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L-Fuzzy Sets and Isomorphic Lattices: Are All the "New" Results Really New? [†]

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Abstract: We review several generalizations of the concept of fuzzy sets with two- or three-dimensional lattices of truth values and study their relationship. It turns out that, in the two-dimensional case, several of the lattices of truth values considered here are pairwise isomorphic, and so are the corresponding families of fuzzy sets. Therefore, each result for one of these types of fuzzy sets can be directly rewritten for each (isomorphic) type of fuzzy set. Finally we also discuss some questionable notations, in particular, those of "intuitionistic" and "Pythagorean" fuzzy sets.

Keywords: fuzzy set; interval-valued fuzzy set; "intuitionistic" fuzzy set; "pythagorean" fuzzy set; isomorphic lattices; truth values

1. Introduction

In the paper "Fuzzy sets" [1] L. A. Zadeh suggested the unit interval $[0, 1]$ (which we shall denote by $\mathbb{I}$ throughout the paper) as set of truth values for fuzzy sets, in a generalization of Boolean logic and Cantorian set theory where the two-element Boolean algebra $\{0, 1\}$ is used.

Soon after a further generalization was proposed in J. Goguen [2]: to replace the unit interval $\mathbb{I}$ by an abstract set L (in most cases a lattice), noticing that the key feature of the unit interval in this context is its lattice structure. In yet another generalization L. A. Zadeh [3,4] introduced fuzzy sets of type 2 where the value of the membership function is a fuzzy subset of $\mathbb{I}$.

Since then, many more variants and generalizations of the original concept in [1] were presented, most of them being either L-fuzzy sets, type-n fuzzy sets or both. In a recent and extensive "historical account", H. Bustince et al. ([5], Table 1) list a total of 21 variants of fuzzy sets and study their relationships.

In this paper, we will deal with the concepts of (generalizations of) fuzzy sets where the set of truth values is either one-dimensional (the unit interval $\mathbb{I}$), two-dimensional (e.g., a suitable subset of the unit square $\mathbb{I} \times \mathbb{I}$) or three-dimensional (a subset of the unit cube $\mathbb{I}^3$).

The one-dimensional case (where the set of truth values equals $\mathbb{I}$) is exactly the case of fuzzy sets in the sense of [1].

Concerning the two-dimensional case, we mainly consider the following subsets of the unit square $\mathbb{I} \times \mathbb{I}$:

$$
\begin{aligned}
\mathbb{L}^* &= \{(x_1, x_2) \in \mathbb{I} \times \mathbb{I} \mid x_1 + x_2 \leq 1\}, \\
L_2(\mathbb{I}) &= \{(x_1, x_2) \in \mathbb{I} \times \mathbb{I} \mid 0 \leq x_1 \leq x_2 \leq 1\}, \\
P^* &= \{(x_1, x_2) \in \mathbb{I} \times \mathbb{I} \mid x_1^2 + x_2^2 \leq 1\},
\end{aligned}
$$

and the related set of all closed subintervals of the unit interval $\mathbb{I}$:

$$\mathfrak{I}(\mathbb{I}) = \{[x_1, x_2] \subseteq \mathbb{I} \mid 0 \leq x_1 \leq x_2 \leq 1\}.$$

Equipped with suitable orders, these lattices of truth values give rise to several generalizations of fuzzy sets known from the literature: $\mathbb{L}^*$-fuzzy sets, "intuitionistic" fuzzy sets [6,7], grey sets [8,9], vague sets [10], 2-valued sets [11], interval-valued fuzzy sets [4,12–14], and "Pythagorean" fuzzy sets [15].

In the three-dimensional case, the following subsets of the unit cube $\mathbb{I}^3$ will play a major role:

$$\mathbb{D}^* = \{(x_1, x_2, x_3) \in \mathbb{I}^3 \mid x_1 + x_2 + x_3 \leq 1\},$$
$$L_3(\mathbb{I}) = \{(x_1, x_2, x_3) \in \mathbb{I}^3 \mid 0 \leq x_1 \leq x_2 \leq x_3 \leq 1\}.$$

Equipped with suitable orders, these lattices of truth values lead to the concepts of 3-valued sets [11] and picture fuzzy sets [16].

While it is not surprising that lattices of truth values of higher dimension correspond to more complex types of fuzzy sets, it is remarkable that in the two-dimensional case the lattices with the carriers $\mathbb{L}^*$, $L_2(\mathbb{I})$, P^*, and $\mathfrak{I}(\mathbb{I})$ are mutually isomorphic, i.e., the families of fuzzy sets with these truth values have the same lattice-based properties. This implies that mathematical results for one type of fuzzy sets can be carried over in a straightforward way to the other (isomorphic) types. This also suggests that, in a mathematical sense, often only one of these lattices of truth values (and only one of the corresponding types of fuzzy sets) is really needed.

Note that if some algebraic structures are isomorphic, then it is meaningful to consider all of them only if they have different meanings and interpretations.

This is, e.g., the case for the arithmetic mean (on $[-\infty, \infty[$) and for the geometric mean (on $[0, \infty[$). On the other hand, concerning results dealing with such isomorphic structures, it is enough to prove them once and then to transfer them to the other isomorphic structures simply using the appropriate isomorphisms. For example, in the case of the arithmetic and geometric means mentioned here, the additivity of the arithmetic mean is equivalent to the multiplicativity of the geometric mean.

Another example are pairs (a, b) of real numbers which can be interpreted as points in the real plane, as (planar) vectors, as complex numbers, and (if $a \leq b$) as closed sub-intervals of the real line. Most algebraic operations for these objects are defined for the representing pairs of real numbers; in the case of the addition, the exact same formula is used.

We only mention that in the case of three-dimensional sets of truth values, the corresponding lattices (and the families of fuzzy sets based on them) are not isomorphic, which means that they have substantially different properties.

The paper is organized as follows. In Section 2, we discuss the sets of truth values for Cantorian (or crisp) sets and for fuzzy sets and present the essential notions of abstract lattice theory, including the crucial concept of isomorphic lattices. In Section 3, we review the two- and three-dimensional sets of truth values mentioned above and study the isomorphisms between them and between the corresponding families of fuzzy sets. Finally, in Section 4, we discuss some further consequences of lattice isomorphisms as well as some questionable notations appearing in the literature, in particular "intuitionistic" fuzzy sets and "Pythagorean" fuzzy sets.

2. Preliminaries

Let us start with collecting some of the basic and important prerequisites from set theory, fuzzy set theory, and some generalizations thereof.

2.1. Truth Values and Bounded Lattices

The set of truth values in Cantorian set theory [17,18] (and in the underlying Boolean logic [19,20]) is the Boolean algebra $\{0, 1\}$, which we will denote by $\mathbf{2}$ in this paper. Given a universe of discourse, i.e., a non-empty set X, each *Cantorian* (or *crisp*) *subset* A *of* X can be identified with its *indicator function* $\mathbf{1}_A \colon X \to \mathbf{2}$, defined by $\mathbf{1}_A(x) = 1$ if and only if $x \in A$.

In L. A. Zadeh's seminal paper on fuzzy sets [1] (compare also the work of K. Menger [21–23] and D. Klaua [24,25]), the unit interval $[0, 1]$ was proposed as set of *truth values*, thus providing a natural extension of the Boolean case. As usual, a *fuzzy subset* A of the universe of discourse X is described by its *membership function* $\mu_A \colon X \to \mathbb{I}$, and $\mu_A(x)$ is interpreted as the *degree of membership* of the object x in the fuzzy set A. The standard order reversing involution (or double negation) $N_{\mathbb{I}} \colon \mathbb{I} \to \mathbb{I}$ is given by $N_{\mathbb{I}}(x) = 1 - x$.

For the rest of this paper, we will reserve the shortcut $\mathbb{I}$ for the unit interval $[0, 1]$ of the real line $\mathbb{R}$. On each subset of the real line, the order $\leq$ will denote the standard linear order inherited from $\mathbb{R}$.

In a further generalization, J. Goguen [2] suggested to use the elements of an abstract set L as truth values and to describe an *L-fuzzy subset* A of X by means of its membership function $\mu_A \colon X \to L$, where $\mu_A(x)$ stands for the *degree of membership* of the object x in the L-fuzzy set A.

Several important examples for L were discussed in [2], such as *complete lattices* or *complete lattice-ordered semigroups*. There is an extensive literature on L-fuzzy sets dealing with various aspects of algebra, analysis, category theory, topology, and stochastics (see, e.g., [26–44]). For a more recent overview of these and other types and generalizations of fuzzy sets see [5].

In most of these papers the authors work with a *lattice* $(L, \leq_L)$, i.e., a non-empty, partially ordered set $(L, \leq_L)$ such that each finite subset of L has a *meet* (or *greatest lower bound*) and a *join* (or *least upper bound*) in L. If each arbitrary subset of L has a meet and a join then the lattice is called *complete*, and if there exist a *bottom* (or *smallest*) *element* $\mathbf{0}_L$ and a *top* (or *greatest*) *element* $\mathbf{1}_L$ in L, then the lattice is called *bounded*.

For notions and results in the theory of general lattices we refer to the book of G. Birkhoff [45]. There is an equivalent, purely algebraic approach to lattices without referring to a partial order: if $\wedge_L \colon L \times L \to L$ and $\vee_L \colon L \times L \to L$ are two commutative, associative operations on a set L that satisfy the two *absorption laws*, i.e., for all $x, y \in L$ we have $x \wedge_L (x \vee_L y) = x$ and $x \vee_L (x \wedge_L y) = x$, and if we define the binary relation $\leq_L$ on L by $x \leq_L y$ if and only if $x \wedge_L y = x$ (which is equivalent to saying that $x \leq_L y$ if and only if $x \vee_L y = y$), then $\leq_L$ is a partial order on L and $(L, \leq_L)$ is a lattice such that, for each set $\{x, y\} \subseteq L$, the elements $x \wedge_L y$ and $x \vee_L y$ coincide with the meet and the join, respectively, of the set $\{x, y\}$ with respect to the order $\leq_L$.

Clearly, the lattices $(\mathbf{2}, \leq)$ and $(\mathbb{I}, \leq)$ already mentioned are examples of complete bounded lattices: $\mathbf{2}$-fuzzy sets are exactly crisp sets, $\mathbb{I}$-fuzzy sets are the fuzzy sets in the sense of [1].

If $(L_1, \leq_{L_1}), (L_2, \leq_{L_2}), \ldots, (L_n, \leq_{L_n})$ are lattices and $\prod\limits_{i=1}^{n} L_i = L_1 \times L_2 \times \cdots \times L_n$ is the Cartesian product of the underlying sets, then also

$$\left(\prod_{i=1}^{n} L_i, \leq_{\text{comp}}\right), \tag{1}$$

is a lattice, the so-called *product lattice* of $(L_1, \leq_{L_1})$, $(L_2, \leq_{L_2})$, $\ldots$, $(L_n, \leq_{L_n})$, where $\leq_{\text{comp}}$ is the *componentwise* partial order on the Cartesian product $\prod L_i$ given by

$$\begin{aligned} &(x_1, x_2, \ldots, x_n) \leq_{\text{comp}} (y_1, y_2, \ldots, y_n) \\ &\qquad \Longleftrightarrow \quad x_1 \leq_{L_1} y_1 \ \text{ AND } \ x_2 \leq_{L_2} y_2 \ \text{ AND } \ \ldots \ \text{ AND } \ x_n \leq_{L_n} y_n. \end{aligned} \tag{2}$$

The componentwise partial order is not the only partial order that can be defined on $\prod L_i$. An alternative is, for example, the *lexicographical* partial order $\leq_{\text{lexi}}$ given by $(x_1, x_2, \ldots, x_n) \leq_{\text{lexi}} (y_1, y_2, \ldots, y_n)$ if and only if $((x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ or $(x_1, x_2, \ldots, x_n) <_{\text{lexi}} (y_1, y_2, \ldots, y_n))$, where the strict inequality $(x_1, x_2, \ldots, x_n) <_{\mathrm{lexi}} (y_1, y_2, \ldots, y_n)$ holds if and only if there is an $i_0 \in \{1, 2, \ldots, n\}$ such that $x_i = y_i$ for each $i \in \{1, 2, \ldots, i_0 - 1\}$ and $x_{i_0} <_{L_{i_0}} y_{i_0}$.

