Green’s Relations on a Semigroup of Transformations with Restricted Range that Preserves an Equivalence Relation and a Cross-Section

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Abstract: Let \( T(X, Y) \) be the semigroup consisting of all total transformations from \( X \) into a fixed nonempty subset \( Y \) of \( X \). For an equivalence relation \( \rho \) on \( X \), let \( \rho \) be the restriction of \( \rho \) on \( Y, R \) a cross-section of \( Y/\rho \) and define \( T(X, Y, \rho, R) \) to be the set of all total transformations \( \alpha \) from \( X \) into \( Y \) such that \( \alpha \) preserves both \( \rho \) (if \((a, b) \in \rho \), then \((a\alpha, b\alpha) \in \rho \)) and \( R \) (if \( r \in R \), then \( r\alpha \in R \)). \( T(X, Y, \rho, R) \) is then a subsemigroup of \( T(X, Y) \). In this paper, we give descriptions of Green’s relations on \( T(X, Y, \rho, R) \), and these results extend the results on \( T(X, Y) \) and \( T(X, \rho, R) \) when taking \( \rho \) to be the identity relation and \( Y = X \), respectively.

Keywords: transformation semigroup; Green’s relations; equivalence relation; cross-section

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1. Introduction

Let \( X \) be a nonempty set and \( T(X) \) denote the semigroup containing all full transformations from \( X \) into itself with the composition. It is well-known that \( T(X) \) is a regular semigroup, as shown in Reference [1]. Various subsemigroups of \( T(X) \) have been investigated in different years. One of the subsemigroups of \( T(X) \) is related to an equivalence relation \( \rho \) on \( X \) and a cross-section \( R \) of the partition \( X/\rho \) (i.e., each \( \rho \)-class contains exactly one element of \( R \)), namely \( T(X, \rho, R) \), which was first considered by Araújo and Konieczny in 2003 [2], and is defined by

\[
T(X, \rho, R) = \{a \in T(X) : Ra \subseteq R \text{ and } (a, b) \in \rho \implies (a\alpha, b\alpha) \in \rho\},
\]

where \( Z\alpha = \{za : z \in Z\} \). They studied automorphism groups of centralizers of idempotents. Moreover, they also determined Green’s relations and described the regular elements of \( T(X, \rho, R) \) in 2004 [3].

Let \( Y \) be a nonempty subset of the set \( X \). Consider another subsemigroup of \( T(X) \), which was first introduced by Symons [4] in 1975, called \( T(X, Y) \), defined by

\[
T(X, Y) = \{a \in T(X) : Xa \subseteq Y\},
\]

when \( Xa \) denotes the image of \( a \). He described all the automorphisms of \( T(X, Y) \) and also determined when \( T(X_1, Y_1) \) is isomorphic to \( T(X_2, Y_2) \). In 2009, Sanwong, Singha and Sullivan [5] described all the maximal and minimal congruences on \( T(X, Y) \). Later, in Reference [6], Sanwong and Sommanee studied other algebraic properties of \( T(X, Y) \). They gave necessary and sufficient conditions for \( T(X, Y) \).
to be regular and also determined Green’s relations on \( T(X, Y) \). Furthermore, they obtained a class of maximal inverse subsemigroups of \( T(X, Y) \) and proved that the set

\[
F(X, Y) = \{a \in T(X, Y) : Xa \subseteq Ya \}
\]

contains all regular elements in \( T(X, Y) \), and is the largest regular subsemigroup of \( T(X, Y) \).

From now on, we study the subsemigroup \( T(X, Y, \rho, R) \) of \( T(X, Y) \) defined by

\[
T(X, Y, \rho, R) = \{a \in T(X, Y) :Ra \subseteq R \text{ and } (a, b) \in \rho \Rightarrow (aa, ba) \in \rho\},
\]

where \( \rho \) is an equivalence relation on \( X \) and \( R \) is a cross-section of the partition \( Y/\hat{\rho} \) in which \( \hat{\rho} = \rho \cap (Y \times Y) \). If \( Y = X \), then \( T(X, Y, \rho, R) = T(X, \rho, R) \); and if \( \rho \) is the identity relation, then \( T(X, Y, \rho, R) = T(X, Y) \), so we may regard \( T(X, Y, \rho, R) \) as a generalization of \( T(X, \rho, R) \) and \( T(X, Y) \).

Green’s relations play a role in semigroup theory, and the aim of this paper is to characterize Green’s relations on \( T(X, Y, \rho, R) \). As consequences, we obtain Green’s relations on \( T(X, \rho, R) \) and \( T(X, Y) \) as corollaries.

2. Preliminaries and Notations

For any semigroup \( S \), let \( S^1 \) be a semigroup obtained from \( S \) by adjoining an identity if \( S \) has no identity and letting \( S^1 = S \) if it already contains an identity. Green’s relations of \( S \) are equivalence relations on the set \( S \) which were first defined by Green. According to such definitions, we define the \( L \)-relation as follows. For any \( a, b \in S \),

\[
aLb \text{ if and only if } S^1a = S^1b,
\]

or equivalently, \( aLb \) if and only if \( a = xb \) and \( b = ya \) for some \( x, y \in S^1 \).

Furthermore, we dually define the \( R \)-relation as follows.

\[
aRb \text{ if and only if } aS^1 = bS^1,
\]

or equivalently, \( aRb \) if and only if \( a = bx \) and \( b = ay \) for some \( x, y \in S^1 \).

Moreover, we define the \( J \)-relation as follows.

\[
aJb \text{ if and only if } S^1aS^1 = S^1bS^1,
\]

or equivalently, \( aJb \) if and only if \( a = xby \) and \( b = uav \) for some \( x, y, u, v \in S^1 \).

Finally, we define \( H = L \cap R \) and \( D = L \circ R \), where \( \circ \) is the composition of relations. Since the relations \( L \) and \( R \) commute, it follows that \( L \circ R = R \circ L \).

In this paper, we write functions on the right; in particular, this means that for a composition \( ab \), \( a \) is applied first. Furthermore, the cardinality of a set \( A \) is denoted by \( |A| \).

For each \( a \in T(X) \), we denote by \( \ker(a) \) the kernel of \( a \), the set of ordered pairs in \( X \times X \) having the same image under \( a \), that is,

\[
\ker(a) = \{(a, b) \in X \times X : aa = ba\}.
\]

Moreover, the symbol \( \pi(a) \) denotes the partition of \( X \) induced by the map \( a \), namely

\[
\pi(a) = \{xa^{-1} : x \in Xa\}.
\]

We observe that \( \ker(a) \) is an equivalence relation on \( X \) in which the partition \( X/\ker(a) \) and \( \pi(a) \) coincide. Moreover, for all \( a, b \in T(X) \), we have \( \ker(a) = \ker(b) \) if and only if \( \pi(a) = \pi(b) \).
In addition, if \( \rho \) is an equivalence relation on the set \( X \) and \( a, b \in X \), we sometimes write \( a \rho b \) instead of \( (a, b) \in \rho \), and define \( a \rho \) to be the equivalence class that contains \( a \), that is, \( a \rho = \{ b \in X : b \rho a \} \).

For the subsemigroup \( T(X, Y, \rho, R) \) of \( T(X) \) where \( \rho \) is an equivalence relation on \( X, Y \) is a nonempty subset of \( X \) and \( R \) is a cross-section of \( Y/\hat{\rho} \) in which \( \hat{\rho} = \rho \cap (Y \times Y) \), we see that if \( a \in X \) and \( a \rho \cap Y \neq \emptyset \), then there exists a unique \( r \in R \) such that \( a \rho r \) and we denote this element by \( ra \). Furthermore, we observe that \( F(X, Y) \cap T(X, Y, \rho, R) \) contains all constant maps whose images belong to \( R \). This implies that \( F(X, Y) \cap T(X, Y, \rho, R) \) is a subsemigroup of \( T(X, Y, \rho, R) \), which will be denoted by \( F \).

