y_k\|_2 = \frac{\sqrt{1+4\rho(x,y_k)}-1}{2}.$$

Hence:

$$\rho(T^{k+1}y_0, T^{4n+k}x) \leq \frac{\rho(x,y_k)}{\sqrt{1+4\rho(x,y_k)}}.$$

The function $\frac{t}{\sqrt{1+4t}}$ is a continuous function in the interval $[0,+\infty)$. From $\frac{\varepsilon}{\sqrt{1+4\varepsilon}} < \varepsilon$, we get the condition that there exists $\delta(\varepsilon) > 0$ such that the inequality $\rho(T^{k+1}y_0, T^{4n+k}x) \leq \frac{\rho(x,y_k)}{\sqrt{1+4\rho(x,y_k)}} < \varepsilon$ holds whenever the inequality holds $\rho(T^k y_0, T^{4n+k-1}x) = \rho(y_k, x) < \varepsilon + \delta(\varepsilon)$. Consequently, T is a 4-cyclic orbital Meir–Keeler contraction, and x is the unique fixed point.

6. Conclusions

In this paper, the contraction condition of the p-cyclic orbital M-K contraction map was not imposed on all pairs of points of the partial metric space. Even if the contraction condition holds for just one point in the space, it is possible to obtain a unique fixed point/unique best proximity point, which is the limit of the sequence of iterates of that point. So far, the best proximity points have not been obtained for contraction maps defined on a partial metric space. In this paper, the notion of the partial metric space with property UC was introduced, and the best proximity point for p-cyclic orbital M-K contraction was obtained.

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