Obviously, whenever $(L_1, \leq_{L_1}), (L_2, \leq_{L_2}), \ldots, (L_n, \leq_{L_n})$ are lattices then also

$$\left(\prod_{i=1}^{n} L_i, \leq_{\mathrm{lexi}}\right)$$

is a lattice. Moreover, if each of the partial orders $\leq_{L_1}, \leq_{L_2}, \ldots, \leq_{L_n}$ is linear, then $\leq_{\mathrm{lexi}}$ is also a linear order. Note that this is not the case for $\leq_{\mathrm{comp}}$ whenever $n > 1$ and at least two of the sets $L_1, L_2, \ldots, L_n$ contain two or more elements. To take the simplest example: the lattice $(\mathbf{2} \times \mathbf{2}, \leq_{\mathrm{lexi}})$ is a chain, i.e., $(0,0) <_{\mathrm{lexi}} (0,1) <_{\mathrm{lexi}} (1,0) <_{\mathrm{lexi}} (1,0)$, but in the product lattice $(\mathbf{2} \times \mathbf{2}, \leq_{\mathrm{comp}})$ the elements $(0,1)$ and $(1,0)$ are incomparable with respect to $\leq_{\mathrm{comp}}$.

We only mention that also the product of infinitely many lattices may be a lattice. As an example, if $(L, \leq_L)$ is a lattice and X a non-empty set, then the set L^X of all functions from X to L, equipped with the componentwise partial order $\leq_{\mathrm{comp}}$, is again a lattice. Recall that, for functions $f, g \colon X \to L$, the componentwise partial order $\leq_{\mathrm{comp}}$ is defined by $f \leq_{\mathrm{comp}} g$ if and only if $f(x) \leq_L g(x)$ for all $x \in X$. If no confusion is possible, we simply shall write $f \leq_L g$ rather than $f \leq_{\mathrm{comp}} g$.

2.2. Isomorphic Lattices: Some General Consequences

For two partially ordered sets $(L_1, \leq_{L_1})$ and $(L_2, \leq_{L_2})$, a function $\varphi \colon L_1 \to L_2$ is called an *order homomorphism* if it preserves the monotonicity, i.e., if $x \leq_{L_1} y$ implies $\varphi(x) \leq_{L_2} \varphi(y)$.

If $(L_1, \leq_{L_1})$ and $(L_2, \leq_{L_2})$ are two lattices then a function $\varphi \colon L_1 \to L_2$ is called a *lattice homomorphism* if it preserves finite meets and joins, i.e., if for all $x, y \in L_1$

$$\varphi(x \wedge_{L_1} y) = \varphi(x) \wedge_{L_2} \varphi(y) \qquad \text{and} \qquad \varphi(x \vee_{L_1} y) = \varphi(x) \vee_{L_2} \varphi(y). \tag{3}$$

Each lattice homomorphism is an order homomorphism, but the converse is not true in general. A lattice homomorphism $\varphi \colon L_1 \to L_2$ is called an *embedding* if it is injective, an *epimorphism* if it is surjective, and an *isomorphism* if it is bijective, i.e., if it is both an embedding and an epimorphism.

If a function $\varphi \colon L_1 \to L_2$ is an embedding from a lattice $(L_1, \leq_{L_1})$ into a lattice $(L_2, \leq_{L_2})$ then the set $\{\varphi(x) \mid x \in L_1\}$ (equipped with the partial order inherited from $(L_2, \leq_{L_2})$) forms a sublattice of $(L_2, \leq_{L_2})$ which is isomorphic to $(L_1, \leq_{L_1})$. If $(L_1, \leq_{L_1})$ is bounded or complete, so is this sublattice of $(L_2, \leq_{L_2})$. Conversely, if $(L_1, \leq_{L_1})$ is a sublattice of $(L_2, \leq_{L_2})$ then $(L_1, \leq_{L_1})$ trivially can be embedded into $(L_2, \leq_{L_2})$ (the identity function $\mathrm{id}_{L_1} \colon L_1 \to L_2$ provides an embedding).

The word "isomorphic" is derived from the composition of the two Greek words "isōs" (meaning *similar*, *equal*, *corresponding*) and "morphē" (meaning *shape, structure*), so it means having *the same shape* or *the same structure*.

If two lattices $(L_1, \leq_{L_1})$ and $(L_2, \leq_{L_2})$ are isomorphic this means that they have the same mathematical structure in the sense that there is a bijective function $\varphi \colon L_1 \to L_2$ that preserves the order as well as finite meets and joins, compare (3).

However, being isomorphic does not necessarily mean to be identical, for example (not in the lattice framework), consider the *arithmetic mean* on $[-\infty, \infty[$ and the *geometric mean* on $[0, \infty[$ [46,47] which are isomorphic aggregation functions on $\mathbb{R}^n$, but they have some different properties and they are used for different purposes.

If $(L_1, \leq_{L_1})$ and $(L_2, \leq_{L_2})$ are isomorphic and if $(L_1, \leq_{L_1})$ has additional order theoretical properties, these properties automatically carry over to the lattice $(L_2, \leq_{L_2})$.

For instance, if the lattice $(L_1, \leq_{L_1})$ is complete so is $(L_2, \leq_{L_2})$. Or, if the lattice $(L_1, \leq_{L_1})$ is bounded (with bottom element $\mathbf{0}_{L_1}$ and top element $\mathbf{1}_{L_1}$) then also $(L_2, \leq_{L_2})$ is bounded, and the bottom and top elements of $(L_2, \leq_{L_2})$ are obtained via $\mathbf{0}_{L_2} = \varphi(\mathbf{0}_{L_1})$ and $\mathbf{1}_{L_2} = \varphi(\mathbf{1}_{L_1})$.

Moreover, it is well-known that corresponding constructs over isomorphic structures are again isomorphic. Here are some particularly interesting cases:

Remark 1. *Suppose that* $(L_1, \leq_{L_1})$ *and* $(L_2, \leq_{L_2})$ *are isomorphic lattices and that* $\varphi\colon L_1 \to L_2$ *is a lattice isomophism between* $(L_1, \leq_{L_1})$ *and* $(L_2, \leq_{L_2})$*.*

(i) *If* $f\colon L_1 \to L_1$ *is a function then the composite function* $\varphi \circ f \circ \varphi^{-1}\colon L_2 \to L_2$ *has the same order theoretical properties as* f*.*
(ii) *If* $F\colon L_1 \times L_1 \to L_1$ *is a binary operation on* L_1 *and if we define* $\big(\varphi^{-1}, \varphi^{-1}\big)\colon L_2 \times L_2 \to L_2 \times L_2$ *by* $\big(\varphi^{-1}, \varphi^{-1}\big)((x, y)) = \big(\varphi^{-1}(x), \varphi^{-1}(y)\big)$*, then the function* $\varphi \circ F \circ \big(\varphi^{-1}, \varphi^{-1}\big)\colon L_2 \times L_2 \to L_2$ *is a binary operation on* L_2 *with the same order theoretical properties as* F*.*
(iii) *If* $\mathbf{A}_1\colon (L_1)^n \to L_1$ *is an n-ary operation on* L_1 *then, as a straightforward generalization, the composite function* $\varphi \circ \mathbf{A}_1 \circ (\varphi^{-1}, \varphi^{-1}, \ldots, \varphi^{-1})\colon (L_2)^n \to L_2$ *given by*

$$\varphi \circ \mathbf{A}_1 \circ (\varphi^{-1}, \varphi^{-1}, \ldots, \varphi^{-1})(x_1, x_2, \ldots, x_n) = \varphi\Big(\mathbf{A}_1(\varphi^{-1}(x_1), \varphi^{-1}(x_2), \ldots, \varphi^{-1}(x_n))\Big),$$

is an n-ary operation on L_2 *with the same order theoretical properties as* $\mathbf{A}_1$*.*

As a consequence of Remark 1, many structures used in fuzzy set theory can be carried over to any isomorphic lattice, for example, order reversing involutions or residua [45], which are used in BL-logics [48–62]. The same is true for many connectives (mostly on the unit interval $\mathbb{I}$ but also on more general and more abstract structures (see, e.g., [63,64])) for many-valued logics such as triangular norms and conorms (t-norms and t-conorms for short), going back to K. Menger [65] and B. Schweizer and A. Sklar [66–68] (see also [69–73]), uninorms [74], and nullnorms [75]. Another example are aggregation functions which have been extensively studied on the unit interval $\mathbb{I}$ in, e.g., [46,47,76–78].

Example 1. *Let* $(L_1, \leq_{L_1})$ *and* $(L_2, \leq_{L_2})$ *be isomorphic bounded lattices, suppose that* $\varphi\colon L_1 \to L_2$ *is a lattice isomorphism between* $(L_1, \leq_{L_1})$ *and* $(L_2, \leq_{L_2})$*, and denote the bottom and top elements of* $(L_1, \leq_{L_1})$ *by* $\mathbf{0}_{L_1}$ *and* $\mathbf{1}_{L_1}$*, respectively.*