An element \( a \) in a semigroup \( S \) is said to be regular if there exists \( x \in S \) such that \( a = axa \); and \( S \) is a regular semigroup if every element of \( S \) is regular.

In general, \( T(X, Y, \rho, R) \) is not a regular semigroup, so we cannot apply Hall’s Theorem to find the \( \mathcal{L} \)-relation and the \( \mathcal{R} \)-relation on \( T(X, Y, \rho, R) \).

Now, we give an example of a non-regular element in \( T(X, Y, \rho, R) \). Let \( X = \{1, 2, 3, 4, 5\} \), \( Y = \{3, 4, 5\} \), \( X/\rho = \{\{1, 2\}, \{3, 4, 5\}\} \), \( Y/\hat{\rho} = \{\{3, 4, 5\}\} \) and \( R = \{3\} \). Define \( \alpha \in T(X, Y, \rho, R) \) by

\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
4 & 3 & 3 & 5 & 5
\end{pmatrix}.
\]

Suppose that \( \alpha \) is regular. Then \( \alpha = \alpha \beta \alpha \) for some \( \beta \in T(X, Y, \rho, R) \). We see that \( 4 = 1\alpha = 1(\alpha \beta \alpha) = (4\beta)\alpha \), which implies that \( 1 = 4\beta \in Y \), a contradiction.

Throughout this paper, the set \( X \) we study can be a finite or an infinite set. For convenience, we will denote \( T(X, Y, \rho, R) \) by \( T \).

3. Green’s Relations on \( T(X, Y, \rho, R) \)

Unlike \( T(X, \rho, R) \), in general \( T \) has no identity, as shown in the following example.

**Example 1.** Let \( X = \{1, 2, 3, 4, 5, 6\} \), \( Y = \{1, 3\} \), \( X/\rho = \{\{1, 2\}, \{3, 4, 5, 6\}\} \), \( Y/\hat{\rho} = \{\{1\}, \{3\}\} \) and \( R = \{1, 3\} \). Suppose that \( \varepsilon \) is an identity element in \( T \). Consider \( \alpha \in T \) defined by

\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 3 & 3
\end{pmatrix}.
\]

We see that \( 5\varepsilon \alpha = 5(\varepsilon \alpha) = 5\alpha = 3 \), which implies that \( 5\varepsilon \in \{5, 6\} \). This leads to a contradiction, since both 5 and 6 are not in \( Y \).

Therefore, we use the semigroup \( T \) with identity adjoined, given by \( T^\dagger \), in studying its Green’s relations.

From now on, the notation \( \mathcal{L}_\alpha (R_\alpha, H_\alpha, D_\alpha) \) denote the set of all elements of \( T \) which are \( \mathcal{L} \)-related (\( R \)-related, \( H \)-related, \( D \)-related) to \( \alpha \), where \( \alpha \in T \).

Let \( \mathcal{A} \) and \( \mathcal{B} \) be families of sets. If for each set \( A \in \mathcal{A} \) there is a set \( B \in \mathcal{B} \) such that \( A \subseteq B \), we say that \( \mathcal{A} \) refines \( \mathcal{B} \), denoted by \( \mathcal{A} \hookrightarrow \mathcal{B} \).

In what follows, most of the notation used are taken from Reference [3]. For each \( \alpha \in T \), we denote by \( \mathcal{F}_\alpha \) the family \( \{(xp)\alpha : x \in X\} \) and \( \mathcal{V}_\alpha \) the family \( \{(r\hat{\rho})\alpha : r \in R\} \). Furthermore, we define \( \mathcal{V}_\alpha = \{(xp)\alpha^{-1} : x \in X \text{ and } (xp)\alpha^{-1} \neq \emptyset\} \). In fact, we see that \( \mathcal{V}_\alpha = \{(r\hat{\rho})\alpha^{-1} : r \in R \text{ and } (r\hat{\rho})\alpha^{-1} \neq \emptyset\} \).

The following example describes the above notation.

**Example 2.** Let \( X = \{1, 2, 3, 4, 5, 6, 7, 8\} \), \( Y = \{1, 2, 3, 4, 5\} \), \( X/\rho = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}\} \), \( Y/\hat{\rho} = \{\{1, 2, 3\}, \{4, 5\}\} \) and \( R = \{1, 4\} \). Define \( \alpha \in T \) by

\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 5 & 5 & 1 & 2 & 3 & 3
\end{pmatrix}.
\]
**Theorem 1.** We choose the following lemma is needed.

**Example 3.** Let \( X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), \( Y = \{1, 2, 3, 4, 5, 6, 7\} \), \( X/\rho = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), \( Y/\beta = \{1, 2, 3, 4, 5, 6, 7\} \) and \( R = \{1, 4, 6\} \). Define \( \alpha, \beta_1, \beta_2, \gamma \in T \) as follows.

\[
\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 2 & 6 & 7 & 1 & 2 & 2 & 7 & 7 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 1 & 3 & 2 & 6 & 7 \end{pmatrix},
\]

\[
\beta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 2 & 6 & 7 & 4 & 5 & 5 & 3 & 3 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 2 & 4 & 5 & 1 & 2 & 2 & 5 & 5 \end{pmatrix}.
\]

We see that \( \alpha \neq \gamma \beta_1 \) for all \( \gamma' \in T \), for if \( \alpha = \gamma \beta_1 \) for some \( \gamma' \in T \), then \( 7 \alpha = 9 \alpha = 9 \gamma' \beta_1 \), and thus \( 10 = 9 \gamma' \in Y \), a contradiction. However, \( \alpha = \{1, 2, 6, 7\} \mapsto \{1, 2, 3, 4, 5, 6, 7\} = \gamma \beta_1 \). But \( \alpha = \gamma \beta_2 \) since \( \alpha \rightarrow \gamma \beta_2 = \{1, 2, 6, 7\} \).

**Theorem 1.** Let \( \alpha, \beta \in T \). Then \( \alpha = \gamma \beta \) for some \( \gamma \in T^I \) if and only if \( \alpha = \beta \) or \( \alpha \rightarrow \gamma \beta \). Consequently, \( \alpha \in L \beta \) if and only if \( \alpha = \beta \) or \( \alpha \rightarrow \gamma \beta \) and \( \gamma \rightarrow \gamma \alpha \).

**Proof.** Assume that \( \alpha = \gamma \beta \) for some \( \gamma \in T^I \). Suppose that \( \alpha \neq \beta \). Thus \( \gamma \neq 1 \). We prove that \( \gamma \alpha \rightarrow \gamma \beta \). Let \( \alpha \in \gamma \alpha \). Then \( \gamma (x) \alpha = ((x)\gamma) \beta \) for some \( x \in X \) and \( (x)\gamma \subseteq \gamma' \beta \) for some \( \gamma' \in \gamma \). Therefore \( \alpha \subseteq (x)\gamma \beta \), and thus \( \alpha \rightarrow \gamma \beta \).

Conversely, assume that \( \alpha = \beta \) or \( \alpha \rightarrow \gamma \beta \). If \( \alpha = \beta \), then \( \alpha = 1 \beta \), where \( \gamma = 1 \in T^I \). For the case \( \alpha \rightarrow \gamma \beta \), we define \( \gamma \) on each \( p \)-class as follows. Let \( x \gamma \beta \in \gamma \alpha \). Then \( (x) \alpha \subseteq (x)\gamma \beta \subseteq \beta' \beta \) for some \( r \), \( s \in R \) (since \( \alpha \rightarrow \gamma \beta \)). So, for each \( a \in x \gamma \beta \), we choose \( b_a \in r \beta \) such that \( (x) \alpha = b_a \gamma \beta \) (if \( a = r \in R \), we choose \( b_a = r \) since \( t \alpha = s \beta \) and define \( \gamma' \beta = b_a \). From \( b_a \in r \beta \) for all \( a \in x \gamma \beta \), we obtain \( (x) \gamma \beta \subseteq \beta' \beta \). By the definition of \( \gamma \), \( r \gamma = b_1 = r \). Since \( x \gamma \beta \) is arbitrary, we conclude that \( \gamma \in T \). To see that \( \alpha = \gamma \beta \), let \( a \in X \). Then \( \gamma' \beta = (a) \gamma' \beta = b_a \beta = a \alpha \), and so \( \alpha = \gamma \beta \).