(i) *Let* $N_{L_1}\colon L_1 \to L_1$ *be an order reversing involution (or double negation) on* L_1*, i.e.,* $x \leq_{L_1} y$ *implies* $N_{L_1}(y) \leq_{L_1} N_{L_1}(x)$*, and* $N_{L_1} \circ N_{L_1} = id_{L_1}$*. Then the function* $\varphi \circ \mathbb{N}_{L_1} \circ \varphi^{-1}$ *is an order reversing involution on* L_2*, and the complemented lattice* $\big(L_2, \leq_2, \varphi \circ \mathbb{N}_{L_1} \circ \varphi^{-1}\big)$ *is isomorphic to* $(L_1, \leq_1, N_{L_1})$*.*
(ii) *Let* $(L_1, \leq_{L_1}, *_1, e_1, \to_1, \leftarrow_1)$ *be a residuated lattice, i.e.,* $(L_1, *_1)$ *is a (not necessarily commutative) monoid with neutral element* e_1*, and for the residua* $\to_1, \leftarrow_1\colon L_1 \times L_1 \to L_1$ *we have that for all* $x, y, z \in L_1$ *the assertion* $(x *_1 y) \leq_{L_1} z$ *is equivalent to both* $y \leq_{L_1} (x \to_1 z)$ *and* $x \leq_{L_1} (z \leftarrow_1 y)$*. Then*

$$\Big(L_2, \leq_{L_2}, \varphi \circ *_1 \circ \big(\varphi^{-1}, \varphi^{-1}\big), \varphi(e_1), \varphi \circ {\to_1} \circ \big(\varphi^{-1}, \varphi^{-1}\big), \varphi \circ {\leftarrow_1} \circ \big(\varphi^{-1}, \varphi^{-1}\big)\Big)$$

is an isomorphic residuated lattice.
(iii) *Let* $T_1\colon L_1 \times L_1 \to L_1$ *be a triangular norm on* L_1*, i.e.,* T_1 *is an associative, commutative order homomorphism with neutral element* $\mathbf{1}_{L_1}$*. Then the function* $\varphi \circ T_1 \circ \big(\varphi^{-1}, \varphi^{-1}\big)$ *is a triangular norm on* L_2*.*
(iv) *Let* $S_1\colon L_1 \times L_1 \to L_1$ *be a triangular conorm on* L_1*, i.e.,* S_1 *is an associative, commutative order homomorphism with neutral element* $\mathbf{0}_{L_1}$*. Then the function* $\varphi \circ S_1 \circ \big(\varphi^{-1}, \varphi^{-1}\big)$ *is a triangular conorm on* L_2*.*
(v) *Let* $U_1\colon L_1 \times L_1 \to L_1$ *be a uninorm on* L_1*, i.e.,* U_1 *is an associative, commutative order homomorphism with neutral element* $e \in L_1$ *such that* $\mathbf{0}_{L_1} <_{L_1} e <_{L_1} \mathbf{1}_{L_1}$*. Then the function* $\varphi \circ U_1 \circ \big(\varphi^{-1}, \varphi^{-1}\big)$ *is a uninorm on* L_2 *with neutral element* $\varphi(e)$*.*
(vi) *Let* $V_1\colon L_1 \times L_1 \to L_1$ *be a nullnorm on* L_1*, i.e.,* V_1 *is an associative, commutative order homomorphism such that there is an* $a \in L_1$ *with* $\mathbf{0}_{L_1} <_{L_1} a <_{L_1} \mathbf{1}_{L_1}$ *such that for all* $x \leq_{L_1} a$ *we have* $V_1((x, \mathbf{0}_{L_1})) = x$*, and for all* $x \geq_{L_1} a$ *we have* $V_1((x, \mathbf{1}_{L_1})) = x$*. Then the function* $\varphi \circ V_1 \circ \big(\varphi^{-1}, \varphi^{-1}\big)$ *is a nullnorm on* L_2*.*

(vii) *Let* $\mathbf{A}_1\colon (L_1)^n \to L_1$ *be an n-ary aggregation function on* L_1, *i.e.,* $\mathbf{A}_1$ *is an order homomorphism which satisfies* $\mathbf{A}_1(\mathbf{0}_{L_1}, \mathbf{0}_{L_1}, \dots, \mathbf{0}_{L_1}) = \mathbf{0}_{L_1}$ *and* $\mathbf{A}_1(\mathbf{1}_{L_1}, \mathbf{1}_{L_1}, \dots, \mathbf{1}_{L_1}) = \mathbf{1}_{L_1}$. *Then the function* $\varphi \circ \mathbf{A}_1 \circ (\varphi^{-1}, \varphi^{-1}, \dots, \varphi^{-1})$ *is an n-ary aggregation function on* L_2.

3. Some Generalizations of Truth Values and Fuzzy Sets

In this section we first review the lattices of truth values for crisp sets and for fuzzy sets as introduced in [1], followed by a detailed description of various generalizations thereof by means of sets of truth values of dimension two and higher.

3.1. The Classical Cases: Crisp and Fuzzy Sets

Now we shall consider different lattices of types of truth values and, for a fixed non-empty universe of discourse X, the corresponding classes of (fuzzy) subsets of X.

Recall that if the set of truth values is the classical Boolean algebra $\{0, 1\}$ (denoted in this paper simply by $\mathbf{2}$), then the corresponding set of all *crisp* (or *Cantorian*) *subsets of* X will be denoted by $\mathcal{P}(X)$ (called the *power set of* X). Each crisp subset A of X can be identified with its *characteristic function* $\mathbf{1}_A\colon X \to \mathbf{2}$, which is defined by $\mathbf{1}_A(x) = 1$ if and only if $x \in A$. There are exactly two constant characteristic functions: $\mathbf{1}_\emptyset\colon X \to \mathbf{2}$ maps every $x \in X$ to 0, and $\mathbf{1}_X\colon X \to \mathbf{2}$ maps every $x \in X$ to 1.

Obviously, we have $A \subseteq B$ if and only if $\mathbf{1}_A \leq \mathbf{1}_B$, i.e., $\mathbf{1}_A(x) \leq \mathbf{1}_B(x)$ for all $x \in X$, and $(\mathcal{P}(X), \subseteq)$ is a complete bounded lattice with bottom element $\emptyset$ and top element X, i.e., $(\mathcal{P}(X), \subseteq)$ is isomorphic to the product lattice $(\mathbf{2}^X, \leq)$, where $\mathbf{2}^X$ is the set of all functions from X to $\mathbf{2}$, and $\leq$ is the componentwise standard order.

Switching to the unit interval (denoted by $\mathbb{I}$) as set of truth values in the sense of [1], the set of all *fuzzy subsets of* X will be denoted by $\mathcal{F}(X)$. As usual, each fuzzy subset $A \in \mathcal{F}(X)$ is characterized by its *membership function* $\mu_A\colon X \to \mathbb{I}$, where $\mu_A(x) \in \mathbb{I}$ describes the *degree of membership* of the object $x \in X$ in the fuzzy set A.

For fuzzy sets $A, B \in \mathcal{F}(X)$ we have $A \subseteq B$ if and only if $\mu_A \leq \mu_B$, i.e., $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$. Therefore, $(\mathcal{F}(X), \subseteq)$ is a complete bounded lattice with bottom element $\emptyset$ and top element X, i.e., $(\mathcal{F}(X), \subseteq)$ is isomorphic to $(\mathbb{I}^X, \leq)$, where $\mathbb{I}^X$ is the set of all functions from X to $\mathbb{I}$.

Only for the sake of completeness we mention that the bottom and top elements in $\mathcal{F}(X)$ are also denoted by $\emptyset$ and X, and they correspond to the membership functions $\mu_\emptyset = \mathbf{1}_\emptyset$ and $\mu_X = \mathbf{1}_X$, respectively.

The lattice $(\mathcal{P}(X), \subseteq)$ of crisp subsets of X can be embedded into the lattice $(\mathcal{F}(X), \subseteq)$ of fuzzy sets of X: the function $\mathrm{emb}_{\mathcal{P}(X)}\colon \mathcal{P}(X) \to \mathcal{F}(X)$ given by $\mu_{\mathrm{emb}_{\mathcal{P}(X)}(A)} = \mathbf{1}_A$, i.e., the membership function of $\mathrm{emb}_{\mathcal{P}(X)}(A)$ is just the characteristic function of A, provides a natural embedding.

The membership function $\mu_{A^{\complement}}\colon X \to \mathbb{I}$ of the *complement* $A^{\complement}$ of a fuzzy set $A \in \mathcal{F}(X)$ is given by $\mu_{A^{\complement}}(x) = N_{\mathbb{I}}(\mu_A(x)) = 1 - \mu_A(x)$.

For a fuzzy set $A \in \mathcal{F}(X)$ and $\alpha \in \mathbb{I}$, the α*-cut* (or α*-level set*) of A is defined as the crisp set $[A]_\alpha \in \mathcal{P}(X)$ given by

$$[A]_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

The 1-cut $[A]_1 = \{x \in X \mid \mu_A(x) = 1\}$ of a fuzzy set $A \in \mathcal{F}(X)$ is often called the *kernel* of A, and the crisp set $\{x \in X \mid \mu_A(x) > 0\}$ usually is called the *support* of the fuzzy set A.

The family $([A]_\alpha)_{\alpha \in \mathbb{I}}$ of α-cuts of a fuzzy subset A of X carries the same information as the membership function $\mu_A\colon X \to \mathbb{I}$ in the sense that it is possible to reconstruct the membership function μ_A from the family of α-cuts of A: for all $x \in X$ we have [27,79]

$$\mu_A(x) = \sup\big\{\min\big(\alpha, \mathbf{1}_{[A]_\alpha}(x)\big) \bigm| \alpha \in \mathbb{I}\big\}.$$

We only mention that this is no more possible if the unit interval $\mathbb{I}$ is replaced by some lattice L which is not a chain.

3.2. Generalizations: The Two-Dimensional Case

A simple example of a two-dimensional lattice is $(\mathbb{I} \times \mathbb{I}, \leq_{\text{comp}})$ as defined by (1) and (2), i.e., the unit square of the real plane $\mathbb{R}^2$. In [63], triangular norms on this lattice (and on other product lattices) were studied. The standard order reversing involution $N_{\mathbb{I}\times\mathbb{I}} \colon \mathbb{I} \times \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ in $(\mathbb{I} \times \mathbb{I}, \leq_{\text{comp}})$ is given by

$$N_{\mathbb{I}\times\mathbb{I}}((x,y)) = (1-y, 1-x). \tag{4}$$

This product lattice was considered in several expert systems [80–82]. There, the first coordinate was interpreted as a degree of positive information (*measure of belief*), and the second coordinate as a degree of negative information (*measure of disbelief*). Note that though several operations for this structure were considered in the literature (for a nice overview see [83]), a deeper algebraic investigation is still missing in this case.

To the best of our knowledge, K. T. Atanassov [6,7,84] (compare [85,86]) was the first to consider both the *degree of membership* and the *degree of non-membership* when using and studying the bounded lattice $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ of truth values given by (5) and (6). Unfortunately, he called the corresponding $\mathbb{L}^*$-fuzzy sets *"intuitionistic" fuzzy sets* because of the lack of the *law of excluded middle* (for a critical discussion of this terminology see Section 4.2):

$$\mathbb{L}^* = \{(x_1, x_2) \in \mathbb{I} \times \mathbb{I} \mid x_1 + x_2 \leq 1\}, \tag{5}$$

$$(x_1, x_2) \leq_{\mathbb{L}^*} (y_1, y_2) \quad \Longleftrightarrow \quad x_1 \leq y_1 \text{ AND } x_2 \geq y_2. \tag{6}$$

Obviously, $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ is a complete bounded lattice: $\mathbf{0}_{\mathbb{L}^*} = (0,1)$ and $\mathbf{1}_{\mathbb{L}^*} = (1,0)$ are the bottom and top elements of $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, respectively, and the meet $\wedge_{\mathbb{L}^*}$ and the join $\vee_{\mathbb{L}^*}$ in $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ are given by

$$(x_1, x_2) \wedge_{\mathbb{L}^*} (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2)),$$
$$(x_1, x_2) \vee_{\mathbb{L}^*} (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2)).$$

Moreover, $(\mathbb{I}, \leq)$ can be embedded in a natural way into $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$: the function $\text{emb}_{\mathbb{I}} \colon \mathbb{I} \to \mathbb{L}^*$ given by $\text{emb}_{\mathbb{I}}(x) = (x, 1-x)$ is an embedding. Observe that there are also other embeddings of $(\mathbb{I}, \leq)$ into $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, e.g., $\varphi \colon \mathbb{I} \to \mathbb{L}^*$ given by $\varphi(x) = (x, 0)$.