If we replace \( Y \) with \( X \) in Theorem 1, then \( T = T(X, \rho, R) \), and \( \gamma \alpha = \alpha \) for all \( \alpha \in T \). Therefore, we have the \( L \) relation on \( T(X, \rho, R) \).

**Corollary 1.** [3] [Theorem 2.5] Let \( \alpha, \beta \in T(X, \rho, R) \). Then \( \alpha \in L \beta \) if and only if \( \alpha \rightarrow \beta \) and \( \beta \rightarrow \alpha \).

Similarly to \( T(X, Y) \), there are two types of \( L \)-classes on \( T \). In order to describe these \( L \)-classes, the following lemma is needed.
Lemma 1. Let $\alpha, \beta \in T$ be such that $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$. Then $\alpha, \beta \in F$ and $X\alpha = X\beta$.

Proof. For each $aa \in Xa$, we have $aa \in (ap)\alpha \subseteq (rf)\beta$ for some $r \in R$, since $\alpha \rightarrow \beta$. Thus, $Xa \subseteq Y\beta$. Similarly, $X\beta \subseteq Ya$, since $\beta \rightarrow \alpha$. It follows that $X\alpha \subseteq Y\beta \subseteq X\beta \subseteq Ya \subseteq Xa$, and thus $\alpha, \beta \in F$ and $Xa = X\beta$. □

Corollary 2. For $a \in T$, the following statements hold.

(i) If $a \in F$, then $L_a = \{a\} \cup \{\beta \in F : \alpha \rightarrow \beta \text{ and } \beta \rightarrow \alpha\}.

(ii) If $a \in T \setminus F$, then $L_a = \{a\}.$

Proof. Let $a$ be any element in $T$ and let $\beta \in L_a$. Then $aL\beta$, which implies that $a = \beta$; or $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ by Theorem 1.

(i) Assume that $a \in F$. It is clear by Theorem 1 that $\{a\} \cup \{\beta \in F : \alpha \rightarrow \beta \text{ and } \beta \rightarrow \alpha\} \subseteq L_a$.

To prove the other containment, we consider when $\beta \neq a$. Since $\beta L\alpha$, we obtain $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ by Theorem 1.

By Lemma 1, we obtain $\beta \in F$. Thus, we have $L_a \subseteq \{a\} \cup \{\beta \in F : \alpha \rightarrow \beta \text{ and } \beta \rightarrow \alpha\}.$

(ii) Assume that $L_a \neq \{a\}$. Then there is $\gamma \neq a$ such that $\gamma L\alpha$, and $\alpha \rightarrow \gamma$ and $\gamma \rightarrow \alpha$.

By Lemma 1, we get $a \in F$. □

As a direct consequence of Corollary 2, we obtain the $L$-relation on $T(X,Y)$ as follows.

Corollary 3. [6] [Theorem 3.2] For $a \in T(X,Y)$, the following statements hold.

(i) If $a \in F(X,Y)$, then $L_a = \{\beta \in F(X,Y) : Xa = X\beta\}.

(ii) If $a \in T(X,Y) \setminus F(X,Y)$, then $L_a = \{a\}.$

Proof. If we replace $\rho$ with the identity relation in Corollary 2, then $T = T(X,Y)$, $F = F(X,Y) \cap T(X,Y) = F(X,Y)$ and $Y = R$. Therefore, (ii) holds. To see that (i) holds, it suffices to prove that for $a, \beta \in F,$

$\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ if and only if $Xa = X\beta$.

By Lemma 1, we have the "only if" part of the above statement. Now, if $a, \beta \in F$ and $Xa = X\beta$, then for each $xa \in Xa$ there exist $x \in X$ and $r \in R$ such that $xa = y\beta = r\beta$, since $Xa \subseteq X\beta$ and $\beta \in F$. Hence, $\alpha = \{(xa) : x \in X\} \subseteq \{(r\beta) : r \in R\} = \beta$. Similarly, by using $X\beta \subseteq Xa$ and $a \in F$, we obtain $\beta \rightarrow \alpha$. □

As we know, $aR\beta$ on $T(X,Y)$ (or $T(X,\rho, R)$) if and only if $\ker(a) = \ker(\beta)$. However, for the semigroup $T$, this is true only on $F$ (see Corollary 5). For $a, \beta$ outside $F$, there are more terminologies involved.

The following example shows that there are $a, \beta \in T$ with $\ker(\beta) \subseteq \ker(a)$ but $a \neq \beta \gamma$ for all $\gamma \in T$.

Example 4. Considering $a, \beta_1$ and $\beta_2$ defined in Example 3, we see that $\ker(\beta_1) \subseteq \ker(a)$ but $a \neq \beta_1 \gamma$ for all $\gamma \in T$, for if $a = \beta_1 \gamma$ for some $\gamma \in T$, then $7 = 9a = 9 \beta_1 \gamma = 6 \gamma \in R$, a contradiction. Moreover, we have $\beta_1 = \{(1,2,3,6,7,8), (4,5), (9,10)\} \leftrightarrow \{(1,2,3,6,7,8), (4,5,9,10)\} = \beta$ but $(R\beta_1^{-1})a = \{1,4,6,9\} \notin R$. In the same way, $\ker(\beta_2) \subseteq \ker(a)$ but $a \neq \beta_2 \gamma$ for all $\gamma' \in T$, for if $a = \beta_2 \gamma'$ for some $\gamma' \in T$, then $(1\gamma', 3\gamma') = (1\beta_2 \gamma', 9\beta_2 \gamma') = (1a, 9a) = (1,7) \notin \rho$, which is a contradiction. Furthermore, $(R\beta_2^{-1})a \subseteq R$ but $\beta_2 = \{(1,2,3,9,10), (4,5), (6,7,8)\} \notin \beta$.

The proof below is completely different from those for $T(X,Y)$ and $T(X,\rho, R)$, especially when proving the existence of such $\gamma \in T$.

Theorem 2. Let $a, \beta \in T$. Then $a = \beta \gamma$ for some $\gamma \in T$ if and only if $\ker(\beta) \subseteq \ker(a), \gamma \beta \rightarrow \alpha \text{ and } (R\beta^{-1})a \subseteq R$. Consequently, $a \in T(X, Y)$ if and only if $\ker(a) = \ker(\beta), \gamma \beta \rightarrow \alpha$ and $(R\beta^{-1})a, (R \alpha^{-1}) \beta \subseteq R$. 

Proof. Assume that $\alpha = \beta \gamma$ for some $\gamma \in T$. If $\gamma = 1$, then $\alpha = \beta$ and the theorem holds. Now, we prove for $\gamma \in T$. Let $a, b \in X$ be such that $a \beta = b \beta$. Then $a \alpha = a \beta \gamma = b \beta \gamma = ba$. Thus, $\ker(\beta) \subseteq \ker(\alpha)$. For each $U \in \mathcal{U}$, $U \beta \subseteq r \beta$ for some $r \in R$ and so $U \alpha = U \beta \gamma \subseteq (r \beta) \gamma \subseteq s \delta$ for some $s \in R$. Thus, $U \subseteq (s \delta) \alpha^{-1} \ni \alpha$. Which implies that $\alpha \beta \mapsto \alpha$. Now, let $c \in (R \beta^{-1}) \alpha$. Then $c = da$ and $d \beta = t$ for some $d \in X$ and $t \in R$. Hence, $c = da = d \beta \gamma = t \gamma \in R$, that is, $(R \beta^{-1}) \alpha \subseteq R$.

Conversely, assume that $\ker(\beta) \subseteq \ker(\alpha), \alpha \beta \mapsto \alpha$ and $(R \beta^{-1}) \alpha \subseteq R$. Let $r_0 \in R$ be fixed, and define $\gamma \in T$ on each $\rho$-class as follows. Let $x \rho \in X/\rho$.

If $x \rho \cap X \beta = \emptyset$, then define $a \gamma = r_0$ for all $a \in x \rho$. Therefore $(x \rho) \gamma = \{r_0\} \subseteq r_\rho$. If $x \rho \cap X \beta \neq \emptyset$, then $x \rho \cap Y \neq \emptyset$. Let $x \rho \cap Y = r_\rho$ for some $r \in R$. We obtain $(r_\rho) \beta^{-1} \neq \emptyset$. Since $\alpha \beta \mapsto \alpha$, it follows that $(r_\rho) \beta^{-1} \subseteq (s \delta) \alpha^{-1}$ for some $s \in R$. Now, let $a \in x \rho$ and consider two cases. If $a \notin X \beta$, then define $a \gamma = s \in s \delta$. If $a \in X \beta$, then $a \in x \rho \cap Y = r_\rho$ and $a = b \beta$ for some $b \in X$. Thus, $b \alpha \beta^{-1} \subseteq (r_\rho) \beta^{-1} \subseteq (s \delta) \alpha^{-1}$ which implies $ba \in s \delta$. Now, we define $a \gamma = ba \in s \delta$ (this is well-defined since $\ker(\beta) \subseteq \ker(\alpha)$). We observe that $(x \rho) \gamma \subseteq s \delta$. To see that $r \gamma = s$, we get by the definition of $\gamma$ that $r \gamma = s$ if $r \not\in X \beta$. For $r \in X \beta$, we have $r = cb$ for some $c \in X$ and then $c \in r \beta^{-1} \subseteq (r \rho) \beta^{-1} \subseteq (s \delta) \alpha^{-1}$, hence $ca \in s \delta \cap (R \beta^{-1}) \alpha \subseteq R$ since $(R \beta^{-1}) \alpha \subseteq R$. Thus, $r \gamma = ca = s$.

To prove that $\alpha = \beta \gamma$, let $b \in X$. Then $b \beta \in X \beta$, and so $(b \beta) \rho \cap X \beta \neq \emptyset$. By the definition of $\gamma$, we obtain $b \beta \gamma = (b \beta) \gamma = ba$. Therefore, $\alpha = \beta \gamma$. \qed

The $\mathcal{R}$-relation on $T(X, Y)$ is as follows.

Corollary 4. [6] [Theorem 3.3] Let $\alpha, \beta \in T(X, Y)$. Then $\alpha \mathcal{R} \beta$ if and only if $\pi(\alpha) = \pi(\beta)$.

Proof. If $\rho$ is the identity relation, then $T = T(X, Y)$, $F = F(X, Y)$ and $R = Y$. Moreover, $\alpha \beta = \{r \beta \alpha^{-1} \ni r \in Y \neq \emptyset\} = \{r \alpha^{-1} \ni r \in X\} = \pi(\alpha)$ for all $\alpha \in T(X, Y)$. In addition, $(R \beta^{-1}) \alpha \subseteq R$ always holds for all $\alpha, \beta \in T(X, Y)$ since $(R \beta^{-1}) \alpha = (Y \beta^{-1}) \alpha = X \alpha \subseteq Y = R$. Since we have $\ker(\alpha) = \ker(\beta)$ if and only if $\pi(\alpha) = \pi(\beta)$, it follows from Theorem 2 that $\alpha \mathcal{R} \beta$ if and only if $\pi(\alpha) = \pi(\beta)$.

As one might expect, there are two types of the $\mathcal{R}$-classes on $T$, the one that lies inside $F$ and the other outside $F$. To see this, we need the two lemmas below.

Lemma 2. Let $\alpha, \beta \in F$ and $\ker(\alpha) = \ker(\beta)$. Then the following statements hold.

(i) $\alpha \beta = \beta \alpha$.

(ii) $(R \beta^{-1}) \alpha \cup (Ra^{-1}) \beta \subseteq R$.

Proof. (i) Let $(r \beta) \alpha^{-1} \in \alpha$. Then $(r \beta) \alpha^{-1} = \bigcup_{i \in I} A_i$, where $A_i$ is a $\rho$-class such that $A_i \alpha \subseteq r \rho$ for all $i \in I$. From $a \in F$, there exists $A_{i_0}$ such that $A_{i_0} \cap Y \neq \emptyset$. Therefore there is $s \in A_{i_0} \cap R$ and $sa \beta = t$ for some $t \in R$, which implies that $A_{i_0} \beta \subseteq t \rho$. We prove that $A_i \beta \subseteq t \rho$ for all $i \in I$. Let $i \in I$.

If $A_i \cap Y \neq \emptyset$, then there exists $u \in A_i \cap R$ and $ua \beta = u \beta$. Therefore, $ua = sa$, and so $ua \beta = s \beta = t \beta \alpha^{-1} \subseteq \alpha$. Thus, $A_i \beta \subseteq t \rho$.

If $A_i \cap Y = \emptyset$, then since $A_i \alpha \subseteq r \rho$, there exists $a \in A_i$ such that $aa \alpha \subseteq r \rho$. From $a \in F$, we obtain that $ba = aa \in r \rho$ for some $b \in Y$. Hence, $b \in (r \rho) \alpha^{-1} = \bigcup_{i \in I} A_i$, that is, $b \in A_i \cap Y \neq \emptyset$ for some $j \in I$, which implies $A_i \beta \subseteq t \rho$. Since $\ker(\alpha) = \ker(\beta)$, we obtain that $a \beta = b \beta \in A_i \beta \subseteq t \rho$, where $a \in A_i$, and it follows that $A_i \beta \subseteq t \rho$.

Therefore, $A_i \beta \subseteq t \rho$ for all $i \in I$, that is, $(r \rho) \alpha^{-1} = \bigcup_{i \in I} A_i \subseteq (t \rho) \beta^{-1}$ and $\alpha \beta \mapsto \alpha \beta$ as required. Similarly, since $\beta \in F$, we obtain $\beta \alpha \mapsto \beta \alpha$. Thus, $\alpha \beta = \beta \alpha$.

(ii) Let $a \in (R \beta^{-1}) \alpha$. Then $a = ba \in X$ and $r \in R$. From $\beta \in F$, we get $b \beta = r = s \beta$ for some $s \in R$. Since $\ker(\alpha) = \ker(\beta)$, we obtain that $a = ba = sa \in R$, and thus $(R \beta^{-1}) \alpha \subseteq R$. Similarly, $(R \alpha^{-1}) \beta \subseteq R$. \qed

Lemma 3. Let $\alpha, \beta \in T$. If $\ker(\alpha) = \ker(\beta)$, then either both $\alpha$ and $\beta$ are in $F$, or neither is in $F$. 
Proof. Assume that ker(α) = ker(β). Suppose that one of α and β is not in F. Without loss of generality, assume that α ∉ F. Then there exists x ∈ X such that xa ≠ ya for all y ∈ Y. Thus, (x, y) ∉ ker(α), which implies that xβ ≠ yβ for all y ∈ Y. Hence, Yβ ⊆ Xβ, which leads to β ∉ F. □

Using Theorem 2, Lemmas 2 and 3, we have the following corollary.

Corollary 5. For α ∈ T, the following statements hold.
(i) If α ∈ F, then R₂ = {β ∈ F : ker(β) = ker(α)}.
(ii) If α ∈ T \ F, then R₂ = {β ∈ T \ F : ker(β) = ker(α), \( β \neq \alpha \) and \( (Rβ⁻¹)α, (Rα⁻¹)β \subseteq R \)}.