Note that the order $\leq_{\mathbb{L}^*}$ is not linear. However, it is possible to construct refinements of $\leq_{\mathbb{L}^*}$ which are linear [87].

Mirroring the set $\mathbb{L}^*$ about the axis passing through the points $(0, 0.5)$ and $(1, 0.5)$ of the unit square $\mathbb{I} \times \mathbb{I}$ one immediately sees that there is some other lattice which is isomorphic to $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$. Both lattices are visualized in Figure 1.

Proposition 1. *The complete bounded lattice $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ is isomorphic to the upper left triangle $L_2(\mathbb{I})$ in $\mathbb{I} \times \mathbb{I}$ (with vertexes $(0,0)$, $(0,1)$ and $(1,1)$), i.e.,*

$$L_2(\mathbb{I}) = \{(x_1, x_2) \in \mathbb{I} \times \mathbb{I} \mid 0 \leq x_1 \leq x_2 \leq 1\}, \tag{7}$$

equipped with the componentwise partial order $\leq_{\text{comp}}$, whose bottom and top elements are $\mathbf{0}_{L_2(\mathbb{I})} = (0,0)$ and $\mathbf{1}_{L_2(\mathbb{I})} = (1,1)$, respectively. A canonical isomorphism between the lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ and $(L_2(\mathbb{I}), \leq_{\text{comp}})$ is provided by the function $\varphi_{\mathbb{L}^*}^{L_2(\mathbb{I})} \colon \mathbb{L}^* \to L_2(\mathbb{I})$ defined by $\varphi_{\mathbb{L}^*}^{L_2(\mathbb{I})}((x_1, x_2)) = (x_1, 1-x_2)$.

It is readily seen that $(L_2(\mathbb{I}), \leq_{\text{comp}})$ is a sublattice of the product lattice $(\mathbb{I} \times \mathbb{I}, \leq_{\text{comp}})$, and the standard order reversing involution $N_{L_2(\mathbb{I})} \colon L_2(\mathbb{I}) \to L_2(\mathbb{I})$ is given by

$$N_{L_2(\mathbb{I})}((x,y)) = (1-y, 1-x) \tag{8}$$

(compare (4)). On the other hand, the lattice $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ is not a sublattice of $(\mathbb{I} \times \mathbb{I}, \leq_{\text{comp}})$, but it can be embedded into $(\mathbb{I} \times \mathbb{I}, \leq_{\text{comp}})$ using, e.g., the lattice monomorphism (as visualized in Figure 2)

$$\mathrm{id}_{L_2(\mathbb{I})} \circ \varphi_{\mathbb{L}^*}^{L_2(\mathbb{I})} \colon \mathbb{L}^* \longrightarrow L_2(\mathbb{I}).$$

Several other lattices "look" different when compared with $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ or seem to address a different context, but in fact they carry the same structural information as $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$.

Well-known examples of this phenomenon are the lattices $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$, providing the basis of interval-valued (or grey) fuzzy sets [4,8,9,12–14], and $(P^*, \leq_{\mathbb{L}^*})$, giving rise to the so-called "Pythagorean" fuzzy sets [15,88,89], both turning out to be isomorphic to the lattice $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$. The following statements can be verified by simply checking the required properties.

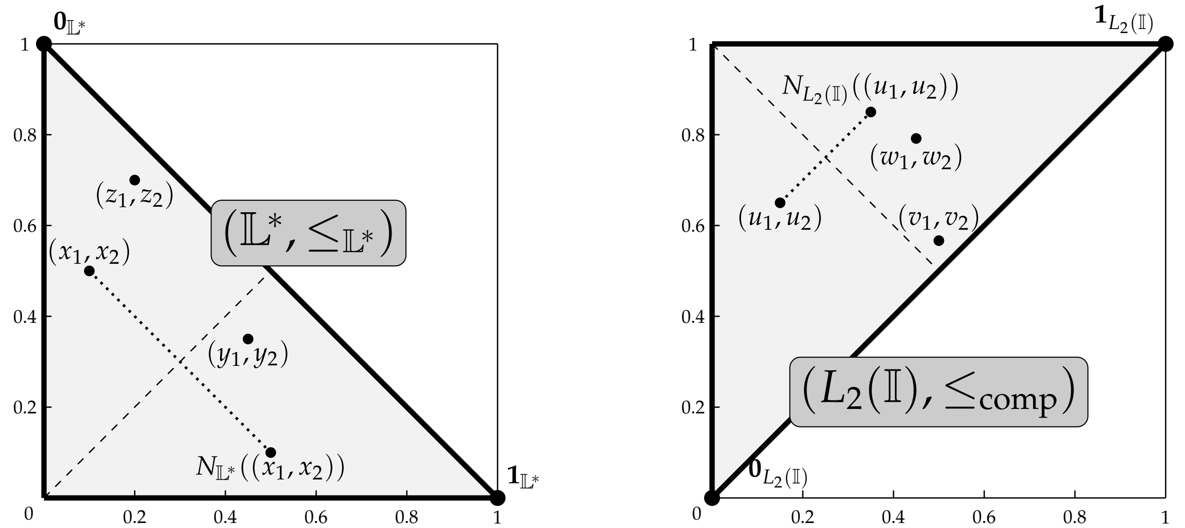

Figure 1. The lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ (**left**) and $(L_2(\mathbb{I}), \leq_{\text{comp}})$ (**right**). Note the difference of two orders: we have $(x_1, x_2) \leq_{\mathbb{L}^*} (y_1, y_2)$, $(z_1, z_2) \leq_{\mathbb{L}^*} (y_1, y_2)$, and $(u_1, u_2) \leq_{\text{comp}} (w_1, w_2)$, but (x_1, x_2) and (z_1, z_2) are not comparable in $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, and (v_1, v_2) is neither comparable to (u_1, u_2) nor to (w_1, w_2) with respect to $\leq_{\text{comp}}$. Also, a hint for the constructions of the order reversing involutions $N_{\mathbb{L}^*}$ and $N_{L_2(\mathbb{I})}$ as reflections through the appropriate diagonal (dashed line) of $\mathbb{I} \times \mathbb{I}$ is given.

Proposition 2. *The complete bounded lattice $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ is isomorphic to the following two lattices:*

(i) *to the lattice $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$ of all closed subintervals of the unit interval $\mathbb{I}$, given by*

$$\mathfrak{I}(\mathbb{I}) = \{[x_1, x_2] \subseteq \mathbb{I} \mid 0 \leq x_1 \leq x_2 \leq 1\}, \tag{9}$$

$$[x_1, x_2] \leq_{\mathfrak{I}(\mathbb{I})} [y_1, y_2] \iff x_1 \leq y_1 \text{ AND } x_2 \leq y_2, \tag{10}$$

with bottom and top elements $\mathbf{0}_{\mathfrak{I}(\mathbb{I})} = [0,0]$ and $\mathbf{1}_{\mathfrak{I}(\mathbb{I})} = [1,1]$, respectively; a canonical example of an isomorphism between $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ and $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$ is provided by the function $\varphi_{\mathbb{L}^*}^{\mathfrak{I}(\mathbb{I})} \colon \mathbb{L}^* \to \mathfrak{I}(\mathbb{I})$ defined by $\varphi_{\mathbb{L}^*}^{\mathfrak{I}(\mathbb{I})}((x_1, x_2)) = [x_1, 1 - x_2]$;

(ii) *to the lattice $(P^*, \leq_{\mathbb{L}^*})$ of all points in the intersection of the unit square $\mathbb{I} \times \mathbb{I}$ and the unit disk with center $(0,0)$, i.e.,*

$$P^* = \{(x_1, x_2) \in \mathbb{I} \times \mathbb{I} \mid x_1^2 + x_2^2 \leq 1\}; \tag{11}$$

a canonical example of a lattice isomorphism from $(P^*, \leq_{\mathbb{L}^*})$ to $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ is provided by the function $\varphi_{P^*}^{\mathbb{L}^*} \colon P^* \to \mathbb{L}^*$ defined by $\varphi_{P^*}^{\mathbb{L}^*}((x_1, x_2)) = (x_1^2, x_2^2)$.

Example 2. *Let us start with the standard order reversing involution $N_{L_2(\mathbb{I})}$ on $(L_2(\mathbb{I}), \leq_{\text{comp}})$ given by (8). The fact that $(L_2(\mathbb{I}), \leq_{\text{comp}})$ is isomorphic to each of the lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, $(P^*, \leq_{\mathbb{L}^*})$, and $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$ (see Propositions 1 and 2) and Example 1(i) allow us to construct the order reversing involutions $N_{\mathbb{L}^*} \colon \mathbb{L}^* \to \mathbb{L}^*$, $N_{P^*} \colon P^* \to P^*$, and $N_{\mathfrak{I}(\mathbb{I})} \colon \mathfrak{I}(\mathbb{I}) \to \mathfrak{I}(\mathbb{I})$ on the lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, $(P^*, \leq_{\mathbb{L}^*})$, and $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$ are given by*

$$N_{\mathbb{L}^*}((x_1, x_2)) = N_{P^*}((x_1, x_2)) = (x_2, x_1), \qquad N_{\mathfrak{I}(\mathbb{I})}([x_1, x_2]) = [1 - x_2, 1 - x_1].$$

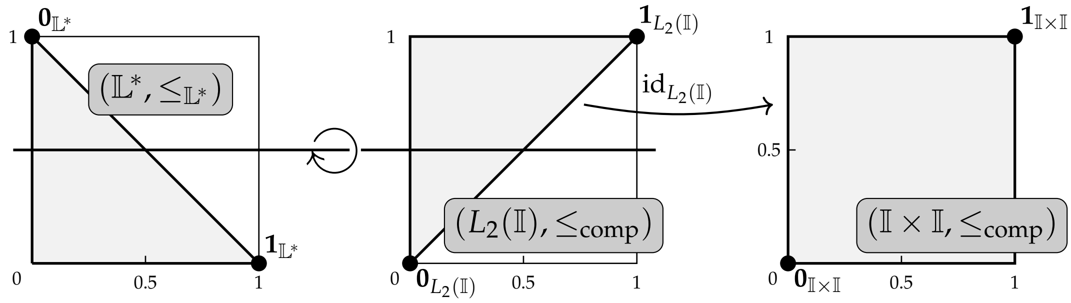

Figure 2. The lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ (**left**), $(L_2(\mathbb{I}), \leq_{\text{comp}})$ (**center**), and $(\mathbb{I} \times \mathbb{I}, \leq_{\text{comp}})$ (**right**). The mirror symmetry between $\mathbb{L}^*$ and $L_2(\mathbb{I})$ shows that $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ and $(L_2(\mathbb{I}), \leq_{\text{comp}})$ are isomorphic, and $(L_2(\mathbb{I}), \leq_{\text{comp}})$ is a sublattice of $(\mathbb{I} \times \mathbb{I}, \leq_{\text{comp}})$.

Given a universe of discourse X, i.e., a non-empty set X, and fixing a bounded lattice $(L, \leq_L)$, we obtain a special type of *L-fuzzy subsets* of X in the sense of [2] and, on the other hand, a particular case of *type-2 fuzzy sets* (also proposed by L. A. Zadeh [3,4]; see [90,91] for some algebraic aspects of truth values for type-2 fuzzy sets).

An *$\mathbb{L}^*$-fuzzy subset* A of X is characterized by its membership function $\mu_A^{\mathbb{L}^*} \colon X \to \mathbb{L}^*$, where the bounded lattice $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ is given by (5) and (6). The bottom and top elements of $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ are $\mathbf{0}_{\mathbb{L}^*} = (0, 1)$ and $\mathbf{1}_{\mathbb{L}^*} = (1, 0)$, respectively.

Over the years, different names for fuzzy sets based on the lattices that are isomorphic to $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ according to Propositions 1 and 2 were used in the literature: in the mid-seventies $\mathfrak{I}(\mathbb{I})$-fuzzy sets were called *interval-valued* in [4,12–14], in the eighties first the name *"intuitionistic"* fuzzy sets was used for $\mathbb{L}^*$-fuzzy sets in [6,7] (compare also [84–86]) and then *grey* sets in [8,9]), and even later *vague* sets in [10] (see also [10,92]). More recently, the name "Pythagorean" fuzzy sets was introduced for P^*-fuzzy sets in [15,88,89].

As a function $\mu_A^{\mathbb{L}^*} \colon X \to \mathbb{L}^* \subseteq \mathbb{I} \times \mathbb{I}$, the membership function $\mu_A^{\mathbb{L}^*}$ has two components $\mu_A, \nu_A \colon X \to \mathbb{I}$ such that for each $x \in X$ we have $\mu_A^{\mathbb{L}^*}(x) = (\mu_A(x), \nu_A(x))$ and $\mu_A(x) + \nu_A(x) \leq 1$.

Both $\mu_A \colon X \to \mathbb{I}$ and $\nu_A \colon X \to \mathbb{I}$ can be seen as membership functions of fuzzy subsets of X, say $A^+, A^- \in \mathscr{F}(X)$, respectively, i.e., for each $x \in X$ we have

$$\mu_{A^+}(x) = \mu_A(x), \quad \mu_{A^-}(x) = \nu_A(x), \quad \text{and} \quad \mu_{A^+}(x) + \mu_{A^-}(x) \leq 1. \tag{12}$$

The value $\mu_{A^+}(x)$ is usually called the *degree of membership* of the object x in the $\mathbb{L}^*$-fuzzy set A, while $\mu_{A^-}(x)$ is said to be the *degree of non-membership* of the object x in the $\mathbb{L}^*$-fuzzy set A.

Denoting the set of all *$\mathbb{L}^*$-fuzzy subsets of X* by $\mathscr{F}_{\mathbb{L}^*}(X)$ and keeping the notations from (12), for each $A \in \mathscr{F}_{\mathbb{L}^*}(X)$ and its membership function $\mu_A^{\mathbb{L}^*} \colon X \to \mathbb{L}^* \subseteq \mathbb{I} \times \mathbb{I}$ we may write

$$\mu_A^{\mathbb{L}^*} = (\mu_A, \nu_A) = (\mu_{A^+}, \mu_{A^-}).$$

As a consequence of (12), for the fuzzy sets A^+ and A^- we have $A^+ \subseteq (A^-)^{\complement}$. In other words, we can identify each $\mathbb{L}^*$-fuzzy subset $A \in \mathscr{F}_{\mathbb{L}^*}(X)$ with a pair of fuzzy sets (A^+, A^-) with $A^+ \subseteq (A^-)^{\complement}$, i.e.,

$$\mathscr{F}_{\mathbb{L}^*}(X) = \{(A^+, A^-) \in \mathscr{F}(X) \times \mathscr{F}(X) \mid A^+ \subseteq (A^-)^{\complement}\},$$

and for two $\mathbb{L}^*$-fuzzy subsets $A = (A^+, A^-)$ and $B = (B^+, B^-)$ of X the assertion $A \subseteq_{\mathbb{L}^*} B$ is equivalent to $A^+ \subseteq B^+$ and $B^- \subseteq A^-$. The complement of an $\mathbb{L}^*$-fuzzy subset $A = (A^+, A^-)$ is the $\mathbb{L}^*$-fuzzy set $A^{\complement} = (A^-, A^+)$.

Then $(\mathscr{F}_{\mathbb{L}^*}(X), \subseteq_{\mathbb{L}^*})$ is a complete bounded lattice with bottom element $\emptyset = (\emptyset, X)$ and top element $X = (X, \emptyset)$, and the lattice $(\mathscr{F}_{\mathbb{L}^*}(X), \subseteq_{\mathbb{L}^*})$ of L^*-fuzzy sets is isomorphic to $({\mathbb{L}^*}^X, \leq_{\mathbb{L}^*})$. Clearly, $(\mathscr{F}(X), \subseteq)$ can be embedded into $(\mathscr{F}_{\mathbb{L}^*}(X), \subseteq_{\mathbb{L}^*})$: a natural embedding is provided by the function $\mathrm{emb}_{\mathscr{F}(X)} \colon \mathscr{F}(X) \to \mathscr{F}_{\mathbb{L}^*}(X)$ defined by $\mathrm{emb}_{\mathscr{F}(X)}(A) = (A, A^{\complement})$.

An *interval-valued fuzzy subset* A of the universe X (introduced independently in [4,12–14], some authors called them *grey* sets [8,9]) is characterized by its membership function $\mu_A \colon X \to \mathfrak{I}(\mathbb{I})$, where $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$ is the bounded lattice of all closed subintervals of the unit interval $\mathbb{I}$ given by (9) and (10). The bottom and top elements of $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$ are then$\mathbf{0}_{\mathfrak{I}(\mathbb{I})} = [0, 0]$ and $\mathbf{1}_{\mathfrak{I}(\mathbb{I})} = [1, 1]$, respectively.

A *"Pythagorean" fuzzy subset* A of the universe X (first considered in [15,88,89]) is characterized by its membership function $\mu_A \colon X \to P^*$, where the bounded lattice $(P^*, \leq_{\mathbb{L}^*})$ is given by (11) and (6). The bottom and top elements of $(P^*, \leq_{\mathbb{L}^*})$ are the same as in $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, i.e., we have $\mathbf{0}_{P^*} = (0, 1)$ and $\mathbf{1}_{P^*} = (1, 0)$.

From Propositions 1 and 2 we know that the four bounded lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$, $(P^*, \leq_{\mathbb{L}^*})$, and $(L_2(\mathbb{I}), \leq_{\mathrm{comp}})$ are isomorphic to each other. As an immediate consequence we obtain the following result.

Proposition 3. *Let X be a universe of discourse. Then we have:*

(i) *The product lattices $((\mathbb{L}^*)^X, \leq_{\mathbb{L}^*})$, $((\mathfrak{I}(\mathbb{I}))^X, \leq_{\mathfrak{I}(\mathbb{I})})$, $((P^*)^X, \leq_{\mathbb{L}^*})$, and $((L_2(\mathbb{I}))^X, \leq_{\mathrm{comp}})$ are isomorphic to each other.*

(ii) *The lattices of all $\mathbb{L}^*$-fuzzy subsets of X, of all "intuitionistic" fuzzy subsets of X, of all interval-valued fuzzy subsets of X, of all "Pythagorean" fuzzy subsets of X, and of all $L_2(\mathbb{I})$-fuzzy subsets of X are isomorphic to each other.*

This means that, mathematically speaking, all the function spaces mentioned in Proposition 3(i) and all the "different" classes of fuzzy subsets of X referred to in Proposition 3(ii) share an identical (lattice) structure. Any differences between them only come from the names used for individual objects, and from the interpretation or meaning of these objects. In other words, since any mathematical result for one of these lattices immediately can be carried over to all isomorphic lattices, in most cases there is no need to use different names for them.

3.3. Generalizations to Higher Dimensions

As a straightforward generalization of the product lattice $(\mathbb{I} \times \mathbb{I}, \leq_{\mathrm{comp}})$, for each $n \in \mathbb{N}$ the *n-dimensional unit cube* $(\mathbb{I}^n, \leq_{\mathrm{comp}})$, i.e., the n-dimensional product of the lattice $(\mathbb{I}, \leq)$, can be defined by means of (1) and (2).

The so-called *"neutrosophic"* sets introduced by F. Smarandache [93] (see also [94–97] are based on the bounded lattices $(\mathbb{I}^3, \leq_{\mathbb{I}^3})$ and $(\mathbb{I}^3, \leq^{\mathbb{I}^3})$, where the orders $\leq_{\mathbb{I}^3}$ and $\leq^{\mathbb{I}^3}$ on the unit cube $\mathbb{I}^3$ are defined by

$$(x_1, x_2, x_3) \leq_{\mathbb{I}^3} (y_1, y_2, y_3) \iff x_1 \leq y_1 \text{ AND } x_2 \leq y_2 \text{ AND } x_3 \geq y_3, \tag{13}$$

$$(x_1, x_2, x_3) \leq^{\mathbb{I}^3} (y_1, y_2, y_3) \iff x_1 \leq y_1 \text{ AND } x_2 \geq y_2 \text{ AND } x_3 \geq y_3. \tag{14}$$

Observe that $\leq_{\mathbb{I}^3}$ is a variant of the order $\leq_{\mathrm{comp}}$: it is defined componentwise, but in the third component the order is reversed. The top element of $(\mathbb{I}^3, \leq_{\mathbb{I}^3})$ is $(1, 1, 0)$, and $(0, 0, 1)$ is its bottom element. Analogous assertions are true for the lattice $(\mathbb{I}^3, \leq^{\mathbb{I}^3})$.

Clearly, the three lattices $(\mathbb{I}^3, \leq_{\mathrm{comp}})$, $(\mathbb{I}^3, \leq_{\mathbb{I}^3})$, and $(\mathbb{I}^3, \leq^{\mathbb{I}^3})$ are mutually isomorphic: the functions $\varphi, \psi \colon \mathbb{I}^3 \to \mathbb{I}^3$ given by $\varphi((x_1, x_2, x_3)) = (x_1, x_2, 1 - x_3)$ and $\psi((x_1, x_2, x_3)) = (x_1, 1 - x_2, x_3)$ are canonical isomorphisms between $(\mathbb{I}^3, \leq_{\text{comp}})$ and $(\mathbb{I}^3, \leq_{\mathbb{I}^3})$, on the one hand, and between $(\mathbb{I}^3, \leq_{\mathbb{I}^3})$ and $(\mathbb{I}^3, \leq^{\mathbb{I}^3})$, on the other hand.

For each $n \in \mathbb{N}$ the *n-fuzzy sets* introduced by B. Bedregal et al. in [11] (see also [98,99]) are based on the bounded lattice $(L_n(\mathbb{I}), \leq_{\text{comp}})$, where the set $L_n(\mathbb{I})$ is a straightforward generalization of $L_2(\mathbb{I})$ defined in (7):

$$L_n(\mathbb{I}) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{I}^n \mid x_1 \leq x_2 \leq \cdots \leq x_n\}. \tag{15}$$

The order $\leq_{\text{comp}}$ on $L_n(\mathbb{I})$ coincides with the restriction of the componentwise order $\leq_{\text{comp}}$ on $\mathbb{I}^n$ to $L_n(\mathbb{I})$, implying that $(L_n(\mathbb{I}), \leq_{\text{comp}})$ is a sublattice of the product lattice $(\mathbb{I}^n, \leq_{\text{comp}})$. As a consequence, we also have the standard order reversing involution $N_{L_n(\mathbb{I})} \colon L_n(\mathbb{I}) \to L_n(\mathbb{I})$ which is defined coordinatewise, i.e., $N_{L_n(\mathbb{I})}((x_1, x_2, \ldots, x_n)) = (1 - x_n, \ldots, 1 - x_2, 1 - x_1)$ (compare (8)). Considering, for $n > 3$, lattices which are isomorphic to $(L_n(\mathbb{I}), \leq_{\text{comp}})$, further generalizations of "neuthrosophic" sets can be introduced.