If Y = X, then F = F(X, Y) ∩ T = T(X) ∩ T(X, ρ, R) = T(X, ρ, R), and so T \ F = ∅. Thus, Corollary 5 gives us a description of the \( R \)-relation on \( T(X, ρ, R) \).

Corollary 6. [3] [Theorem 2.3] Let α, β ∈ T(X, ρ, R). Then α\( R \)β if and only if ker(α) = ker(β).

As direct consequences of Corollaries 2 and 5, we have the \( H \)-relation on T as follows.

Corollary 7. For α ∈ T, the following statements hold.
(i) If α ∈ F, then H₂ = {α} ∪ {β ∈ F : \( \alpha \to \beta \), \( \alpha \to \beta \) and ker(α) = ker(β)}.
(ii) If α ∈ T \ F, then H₂ = \{α\}.

3.2. \( D \)-relation and \( J \)-relation

Let \( \phi : B \to C \) be a function from a set B to a set C. For a family \( \mathcal{A} \) of subsets of B, \( \mathcal{A} \phi \) denotes the family \{\{A\} \phi : A ∈ \mathcal{A}\} of subsets of C.

The main results used for characterizing the Green’s \( D \)-relation on T below are Corollaries 5 and 2. Moreover, the technique for defining such a function \( \phi \) in (i) is taken from Reference [3] [Theorem 2.6].

Theorem 3. For α ∈ T, the following statements hold.
(i) If α ∈ F, then D₂ = \{β ∈ F : ker(β) = ker(α); or there exists a bijection \( \phi : Xα \to Xβ \) such that \( (R ∩ Xα)φ \subseteq R, (\alpha)φ \to (\beta)φ \) and \( β \to (\beta)φ \).
(ii) If α ∈ T \ F, then D₂ = {β ∈ T \ F : ker(β) = ker(α), \( β \neq \alpha \) and \( (Rβ⁻¹)α, (Rα⁻¹)β \subseteq R \)}.

Proof. Let α be any element in T and let β ∈ D₂. Then α\( R \)γ and \( γ \leq β \) for some \( γ \in T \).

(i) Assume that α ∈ F. By Corollary 5, \( γ \in F \) and ker(γ) = ker(α). By Corollary 2, \( β \in F \); also, \( β = γ \) or \( \beta γ \to (\beta)γ \) and \( \beta γ \to \beta \). If \( γ = β \), then ker(β) = ker(α). Now, assume that \( \beta γ \to (\beta)γ \) and \( \beta γ \to \beta \). Define \( \phi : Xα \to Xβ \) by \( \phi(αx) = αγ \). We have \( αγ ∈ (αp)γ \subseteq (Rβ)β \subseteq Xβ \) for some \( r ∈ R \), since \( γ β \to \beta \), so \( αγ \in Xβ \). Since ker(γ) = ker(α), we obtain that \( \phi \) is well-defined and injective. To see that \( \phi \) is surjective, let \( ββ \subseteq Xβ \). Then \( ββ ∈ (bp)β \subseteq (sβ)γ \) for some \( s ∈ R \), since \( β γ \to (\beta)γ \). It follows that \( ββ = cγ \) for some \( c ∈ sβ \), and so \( (αx)φ = cγ \). Since \( γ β \to \beta \), hence \( φ \) is surjective. To show that \( (R ∩ Xα)φ \subseteq R \), let \( t ∈ R ∩ Xα \). Since \( t ∈ Xα \) and \( α ∈ F \), there exists \( p ∈ R \) such that \( t ∈ (bp)α \), thus \( t = pa \) and \( tφ = (pa)φ = pγ ∈ R \). Moreover, by the definition of \( φ, (αx)φ = γ \) and \( (\alpha)φ = γ \). Hence, \( (αx)φ \to (\beta)φ \) and \( βγ \to (\beta)φ \).

Conversely, assume that \( λ ∈ F \). If ker(λ) = ker(α), then λ\( Rα \), and it follows that \( λ ∈ D₂ \). If there exists a bijection \( \phi : Xα \to Xλ \) such that \( (R ∩ Xα)φ \subseteq R, (α)φ \to (\lambda)φ \) and \( λ \to (\lambda)φ \), then we define \( γ : X \to X \) by \( γ = (αx)φ \) for all \( a ∈ X \). From \( (R ∩ Xα)φ \subseteq R \), we obtain that \( rγ = (ra)φ \in R \) for all \( r ∈ R \). Thus, \( Rγ \subseteq R \). To show that \( (\alpha)γ ≤ sβ \) for some \( s ∈ R \), considering \( (\alpha)γ = ((\alpha)p)αφ ∈ (αγ)φ \) and \( (αγ)φ \to (\lambda)γ \), we get \( (α)γ ≤ (tφ)γ ≤ (sβ)γ \) for some \( t, s ∈ R \). Thus, \( (\alpha)γ ≤ sβ \), that is, \( γ ∈ T \). To see that \( γ ∈ F \), let \( a ∈ X \). Then \( aγ = (αa)φ = (ba)φ = bγ \) for some \( b ∈ Y \), since \( a ∈ F \). It follows that \( γ ∈ (F(X, Y) ∩ T) = F \). Since \( φ \) is an injective map, we obtain that ker(γ) = ker(α). By the definition of
\( \gamma, v \gamma = (v \alpha) \beta \) and \( v^{-1} \gamma = (v^{-1}) \alpha \beta \). Hence, \( v \gamma \mapsto v \lambda \) and \( v \lambda \mapsto v^{-1} \gamma \). By Corollaries 5 and 2, \( \alpha R \gamma \) and \( \gamma L \lambda \).

Therefore, \( \lambda \in D_a \).

(ii) Assume that \( \alpha \in T \setminus F \). Corollaries 5 and 2 imply that \( \beta = \gamma \in T \setminus F \). Thus, \( \alpha \mathcal{R} \beta \). Again by Corollary 5, we have \( \ker(\beta) = \ker(\alpha), v \beta = v \alpha \) and \( (R \beta^{-1}) \alpha, (R \alpha^{-1}) \beta \subseteq R \). Therefore, \( D_a \subseteq \{ \beta \in T \setminus F : \ker(\beta) = \ker(\alpha), v \beta = v \alpha \) and \( (R \beta^{-1}) \alpha, (R \alpha^{-1}) \beta \subseteq R \} \). The other containment is clear since \( R \subseteq D \). \( \square \)

The two corollaries below are the \( D \)-relations on \( T(X, \rho, R) \) and \( T(X, Y) \), respectively.

**Corollary 8.** [3] [Theorem 2.6] Let \( \alpha, \beta \in T(X, \rho, R) \). Then \( \alpha \mathcal{D} \beta \) if and only if there is a bijection \( \phi : X\alpha \rightarrow X\beta \) such that \( (R \cap X\alpha) \phi \subseteq R, (\alpha \phi) \mapsto v \beta \) and \( (\alpha \phi) \mapsto (\alpha \phi) \).

**Proof.** If we replace \( Y \) with \( X \) in Theorem 3, then \( T \setminus F = \emptyset \). Therefore we have that: For \( \alpha \in T(X, \rho, R) \), \( D_a = \{ \beta \in T(X, \rho, R) : \ker(\beta) \ni \ker(\alpha); \) or there is a bijection \( \phi : X\alpha \rightarrow X\beta \) such that \( (R \cap X\alpha) \phi \subseteq R, (\alpha \phi) \mapsto v \beta \) and \( (\alpha \phi) \mapsto (\alpha \phi) \).