B. C. Cuong and V. Kreinovich [16] proposed the concept of so-called *picture fuzzy sets* which are based on the set $\mathbb{D}^* \subseteq \mathbb{I}^3$ of truth values given by

$$\mathbb{D}^* = \{(x_1, x_2, x_3) \in \mathbb{I}^3 \mid x_1 + x_2 + x_3 \leq 1\}. \tag{16}$$

The motivation for the set $\mathbb{D}^*$ came from a simple voting scenario where each voter can act in one of the four following ways: to vote for the nominated candidate (the proportion of these voters being equal to x_1), to vote against the candidate (described by x_2), to have no preference and to abstain so this vote will not be counted (described by x_3), or to be absent (described by $1 - x_1 - x_2 - x_3$).

In the original proposal [16] the set $\mathbb{D}^*$ was equipped with the partial order $\leq_{\mathbb{I}^3}$ given by (13), as inherited from the lattice $(\mathbb{I}^3, \leq_{\mathbb{I}^3})$. As $\{(x_1, x_3) \in \mathbb{I} \times \mathbb{I} \mid (x_1, 0, x_3) \in \mathbb{D}^*\} = \mathbb{L}^*$ and (6), we may also write $(x_1, x_2, x_3) \leq_{\mathbb{I}^3} (y_1, y_2, y_3)$ if and only if $(x_1, x_3) \leq_{\mathbb{L}^*} (y_1, y_3)$ and $x_2 \leq y_2$. However, $(\mathbb{D}^*, \leq_{\mathbb{I}^3})$ is not a lattice, but only a meet-semilattice with bottom element $\mathbf{0}_{\mathbb{D}^*} = (0, 0, 1)$; indeed, the set $\{(1, 0, 0), (0, 1, 0)\}$ has no join in $\mathbb{D}^*$ with respect to $\leq_{\mathrm{P}}$ (to be more precise, the semi-lattice $(\mathbb{D}^*, \leq_{\mathrm{P}})$ has infinitely many pairwise incomparable maximal elements of the form $(a, 1 - a, 0)$ with $a \in \mathbb{I}$).

Therefore, (without modifications) it is impossible [100] to introduce logical operations such as t-norms or t-conorms [69] and, in general, aggregation functions [46] on $(\mathbb{D}^*, \leq_{\mathbb{I}^3})$.

As a consequence, the order $\leq_{\mathbb{I}^3}$ on $\mathbb{D}^*$ was replaced by the following partial order $\leq_{\mathbb{D}^*}$ on $\mathbb{D}^*$ (compare [16,100–102]) which is a refinement of $\leq_{\mathbb{I}^3}$:

$$\begin{aligned} &(x_1, x_2, x_3) \leq_{\mathbb{D}^*} (y_1, y_2, y_3) \\ &\qquad \iff \quad (x_1, x_3) <_{\mathbb{L}^*} (y_1, y_3) \ \text{OR} \ \Big((x_1, x_3) = (y_1, y_3) \ \text{AND} \ x_2 \leq y_2\Big). \end{aligned} \tag{17}$$

Note that the order $\leq_{\mathbb{D}^*}$ can be seen as a kind of lexicographical order related to two orders: to the order $\leq_{\mathbb{L}^*}$ on $\mathbb{L}^*$ and to the standard order $\leq$ on $\mathbb{I}$.

It is easy to see that $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$ is a bounded lattice with bottom element $\mathbf{0}_{\mathbb{D}^*} = (0, 0, 1)$ and top element $\mathbf{1}_{\mathbb{D}^*} = (1, 0, 0)$. This allows aggregation functions (as studied on the unit interval $\mathbb{I}$ in, e.g., [46,47,76–78]) to be introduced on $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$. Observe also that the lattice $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$ was considered in recent applications of picture fuzzy sets [103,104].

We only recall [105] that the join $\vee_{\leq_{\mathbb{D}^*}}$ and the meet $\wedge_{\leq_{\mathbb{D}^*}}$ in the lattice $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$ are given by

$$(x_1, x_2, x_3) \vee_{\leq_{\mathbb{D}^*}} (y_1, y_2, y_3) = \begin{cases} (x_1, x_2, x_3) & \text{if } (x_1, x_2, x_3) \geq_{\mathbb{D}^*} (y_1, y_2, y_3), \\ (y_1, y_2, y_3) & \text{if } (x_1, x_2, x_3) \leq_{\mathbb{D}^*} (y_1, y_2, y_3), \\ (\max(x_1, y_1), 0, \min(x_3, y_3)) & \text{otherwise,} \end{cases}$$

$$(x_1, x_2, x_3) \wedge_{\leq_{\mathbb{D}^*}} (y_1, y_2, y_3) = \begin{cases} (x_1, x_2, x_3) & \text{if } (x_1, x_2, x_3) \leq_{\mathbb{D}^*} (y_1, y_2, y_3), \\ (y_1, y_2, y_3) & \text{if } (x_1, x_2, x_3) \geq_{\mathbb{D}^*} (y_1, y_2, y_3), \\ (\min(x_1, y_1), 1 - \min(x_1, y_1) & \\ \quad - \max(x_3, y_3), \max(x_3, y_3)) & \text{otherwise,} \end{cases}$$

and the standard order reversing involution $N_{\mathbb{D}^*} : \mathbb{D}^* \to \mathbb{D}^*$ by $N_{\mathbb{D}^*}((x_1, x_2, x_3)) = (x_3, x_2, x_1)$.

From the definition of $\leq_{\mathbb{D}^*}$ in (17) it is obvious that $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ can be embedded in a natural way into $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$; an example of an embedding is given by

$$\begin{aligned} \mathrm{emb}_{\mathbb{L}^*} \colon \quad \mathbb{L}^* &\longrightarrow \mathbb{D}^* \\ (x_1, x_2) &\longmapsto (x_1, 0, x_2). \end{aligned} \tag{18}$$

Let us now have a look at the relationship between the lattice $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$ and the lattice $(L_3(\mathbb{I}), \leq_{\mathrm{comp}})$ given by (2) and (15). It is not difficult to see that the function $\psi \colon \mathbb{D}^* \to L_3(\mathbb{I})$ given by $\psi((x_1, x_2, x_3)) = (x_1, x_1 + x_2, 1 - x_3)$ is a bijection, its inverse $\psi^{-1} \colon L_3(\mathbb{I}) \to \mathbb{D}^*$ being given by $\psi^{-1}((x_1, x_2, x_3)) = (x_1, x_2 - x_1, 1 - x_3)$.

Observe that that the bijection ψ is not order preserving: we have $(0.2, 0.5, 0) \leq_{\mathbb{D}^*} (0.3, 0, 0)$, but $\psi((0.2, 0.5, 0)) = (0.2, 0.7, 1)$ and $\psi((0.2, 0.5, 0)) = (0.3, 0.3, 1)$ are incomparable with respect to $\leq_{\mathrm{comp}}$.

From ([105], Propositions 1 and 2) we have the following result:

Proposition 4. *The lattices* $(L_3(\mathbb{I}), \leq_{\mathrm{comp}})$ *and* $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$ *are not isomorphic. However, we have*

(i) *The lattice* $(L_3(\mathbb{I}), \leq_{\mathrm{comp}})$ *is isomorphic to the lattice* $(\mathbb{D}^*, \leq_{\mathbb{D}_3^*})$ *with top element* $(1, 0, 0)$ *and bottom element* $(0, 0, 1)$*, where the order* $\leq_{\mathbb{D}_3^*}$ *is given by*

$$(x_1, x_2, x_3) \leq_{\mathbb{D}_3^*} (y_1, y_2, y_3) \quad \Longleftrightarrow \quad x_1 \leq y_1 \text{ AND } x_1 + x_2 \leq y_1 + y_2 \text{ AND } x_3 \geq y_3.$$

(ii) *The lattice* $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$ *is isomorphic to the lattice* $(L_3(\mathbb{I}), \leq_{L_3(\mathbb{I})})$ *with top element* $(1, 1, 1)$ *and bottom element* $(0, 0, 0)$*, where the order* $\leq_{L_3(\mathbb{I})}$ *is given by*

$$\begin{aligned} &(x_1, x_2, x_3) \leq_{L_3(\mathbb{I})} (y_1, y_2, y_3) \\ &\qquad \Longleftrightarrow \quad (x_1, x_3) <_{\mathrm{comp}} (y_1, y_3) \text{ OR } ((x_1, x_3) = (y_1, y_3) \text{ AND } x_2 - x_1 \leq y_2 - y_1). \end{aligned}$$

In summary, if a universe of discourse X is fixed, then a *picture* fuzzy subset A of X is based on the bounded lattice $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$ defined in (16) and (17). It is characterized by its membership function $\mu_A^{\mathbb{D}^*} \colon X \to \mathbb{D}^*$ [16,100,106–109] where $\mu_A^{\mathbb{D}^*}(x) = (\mu_{A_1}(x), \mu_{A_2}(x), \mu_{A_3}(x)) \in \mathbb{D}^*$ for some functions $\mu_{A_1}, \mu_{A_2}, \mu_{A_3} \colon X \to \mathbb{I}$.

Clearly, the function $\mu_{A_1}\colon X \to \mathbb{I}$ can be interpreted as the membership function of the fuzzy set $A_1 \in \mathscr{F}(X)$ and, analogously, $\mu_{A_2}\colon X \to \mathbb{I}$ and $\mu_{A_3}\colon X \to \mathbb{I}$ as membership functions of the fuzzy sets A_2 and A_3, respectively. In other words, for each picture fuzzy set A we may write $A = (A_1, A_2, A_3)$.

In this context, the values $\mu_{A_1}(x)$, $\mu_{A_2}(x)$ and $\mu_{A_3}(x)$ are called the *degree of positive membership*, the *degree of neutral membership*, and the *degree of negative membership* of the object x in the picture fuzzy set A, respectively. The value $1 - (\mu_{A_1}(x) + \mu_{A_2}(x) + \mu_{A_3}(x)) \in \mathbb{I}$ is called the *degree of refusal membership* of the object x in A.

If X is a fixed universe of discourse, then we denote the set of all picture fuzzy subsets of X by $\mathscr{F}_{\mathbb{D}^*}(X)$. Obviously, for two picture fuzzy sets $A, B \in \mathscr{F}_{\mathbb{D}^*}(X)$ the assertion $A \subseteq_{\mathbb{D}^*} B$ is equivalent to $(\mu_{A_1}, \mu_{A_2}, \mu_{A_3}) \leq_{\mathbb{D}^*} (\mu_{B_1}, \mu_{B_2}, \mu_{B_3})$, i.e., $(\alpha_A(x), \beta_A(x), \gamma_A(x)) \leq_{\mathbb{D}^*} (\alpha_B(x), \beta_B(x), \gamma_B(x))$ for all $x \in X$, and the membership function of the complement $A^{\complement}$ of a picture fuzzy set $A \in \mathscr{F}_{\mathbb{D}^*}(X)$ with membership function $\mu_A^{\mathbb{D}^*} = (\mu_{A_1}, \mu_{A_2}, \mu_{A_3})$ is given by $\mu_{A^{\complement}}^{\mathbb{D}^*} = (\mu_{A_3}, \mu_{A_2}, \mu_{A_1})$.