Now, we assert that \( \ker(\beta) = \ker(\alpha) \) implies that there is a bijection \( \phi : X\alpha \rightarrow X\beta \) such that \( (R \cap X\alpha) \phi \subseteq R, (\alpha \phi) \mapsto v \beta \) and \( (\alpha \phi) \mapsto (\alpha \phi) \). Assume that \( \ker(\beta) = \ker(\alpha) \) and define \( \phi : X\alpha \rightarrow X\beta \) by \( (a \alpha \phi) = a \beta \) for all \( a \in X \). Then \( \phi \) is a well-defined injective map, since \( \ker(\beta) = \ker(\alpha) \). It is obvious that \( \phi \) is surjective. By the definition of \( \phi, [(a \alpha \phi) \phi = (a \beta) \beta \). Thus, \( (\alpha \phi) \mapsto v \beta \) and \( (\alpha \phi) \mapsto (\alpha \phi) \). Finally, \( (R \cap X\alpha) \phi \subseteq R \subseteq R \subseteq R \). Hence, we have our assertion, and, therefore, \( D_a = \{ \beta \in T(X, \rho, R) : \) There is a bijection \( \phi : X\alpha \rightarrow X\beta \) such that \( (R \cap X\alpha) \phi \subseteq R, (\alpha \phi) \mapsto v \beta \) and \( (\alpha \phi) \mapsto (\alpha \phi) \}, \) as required. \( \square \)

**Corollary 9.** [6] [Theorem 3.7] For \( \alpha \in T(X, Y) \), the following statements hold.

(i) If \( \alpha \in F(X, Y) \), then \( D_a = \{ \beta \in F(X, Y) : X\beta = |X\alpha| \} \).

(ii) If \( \alpha \in T(X, Y) \setminus F(X, Y) \), then \( D_a = \{ \beta \in T(X, Y) \setminus F(X, Y) : \pi(\beta) = \pi(\alpha) \} \).

**Proof.** As in the proof of Corollary 4, if we replace \( \rho \) with the identity relation, then (ii) of Theorem 3 is as follows. If \( \alpha \in T(X, Y) \setminus F(X, Y) \), then \( D_a = \{ \beta \in T(X, Y) \setminus F(X, Y) : \pi(\beta) = \pi(\alpha) \} \).

Now, we claim that the conditions \( \ker(\beta) = \ker(\alpha) \); or there is a bijection \( \phi : X\alpha \rightarrow X\beta \) such that \( (R \cap X\alpha) \phi \subseteq R, (\alpha \phi) \mapsto v \beta \) and \( (\alpha \phi) \mapsto (\alpha \phi) \) in (i) of Theorem 3 is equivalent to \( |X\alpha| = |X\beta| \) for all \( \alpha, \beta \in F(X, Y) \). It is clear that the above conditions imply \( |X\alpha| = |X\beta| \). Now, let \( \alpha, \beta \in F(X, Y) \) and \( |X\alpha| = |X\beta| \). Then there is a bijection \( \phi : X\alpha \rightarrow X\beta \) and \( (R \cap X\alpha) \phi \subseteq X\beta \subseteq Y = R \). To see that the remaining conditions hold, we observe that \( (Y \alpha) \phi = (X \alpha) \phi = Y \beta = Y \beta \), since \( \alpha, \beta \in F(X, Y) \). From \( (X \alpha) \phi = Y \beta \) and \( \rho \) as the identity relation, we obtain \( (X \alpha) \phi = \{ (X \alpha) \phi : x \in X \} = \{ (Y \beta) : r \in Y \} \rightarrow v \beta \), hence, \( (\alpha \phi) \mapsto v \beta \). Similarly, from \( (Y \alpha) \phi = X \beta \), we obtain \( v \beta \mapsto (v^{-1}) \phi \). Therefore, we have our claim. \( \square \)

To characterize the \( J \)-relation on \( T \), we need the terminology below. For each \( \alpha \in T \), we define \( R(\alpha) = \{ r \in R : r \beta \cap X\alpha \neq \emptyset \} \).

The following example shows that \( R(\alpha) \) is necessary.

**Example 5.** Let \( X = \{ 1, 2, 3, 4, 5, 6, 7, 8 \} \) and \( Y = \{ 3, 4, 5, 6, 7, 8 \} \). Let \( X/\rho = \{ \{1, 2\}, \{3, 4\}, \{5, 6, 7\}, \{8\} \} \), \( Y/\beta = \{ \{3, 4\}, \{5, 6, 7\}, \{8\} \} \) and \( R = \{3, 5, 8\} \).

Define \( \alpha, \beta \in T \) as follows:

\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 4 & 5 & 6 & 5 & 6 & 6 & 5
\end{pmatrix}
\quad \text{and} \quad
\beta = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
7 & 7 & 5 & 6 & 3 & 4 & 4 & 8
\end{pmatrix}.
\]
Then $R(\alpha) = \{3, 5\} \subseteq X_\alpha$ and $R(\beta) = \{3, 5, 8\} \subseteq X_\beta$. Moreover,

$$\forall \alpha = \{4, \{5, 6\}, \{5\}\} \text{ and } \forall \beta = \{\{\{4, \{5, 6\}, \{8\}\}\}.}

We show further that $\alpha = \lambda \beta \mu$ for some $\lambda, \mu \in T$, but there is no function $\phi : Y_\beta \to X_\alpha$ such that $(R \cap Y_\beta)\phi \subseteq R$ and $\forall \alpha \to (\forall \beta)\phi \to Y/\hat{\beta}$. Define $\lambda, \mu \in T$ by

$$\lambda = \left(\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
7 & 3 & 4 & 3 & 4 & 4 & 4 & 3
\end{array}\right) \text{ and } \mu = \left(\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 3 & 4 & 5 & 6 & 5 & 3
\end{array}\right).$$

We see that $\alpha = \lambda \beta \mu$. Suppose that there is such a function $\phi$, which would imply $\{4\} = \{1, 2\} \alpha \subseteq A\phi \subseteq \{3, 4\}$ for some $\alpha \in \forall \beta$, and $\mathcal{O} \neq (R \cap A)\phi \subseteq A\phi \cap R = \{3\}$, since $\mathcal{O} \neq R \cap A \subseteq Y_\beta$. Thus, $3 \in (R \cap A)\phi \subseteq X_\alpha$, which is a contradiction. However, since $3 \in R(\alpha) \backslash X_\alpha$, we can define $\phi : Y_\beta \to X_\alpha \cup R(\alpha)$ satisfying the conditions $(R \cap Y_\beta)\phi \subseteq R$ and $\forall \alpha \to (\forall \beta)\phi \to Y/\hat{\beta}$ as follows:

$$\phi = \left(\begin{array}{ccccccc}
3 & 4 & 5 & 6 & 8 \\
3 & 4 & 5 & 6 & 5
\end{array}\right).$$

Note that if $Y = X$ or $\rho$ is the identity relation, then $R(\alpha) \subseteq X_\alpha$ for all $\alpha \in T$.

The following result is the key lemma in characterizing the $R$-relation on $T$. The outline of the proof is the same as Theorem 2.7 in [3], but there are differences in detail, for example, to prove the “only if” part, the function $\phi$ has to be defined from $Y_\beta$ into $X_\alpha \cup R(\alpha)$ in order to make $r_\beta \mu$ in (2) well-defined. In addition, because of the restricted range of $T$, the function $\mu$ defined in (3) of the “if” part is greatly different from that defined in Theorem 2.7 [3]. Moreover, each step of the proof, the restricted range is involved.

**Lemma 4.** Let $\alpha, \beta \in T$. Then $\alpha = \lambda \beta \mu$ for some $\lambda, \mu \in T$ if and only if there exists $\phi : Y_\beta \to X_\alpha \cup R(\alpha)$ such that $(R \cap Y_\beta)\phi \subseteq R$ and $\forall \alpha \to (\forall \beta)\phi \to Y/\hat{\beta}$.

**Proof.** Assume that $\alpha = \lambda \beta \mu$ for some $\lambda, \mu \in T$. We define $\phi : Y_\beta \to X_\alpha \cup R(\alpha)$ such that $(R \cap Y_\beta)\phi \subseteq R$ and $\forall \alpha \to (\forall \beta)\phi \to Y/\hat{\beta}$ as follows.