This means that $(\mathscr{F}_{\mathbb{D}^*}(X)), \subseteq_{\mathbb{D}^*})$ is a bounded lattice with bottom element $\varnothing = (\varnothing, \varnothing, X)$ and top element $X = (X, \varnothing, \varnothing)$, and it is isomorphic to the product lattice $((\mathbb{D}^*)^X, \leq_{\text{comp}})$ of all functions from X to $\mathbb{D}^*$ (clearly, $\mu_A^{\mathbb{D}^*} \leq_{\text{comp}} \mu_B^{\mathbb{D}^*}$ means here $\mu_A^{\mathbb{D}^*}(x) \leq_{\mathbb{D}^*} \mu_B^{\mathbb{D}^*}(x)$ for all $x \in X$).

As a consequence, the lattice $(\mathscr{F}_{\mathbb{L}^*}(X)), \subseteq_{\mathbb{L}^*})$ of all $\mathbb{L}^*$-fuzzy subsets of X can be embedded into the lattice $(\mathscr{F}_{\mathbb{D}^*}(X)), \subseteq_{\mathbb{D}^*})$ of all picture fuzzy subsets of X via

$$\begin{aligned} \text{emb}_{\mathscr{F}_{\mathbb{L}^*}(X)}\colon \quad \mathscr{F}_{\mathbb{L}^*}(X) &\longrightarrow \mathscr{F}_{\mathbb{D}^*}(X) \\ (A^+, A^-) &\longmapsto (A^+, \varnothing, A^-), \end{aligned}$$

and, using the embedding $\text{emb}_{\mathbb{L}^*}\colon \mathbb{L}^* \to \mathbb{D}^*$ defined in (18), the product lattice $((\mathbb{L}^*)^X, \leq_{\text{comp}})$ can be embedded into the product lattice $((\mathbb{D}^*)^X, \leq_{\text{comp}})$.

We recognize a chain of subsets of X of increasing generality and complexity: crisp sets $\mathscr{P}(X)$, fuzzy sets $\mathscr{F}(X)$, $\mathbb{L}^*$-fuzzy sets $\mathscr{F}_{\mathbb{L}^*}(X)$, and picture fuzzy sets $\mathscr{F}_{\mathbb{D}^*}(X)$. This corresponds to the increasing complexity and dimensionality of the lattices of truth values $(\mathbf{2}, \leq)$, $(\mathbb{I}, \leq)$, $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, and $(\mathbb{D}^*, \leq_{\mathbb{D}^*})$. The commutative diagram in Figure 3 visualizes the relationship between these types of (fuzzy) sets and their respective membership functions, and also of the corresponding lattices of truth values.

The content of this subsection also makes clear that the situation in the case of three-dimensional sets of truth values is much more complex than for the two-dimensional truth values considered before.

In Proposition 3, we have seen that several classes of fuzzy sets with two-dimensional sets of truth values are isomorphic to each other, while, in the case of three-dimensional truth values, we have given a number of lattices of truth values that are not isomorphic to each other.

Obviously, continuing in the series of generalizations from $\mathbb{I}$ over $\mathbb{L}^*$ to $\mathbb{D}^*$, for any arity $n \in \mathbb{N}$ one can define a carrier

$$\mathbb{D}_n^* = \left\{ (x_1, ..., x_n) \in \mathbb{I}^n \;\middle|\; \sum_{i=1}^{n} x_i \leq 1 \right\}$$

and equip it with some order $\preceq$ such that $(\mathbb{D}_n^*, \preceq)$ is a bounded lattice with top element $(1, 0, ..., 0)$ and bottom element $(0, ..., 0, 1)$. The problematic question is whether such a generalization is meaningful and can be used to model some real problem.

If the arrow $\hookrightarrow$ indicates an embedding, $\twoheadrightarrow$ an epimorphism, and $\hookrightarrow\!\!\!\!\!\twoheadrightarrow$ an isomorphism, and if the homomorphisms are defined by

$$
\begin{array}{ll}
\mathrm{emb}_{\mathbb{I}}(x) = (x, 1-x), & \mathrm{con}_{\mathbb{L}^*}((\alpha_1, \alpha_2)) = (\alpha_1 \cdot \mathbf{1}_X, \alpha_2 \cdot \mathbf{1}_X), \\
\mathrm{emb}_{\mathbb{L}^*}((x_1, x_2)) = (x_1, 0, x_2), & \mathrm{con}_{\mathbb{D}^*}((\alpha_1, \alpha_2, \alpha_3)) = (\alpha_1 \cdot \mathbf{1}_X, \alpha_2 \cdot \mathbf{1}_X, \alpha_3 \cdot \mathbf{1}_X), \\
\mathrm{emb}_{\mathbb{I}^X}(f) = (f, 1-f), & \mathrm{emb}_{(\mathbb{L}^*)^X}((\mu_{A^+}, \mu_{A^-})) = (\mu_{A^+}, \mathbf{1}_{\emptyset}, \mu_{A^-}), \\
\mathrm{emb}_{\mathscr{F}(X)}(A) = (A, \complement A), & \mathrm{emb}_{\mathscr{F}_{\mathbb{L}^*}(X)}((A^+, A^-)) = (A^+, \emptyset, A^-), \\
\pi_\alpha(f) = f(\alpha), & \mathrm{mem}_{\mathbb{I}}(A) = \mu_A, \\
\mathrm{ind}(A) = \mathbf{1}_A, & \mathrm{mem}_{\mathbb{L}^*}(A) = (\mathrm{mem}_{\mathbb{I}}(A^+), \mathrm{mem}_{\mathbb{I}}(A^-)), \\
\mathrm{con}(\alpha) = \alpha \cdot \mathbf{1}_X, & \mathrm{mem}_{\mathbb{D}^*}(A) = (\mathrm{mem}_{\mathbb{I}}(A^+), \mathrm{mem}_{\mathbb{I}}(A^{(n)}), \mathrm{mem}_{\mathbb{I}}(A^-)),
\end{array}
$$

then we obtain the following commutative diagram:

$$
\begin{tikzcd}
\mathbf{2} \arrow[r, hook, "\mathrm{id}_{\mathbf{2}}"] \arrow[d, hook, shift left, "\mathrm{con}"] & \mathbb{I} \arrow[r, hook, "\mathrm{emb}_{\mathbb{I}}"] \arrow[d, hook, shift left, "\mathrm{con}"] & \mathbb{L}^* \arrow[r, hook, "\mathrm{emb}_{\mathbb{L}^*}"] \arrow[d, hook, shift left, "\mathrm{con}_{\mathbb{L}^*}"] & \mathbb{D}^* \arrow[d, hook, shift left, "\mathrm{con}_{\mathbb{D}^*}"] \\
\mathbf{2}^X \arrow[u, two heads, shift left, "\pi_\alpha"] \arrow[r, hook, "\mathrm{id}_{\mathbf{2}^X}"] & \mathbb{I}^X \arrow[u, two heads, shift left, "\pi_\alpha"] \arrow[r, hook, "\mathrm{emb}_{\mathbb{I}^X}"] & (\mathbb{L}^*)^X \arrow[u, two heads, shift left, "\pi_x"] \arrow[r, hook, "\mathrm{emb}_{(\mathbb{L}^*)^X}"] & (\mathbb{D}^*)^X \arrow[u, two heads, shift left, "\pi_x"] \\
\mathscr{P}(X) \arrow[u, hook, two heads, "\mathrm{ind}"] \arrow[r, hook, "\mathrm{id}_{\mathscr{P}(X)}"'] & \mathscr{F}(X) \arrow[u, hook, two heads, "\mathrm{mem}_{\mathbb{I}}"] \arrow[r, hook, "\mathrm{emb}_{\mathscr{F}(X)}"'] & \mathscr{F}_{\mathbb{L}^*}(X) \arrow[u, hook, two heads, "\mathrm{mem}_{\mathbb{L}^*}"'] \arrow[r, hook, "\mathrm{emb}_{\mathscr{F}_{\mathbb{L}^*}(X)}"'] & \mathscr{F}_{\mathbb{D}^*}(X) \arrow[u, hook, two heads, "\mathrm{mem}_{\mathbb{D}^*}"']
\end{tikzcd}
$$

Figure 3. Crisp sets, fuzzy sets, $\mathbb{L}^*$-fuzzy sets, and picture fuzzy sets, and the corresponding sets of truth values.

4. Discussion: Isomorphisms and Questionable Notations

In this section, we first mention some further consequences of isomorphic lattices for the construction of logical and other connectives, and then we argue why, in our opinion, notations like "intuitionistic" fuzzy sets and "Pythagorean" fuzzy sets are questionable and why it would be better to avoid them.

4.1. Isomorphic Lattices: More Consequences

From Propositions 1 and 2, we know that the bounded lattice $(L_2(\mathbb{I}), \leq_{\text{comp}})$ is isomorphic to each of the lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$, and $(P^*, \leq_{\mathbb{L}^*})$.

Many results for and constructions of operations on the lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$, and $(P^*, \leq_{\mathbb{L}^*})$, and, subsequently, for $\mathbb{L}^*$-fuzzy sets ("intuitionistic" fuzzy sets), interval-valued fuzzy sets, and "Pythagorean" fuzzy sets are a consequence of a rather general result for operations on the lattice $(L_2(\mathbb{I}), \leq_{\text{comp}})$ and, because of the isomorphisms given in Propositions 1 and 2, they automatically can be carried over to the isomorphic lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$, and $(P^*, \leq_{\mathbb{L}^*})$.

The following result makes use of the fact that $(L_2(\mathbb{I}), \leq_{\text{comp}})$ is a sublattice of the product lattice $(\mathbb{I} \times \mathbb{I}, \leq_{\text{comp}})$ and is based on [63]. It can be verified in a straightforward way by checking the required properties:

Proposition 5. *Let $F_1, F_2 \colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ be two functions such that $F_1 \leq F_2$, i.e., for each $(x_1, x_2) \in \mathbb{I} \times \mathbb{I}$ we have $F_1((x_1, x_2)) \leq F_2((x_1, x_2))$, and consider the function $F \colon L_2(\mathbb{I}) \times L_2(\mathbb{I}) \to L_2(\mathbb{I})$ given by*

$$F((x_1, x_2), (y_1, y_2)) = (F_1((x_1, y_1)), F_2((x_2, y_2))).$$

Then we have:

(i) *if F_1 and F_2 are two binary aggregation functions then the function F is a binary aggregation function on $L_2(\mathbb{I})$;*
(ii) *if F_1 and F_2 are two triangular norms then the function F is a triangular norm on $L_2(\mathbb{I})$;*
(iii) *if F_1 and F_2 are two triangular conorms then the function F is a triangular conorm on $L_2(\mathbb{I})$;*
(iv) *if F_1 and F_2 are two uninorms then the function F is a uninorm on $L_2(\mathbb{I})$;*
(v) *if F_1 and F_2 are two nullnorms then the function F is a nullnorm on $L_2(\mathbb{I})$.*

Not all t-(co)norms, uninorms and nullnorms on the lattice $(L_2(\mathbb{I}), \leq_{\text{comp}}$ can be obtained by means of Proposition 5, as the following example shows (see [110], Theorem 5):

Example 3. *Let $T \colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ be a t-norm on the unit interval $\mathbb{I}$. Then, for each $\alpha \in \mathbb{I} \setminus \{1\}$, the function $T_\alpha \colon L_2(\mathbb{I}) \times L_2(\mathbb{I}) \to L_2(\mathbb{I})$ defined by*

$$T_\alpha((x_1, x_2), (y_1, y_2)) = (T((x_1, x_2)), \max(T(\alpha, T((y_1, y_2))), T((x_1, y_2)), T((y_1, x_2))))$$

is a t-norm on $L_2(\mathbb{I})$ which cannot be obtained applying Proposition 5.