Fix $r_0 \in X_\alpha \cap R$. Let $a \in Y_\beta$, and define $\phi$ in three steps.

1. If $a \in X(\lambda \beta) \subseteq Y_\beta$, we define $a\phi = a\mu \in X_\alpha$.
2. If $a \in Y_\beta \setminus X(\lambda \beta)$ and $a\phi \cap X(\lambda \beta) \neq \emptyset$, then define $a\phi = r_\beta \mu$.
3. If $a \in Y_\beta \setminus (X(\lambda \beta)$ and $a\phi \cap X(\lambda \beta) = \emptyset$, then define $a\phi = r_0 \in X_\alpha$.

We observe that $r_\beta \mu$ in (2) belongs to $R(\alpha)$, since $r_\beta \mu \in R$ and $r_\beta \mu \hat{\beta} \mu \in X_\alpha$ for some $b \in a\phi \cap X(\lambda \beta)$.

By the definition of $\phi$ and the fact that $R_{\mu} \subseteq R$, we obtain that $(R \cap Y_\beta)\phi \subseteq R$. To see that $\forall \alpha \to (\forall \beta)\phi$, let $(x\phi)\alpha \in \forall \alpha$. Then there exists $r \in R$ such that $(x\phi)\alpha \subseteq r\hat{\beta}$. We have $(x\phi)\alpha = (x\phi)\lambda \beta \mu \subseteq [(r\beta)\beta \cap X(\lambda \beta)] \cap [(r\hat{\beta})\beta \cap X(\lambda \beta)] \subseteq [(r\hat{\beta})\beta \cap X(\lambda \beta)] \subseteq [(r\hat{\beta})\beta \cap X(\lambda \beta)] \subseteq X_\alpha$. Therefore, $[(r\beta)\beta] \subseteq X_\alpha$.

Conversely, assume that there exists $\phi : Y_\beta \to \alpha \cup R(\alpha)$ such that $(R \cap Y_\beta)\phi \subseteq R$ and $\forall \alpha \to (\forall \beta)\phi \to Y/\hat{\beta}$. Let $A \in \alpha$. Then there is a unique $r_\alpha \in R$ such that $A \subseteq r_\alpha \hat{\beta}$ (note that $r_\alpha$ may not belong to $X_\alpha$ but $r_\alpha \in (\alpha \cap R)$). Since $\forall \alpha \to (\forall \beta)\phi \to Y/\hat{\beta}$, there exists $C(\beta) \subseteq \beta$ such that $A \subseteq C(\beta) \subseteq r_\alpha \hat{\beta}$.

From $C(\beta) \subseteq \forall \beta$, there is $s_A \in R$ such that $C(\beta) = (s_A \beta) \beta$. Let $t_A = s_A \beta$. Then $C(\beta) = (s_A \beta) \beta \subseteq t_\beta \hat{\beta}$. For every $a \in A$, we choose $u_a^A \in C(\beta)$ such that $a = u_a^A \beta$ (since $t_\beta \beta = r_A$, we may assume that $u_a^A = t_A$ if $a = r_A$). Let $C(\alpha) = \{u_a^A : a \in \alpha\}$. Then $C(\beta) \subseteq C(\alpha)$. For every $b \in C(\alpha)$, we choose $v_b^A \in C(\beta)$ such that $b = v_b^A \beta$ (since $s_A \beta = t_A$, we may assume that $v_b^A = s_A$ if $b = t_A$).

We aim to define $\lambda, \mu \in T$ such that $\alpha = \lambda \beta \mu$. We first define $\lambda$. Let $x \in X, A = (x\phi)\alpha$ and $a = x\alpha$. Then $x\alpha = a \in A$, so there exists $b = u_a^A$ such that $b = v_b^A \beta$, thus we define $x\beta = v_b^A \beta$. By the definition
of \( \lambda \), \((xp)\lambda \subseteq s_L^p\). If \( p \in xp \cap R \), then \( \alpha' = \alpha x = r_A \in R \cap A \), and so \( b' = u_A^p = t_A \). Hence \( v_\alpha^p = s_A \), which implies that \( p\lambda = v_\alpha^p = s_A \in R \). Thus, \( \lambda \in T \).

To define \( \mu \), fix \( r_0 \in R \) and let \( x \in X \).

(1) If \( x \in C_A^p \) for some \( A \in \text{aut} \), define \( \chi_x = x \phi \).

(2) If \( x \notin C_B^p \) for all \( B \in \text{aut} \) and \( C_A \subseteq xp \) for some \( A \in \text{aut} \), define \( \chi_x = r_A \).

(3) If \( xp \cap C_A = \emptyset \) for all \( A \in \text{aut} \), define \( \chi_x = r_0 \).

To see that the definition of \( \mu \) in (2) does not depend on the choice of \( A \), suppose there are \( A, B \in \text{aut} \) such that \( C_A, C_B \subseteq xp \). Since \( C_A = (s_A \beta)^p \subseteq t_A \beta \subseteq xp \) and \( C_B = (s_B \beta)^p \subseteq t_B \beta \subseteq xp \), we obtain \( t_A = t_B \), and thus \( r_A = t_A \beta = t_B \beta = r_B \). Next, we prove that \( \mu \in T \). By the definition of \( \mu \), we see that \( R_\mu \subseteq R \). Now, if \( xp \cap C_A = \emptyset \) for all \( A \in \text{aut} \), then \( (xp)\mu = \{r_0\} \subseteq r_0 \beta \). For \( C_A \subseteq xp \) for some \( A \in \text{aut} \), we have \( \chi_x = x \phi \in r_B \beta \), since \( x \in C_B^p \) for some \( B \in \text{aut} \) and \( x \mu = r_A \in r_A \beta \) if \( x \notin C_B^p \) for all \( B \in \text{aut} \). Thus, \((xp)\mu \subseteq r_A \beta \).

To prove that \( \mu = \lambda \beta \mu \), let \( x \in X \), \( A = (xp)\lambda \) and \( a = xa \). Let \( b = u_A^\lambda \) (note that \( u_A^\lambda \in C_A^\lambda \) was selected such that \( u_A^\lambda \beta = a \)). By the definitions of \( \lambda \) and \( \mu \), we have \( x \lambda = v_\beta^\lambda \) (recall that \( v_\beta^\lambda \) was chosen so that \( v_\beta^\lambda \beta = b \)) and \( b \mu = u_A^\mu = u_A^\lambda \beta = a = xa \). Thus, \( x \lambda \beta \mu = v_\beta^\lambda \beta \mu = b \mu = xa \), as required.

If we take \( Y = X \) in Lemma 4, then \( T = T(X, \rho, R) \), which contains an identity element, the identity map. Thus, we obtain the \( J \)-relation on \( T(X, \rho, R) \).

**Corollary 10.** [3] [Theorem 2.8] Let \( \alpha, \beta \in T(X, \rho, R) \). Then \( \alpha \beta \) if and only if there exist \( \phi : X \beta \rightarrow X \alpha \) and \( \varphi : X \alpha \rightarrow X \beta \) such that \((R \cap X \beta) \phi \subseteq R \), \( \varphi \lambda \rightarrow (\beta \phi) \lambda \rightarrow X / \rho \); also \((R \cap X \alpha) \varphi \subseteq R \), \( \beta \lambda \rightarrow (\alpha \varphi) \lambda \rightarrow X / \rho \).

Now, we are ready to prove the \( J \)-relation on \( T \).

**Theorem 4.** Let \( \alpha, \beta \in T \). Then \( \alpha \beta \) if and only if one of the following conditions holds:

(i) \( \ker(\alpha) = \ker(\beta) \), \( \varphi \alpha = \varphi \beta \) and \((R \cap X \alpha) \beta \subseteq R \);

(ii) there exist \( \phi : Y \beta \rightarrow X \alpha \) such that \((R \cap X \beta) \beta \subseteq R \), \( \lambda \beta \rightarrow (\nu \beta) \lambda \rightarrow Y / \rho \) and \( \nu \alpha \rightarrow (\lambda \beta) \alpha \rightarrow X / \rho \).

**Proof.** Assume that \( \alpha \beta \). Then \( \alpha = \sigma \delta \beta \) and \( \beta = \sigma' \alpha \delta' \) for some \( \sigma, \sigma', \delta, \delta' \in T^1 \). If \( \sigma = 1 = \sigma' \), then \( \alpha = \beta \delta \) and \( \beta = \alpha \delta' \), which implies that \( \alpha \in R \beta \), and so \( \ker(\alpha) = \ker(\beta) \), \( \varphi \alpha = \varphi \beta \) and \((R \cap X \alpha) \beta \subseteq R \). If \( \delta = 1 = \delta' \), then \( \alpha = \sigma \delta \) and \( \beta = \sigma' \alpha \), which implies that \( \alpha \in L \beta \), and so \( \alpha = \beta \); or \( \alpha \in \varphi \beta \) and \( \varphi \beta \rightarrow \varphi \lambda \). If \( \alpha = \beta \), then (i) holds. If \( \alpha \in \varphi \beta \) and \( \beta \in \varphi \alpha \), then we define \( \phi \) and \( \varphi \) to be the identity maps on \( Y \beta \) and \( X \alpha \), respectively. It follows that \((R \cap X \beta) \beta \subseteq R \), \( \lambda \beta \rightarrow (\nu \beta) \lambda \rightarrow Y / \rho \), \( (R \cap Y \alpha) \varphi \subseteq R \) and \( \beta \lambda \rightarrow (\nu \alpha) \varphi \rightarrow Y / \rho \). That is, (ii) holds. For the other cases, we can conclude that \( \alpha = \lambda \beta \mu \) and \( \beta = \lambda' \alpha \mu' \) for some \( \lambda, \lambda', \mu, \mu' \in T \). Therefore, \( \alpha \beta \) if and only if (i) holds in all the remaining cases.

Conversely, assume that the statement holds. If \( \ker(\alpha) = \ker(\beta) \), \( \varphi \alpha = \varphi \beta \) and \((R \cap X \alpha) \beta \subseteq R \), then \( \alpha \beta \) and \( \beta \lambda \mu \in T \). If there exist \( \phi : Y \beta \rightarrow X \alpha \) such that \( (R \cap X \beta) \beta \subseteq R \), \( \lambda \beta \rightarrow (\nu \beta) \lambda \rightarrow Y / \rho \) and \( \nu \alpha \rightarrow (\lambda \beta) \alpha \rightarrow X / \rho \), then \( \alpha = \lambda \beta \mu \) and \( \beta = \lambda' \alpha \mu' \) for some \( \lambda, \lambda', \mu, \mu' \in T \) by Lemma 4. Therefore, \( a \beta \), as required.

By setting \( \beta \) to be the identity relation in Theorem 4, we obtain the \( J \)-relation on \( T(X, Y) \) as follows.

**Corollary 11.** [6] [Theorem 3.9] Let \( \alpha, \beta \in T(X, Y) \). Then \( \alpha \beta \) if and only if \( \pi(\alpha) = \pi(\beta) \) or \( \pi(X) = \pi(A) = |Y| = |X| \).

**Proof.** If \( \rho \) in Theorem 4 is the identity relation, then \( T = T(X, Y) \). By the same proof as given for Corollary 4, we have that (i) of Theorem 4 is equivalent to \( \pi(\alpha) = \pi(\beta) \). Now, we claim that
(ii) is equivalent to $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$. If (ii) holds, then $\phi : Y\beta \to X\alpha$ is onto, since $\star\alpha = \{(x\alpha) : x \in X\} \mapsto \{((r\beta)\phi) : r \in Y\} = (\star\beta)\phi$ implies that for each $x\alpha \in X\alpha$, $x\alpha = (r\beta)\phi$ for some $r \in Y$. Similarly, from $\phi : Y\alpha \to X\beta$ with $\star\beta \mapsto (\star\alpha)\phi$, we obtain that $\phi$ is onto. Thus, $|X\alpha| \leq |Y\beta|$ and $|X\beta| \leq |Y\alpha|$, and so $|Y\alpha| \leq |X\alpha| \leq |Y\beta| \leq |X\beta| \leq |Y\alpha|$. Therefore, $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$. Conversely, if $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$, then there exist bijections $\phi : Y\beta \to X\alpha$ and $\phi : Y\alpha \to X\beta$. Since $\phi$ is onto, it follows that $\star\alpha = \{(x\alpha) : x \in X\} \mapsto \{((r\beta)\phi) : r \in Y\} = (\star\beta)\phi \mapsto \{y : y \in Y\} = Y/\hat{\beta}$. Similarly, as $\phi$ is onto, we have $\phi \mapsto (\star\alpha)\phi \mapsto Y/\hat{\alpha}$. Moreover, $(R \cap Y\beta)\phi \subseteq (Y\beta)\phi = X\alpha \subseteq Y = R$, and in the same way, $(R \cap Y\alpha)\phi \subseteq R$. Therefore, we have our claim. □

Recall that $D \subseteq J$ on any semigroup and $D = J$ on $T(X)$, but in $T$ it is not always true, so we end this section with an example showing that $D \neq J$ on $T$.

**Example 6.** Let $X$ be the set of all positive integers and $Y = X \setminus \{1, 2\}$. Let $X/\rho = \{\{1, 2\}, \{3, 4, 5\}\} \cup \{2n + 4, 2n + 5 : n \in X\}$, $Y/\hat{\beta} = \{\{3, 4, 5\}\} \cup \{2n + 4, 2n + 5 : n \in X\}$ and $R = \{3, 6, 8, 10, \ldots\}$.

Then we define

$$\alpha = \left(\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \ldots
\end{array}\right)$$

$$\beta = \left(\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \ldots
9 & 9 & 6 & 7 & 3 & 4 & 3 & 5 & 6 & 7 & 10 & 11 & \ldots.
\end{array}\right).$$

Thus, $\alpha, \beta \in T\setminus F$ and

$\star\alpha = \{\{4\}, \{8, 9, \{12, 13\}, \{16, 17\}, \{20, 21\}, \{24, 25, \ldots\},
\star\beta = \{\{9\}, \{6, 7\}, \{3, 4\}, \{3, 5\}, \{10, 11\}, \{14, 15\}, \{18, 19\}, \ldots\},
\star\beta = \{\{6, 7\}, \{3, 4\}, \{3, 5\}, \{10, 11\}, \{14, 15\}, \{18, 19\}, \ldots\}.$

It is clear that $\ker(\alpha) \neq \ker(\beta)$. Therefore, $\alpha$ and $\beta$ are not $D$-related by Theorem 3. However, we can define $\phi : Y\beta \to X\alpha \cup R(\alpha)$ and $\phi : Y\alpha \to X\beta \cup R(\beta)$ as follows:

$$((3, 4)\phi = \{3, 4\}, \quad (3, 5)\phi = \{3, 4\}, \quad (6, 7)\phi = \{8, 9\},$$

$$((10, 11)\phi = \{12, 13\}, \quad ((14, 15)\phi = \{16, 17\}, \quad ((18, 19)\phi = \{20, 21\}, \ldots)$$

$$((8, 9)\phi = \{8, 9\}, \quad ((12, 13)\phi = \{6, 7\}, \quad ((16, 17)\phi = \{3, 4\},$$

$$((20, 21)\phi = \{3, 5\}, \quad ((24, 25)\phi = \{10, 11\}, \quad ((28, 29)\phi = \{14, 15\}, \ldots).$$

Both $\phi$ and $\varphi$ satisfy the required properties of Theorem 4. Therefore, $\alpha$ and $\beta$ are $J$-related.

However, if $X$ is a finite set, then $T$ is a finite semigroup and it is periodic. Hence $D = J$ in this case.

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References


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