The characterization of those connectives on $L_2(\mathbb{I}), \leq_{\text{comp}})$ is an interesting problem that has been investigated in several papers (e.g., in [110–120]). Again, each of these results is automatically valid for connectives on the isomorphic lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, $(\mathfrak{J}(\mathbb{I}), \leq_{\mathfrak{J}(\mathbb{I})})$, and $(P^*, \leq_{\mathbb{L}^*})$.

The result of Proposition 5(i) can be carried over to the n-dimensional case in a straightforward way:

Corollary 1. *Let $\mathbf{A}_1, \mathbf{A}_2 \colon \mathbb{I}^n \to \mathbb{I}$ be two n-ary aggregation functions such that $\mathbf{A}_1 \leq \mathbf{A}_2$, i.e., for each $(x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ we have $\mathbf{A}_1((x_1, x_2, \ldots, x_n)) \leq \mathbf{A}_2((x_1, x_2, \ldots, x_n))$. Then also the function $\mathbf{A} \colon (L_2(\mathbb{I}))^n \to L_2(\mathbb{I})$ given by*

$$\mathbf{A}((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = (\mathbf{A}_1(x_1, x_2, \ldots, x_n), \mathbf{A}_2(y_1, y_2, \ldots, y_n))$$

is an n-ary aggregation function on $L_2(\mathbb{I})$.

4.2. The Case of "Intuitionistic" Fuzzy Sets

As already mentioned, $\mathbb{L}^*$-fuzzy sets have been called "intuitionistic" fuzzy sets in [6,7,84] and in a number of other papers (e.g., in [86,92,116,117,121–147]). In ([7], p. 87) K. T. Atanassov points out

> [...] the logical law of the excluded middle is not valid, similarly to the case in intuitionistic mathematics. Herein emerges the name of that set. [...]

Looking at Zadeh's first paper on fuzzy sets [1] one readily sees that the elements of $\mathscr{F}(X)$ also violate the law of the excluded middle if the unit interval $\mathbb{I}$ is equipped with the standard order reversing involution and if the t-norm min and the t-conorm max are used to model intersection and union of elements of $\mathscr{F}(X)$, respectively. In other words, the violation of the law of the excluded middle is no specific feature of the $\mathbb{L}^*$-fuzzy sets.

A short look at the history of mathematics and logic at the beginning of the 20th century shows that the philosophy of *intuitionism* goes back to the work of the Dutch mathematician L. E. J. Brouwer who suggested and discussed (for the first time 1912 in his inaugural address at the University of Amsterdam [148]) a foundation of mathematics independent of the law of excluded middle (see also [149–157]), a proposal eventually leading to a major controversy with the German mathematician D. Hilbert [158–160] (compare also [161]).

There are only a few papers (most remarkably, those by G. Takeuti and S. Titani [162,163]) where the original concept of intuitionistic logic was properly extended to the fuzzy case (see also [164–168])—here the use of the term "intuitionistic" fuzzy set is fully justified (see [169]).

As a consequence, the use of the name "intuitionistic" fuzzy sets in [6,7,84] and in a number of other papers in the same spirit has been criticized (mainly in [169–172]—compare Atanassov's reply [173] where he defended his original naming) because of its lack of relationship with the original concept of intuitionism and intuitionistic logic.

Here are the main arguments against using the term "intuitionistic" fuzzy sets in the context of $\mathbb{L}^*$-fuzzy sets, as given in [169]:

- the mere fact that the law of the excluded middle is violated in the case of $\mathbb{L}^*$-fuzzy sets does not justify to call them "intuitionistic" (also the fuzzy sets in the sense of [1] do not satisfy the law of the excluded middle, in general); moreover (see [53,170,174,175]), the use of an order reversing involution for $\mathbb{L}^*$-fuzzy sets contradicts intuitionistic logic [176]:

  > [...] the connectives of IFS theory violate properties of intuitionistic logic by validating the double negation (involution) axiom [...], which is not valid in intuitionistic logic. (Recall that axioms of intuitionistic logic extended by the axiom of double negation imply classical logic, and thus imply excluded middle [...]

- intuitionistic logic has a close relationship to constructivism:

  > [...] the philosophical ideas behind intuitionism in general, and intuitionistic mathematics and intuitionistic logic in particular have a strong tendency toward constructivist points of view. There are no relationship between these ideas and the basic intuitive ideas of IFS theory [...]

The redundancy of the names "intuitionistic" fuzzy sets, "$\mathbb{L}^*$-fuzzy sets" and "interval-valued fuzzy sets" is also mentioned by J. Gutiérrez García and S. E. Rodabaugh in the abstract of [172]:

> ... (1) the term "intuitionistic" in these contexts is historically inappropriate given the standard mathematical usage of "intuitionistic"; and (2), at *every* level of existence—powerset level, topological fibre level, categorical level—interval-valued sets, [...], and "intuitionistic" fuzzy sets [...] are *redundant* ...

Also in a more recent paper by H. Bustince et al. ([5], p. 189) one can find an extensive discussion of the "terminological problem with the name intuitionistic", and the correctness of the notion chosen in [162,163] is explicitly acknowledged.

To summarize, the name "intuitionistic" in the context of $\mathbb{L}^*$-fuzzy sets is not compatible with the meaning of this term in the history of mathematics, and it would be better to avoid it.

Instead, because of the isomorphism between the lattice $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ and the lattice $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$ of all closed subintervals of the unit interval $\mathbb{I}$, it is only a matter of personal taste and of the meaning given to the corresponding fuzzy sets to use one of the terms "$\mathbb{L}^*$-fuzzy sets" or "interval-valued fuzzy sets".

4.3. The Case of "Pythagorean" Fuzzy Sets

From Propositions 1 and 2 we know that the lattice $(P^*, \leq_{\mathbb{L}^*})$ given by (6) and (11) is isomorphic to each of the lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, $(L_2(\mathbb{I}), \leq_{\text{comp}})$, and $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$.

Recently, in [15,88,89] the term "Pythagorean" fuzzy set was coined and used, which turns out to be a special case of an L-fuzzy set in the sense of [2], to be precise, an L-fuzzy set with P^* as lattice of truth values.

No justification for the choice of the adjective "Pythagorean" in this context was offered. One may only guess that the fact that, in the definition of the set P^* in (11), a sum of two squares occurs, indicating some similarity with the famous formula $a^2 + b^2 = c^2$ for right triangles—usually attributed to the Greek philosopher and mathematician Pythagoras who lived in the sixth century B.C.

The mutual isomorphism between the lattices $(P^*, \leq_{\mathbb{L}^*})$, $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$, $(L_2(\mathbb{I}), \leq_{\text{comp}})$, and $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$ implies that the families of L-fuzzy sets based on these lattices of truth values as well as the families of their corresponding membership functions are also isomorphic, i.e., have the same mathematical structure, as pointed out in Proposition 3. The identity of "Pythagorean" and "intuitionistic" fuzzy sets was also noted in ([5], Corollary 8.1).

Therefore, each mathematical result for $\mathbb{L}^*$-fuzzy sets, interval-valued fuzzy sets, "intuitionistic" fuzzy sets, etc., can be immediately translated into a result for "Pythagorean" fuzzy sets, and vice versa.

In other words, the term "Pythagorean" fuzzy sets is not only a fantasy name with no meaning whatsoever, it is absolutely useless, superfluous and even misleading, because it gives the impression to investigate something new, while isomorphic concepts have been studied already for many years. Therefore, the name "Pythagorean" fuzzy sets should be completely avoided.

Instead, because of the pairwise isomorphism between the lattices $(P^*, \leq_{\mathbb{L}^*})$, $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ and the lattice $(\mathfrak{I}(\mathbb{I}), \leq_{\mathfrak{I}(\mathbb{I})})$ of all closed subintervals of the unit interval $\mathbb{I}$, it is only a matter of personal taste to use one of the synonymous terms "$\mathbb{L}^*$-fuzzy sets" or "interval-valued fuzzy sets"—in any case, this can be done without any problem.

5. Concluding Remarks

As already mentioned, in the case of isomorphic lattices, any result known for one lattice can be rewritten in a straightforward way for each isomorphic lattice.

As a typical situation, recall that $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ and $(L_2(\mathbb{I}), \leq_{\text{comp}})$ are isomorphic lattices. Then, for each aggregation function $\mathbf{A}\colon \mathbb{I}^n \to \mathbb{I}$, the function $\mathbf{A}_{(2)}\colon (L_2(\mathbb{I}))^n \to L_2(\mathbb{I})$ given by

$$\mathbf{A}_{(2)}((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = (\mathbf{A}(x_1, x_2, \ldots, x_n), \mathbf{A}(y_1, y_2, \ldots, y_n))$$

is an aggregation function on $L_2(\mathbb{I})$ (called representable in [85,110–120]), and any properties of $\mathbf{A}$ are inherited by $A_{(2)}$. For example, if $\mathbf{A}$ is a t-norm or t-conorm, uninorm, nullnorm, so is $\mathbf{A}_{(2)}$. If $\mathbf{A}$ is an averaging (conjunctive, disjunctive) aggregation function [46] so is $\mathbf{A}_{(2)}$, etc.

Due to the isomorphism between the lattices $(\mathbb{L}^*, \leq_{\mathbb{L}^*})$ and $(L_2(\mathbb{I}), \leq_{\text{comp}})$ (see Proposition 1), one can easily, for each aggregation function $\mathbf{A}\colon \mathbb{I}^n \to \mathbb{I}$, define the corresponding aggregation function $\mathbf{A}^*\colon (\mathbb{L}^*)^n \to \mathbb{L}^*$ by

$$\mathbf{A}^*((x_1, y_1), \ldots, (x_n, y_n)) = (\mathbf{A}(x_1, \ldots, x_n), 1 - \mathbf{A}(1 - y_1, \ldots, 1 - y_n)).$$

In doing so, it is superfluous to give long and tedious proofs that, whenever $\mathbf{A}$ is a t-norm (t-conorm, uninorm, nullnorm) on $\mathbb{I}$, then $\mathbf{A}^*$ is a t-norm (t-conorm, uninorm, nullnorm) on $\mathbb{L}^*$. Similarly, considering any averaging aggregation function $\mathbf{A}$ [46] (e.g., a weighted quasi-arithmetic mean based on an additive generator of some continuous Archimedean t-norm, e.g., the Einstein t-norm [177]), then evidently also $\mathbf{A}^*$ is an averaging (thus idempotent) aggregation function on $\mathbb{L}^*$.

In the same way, one can easily re-define aggregation functions on the "Pythagorean" lattice $(P^*, \leq_{\mathbb{L}^*})$, and again there is no need of proving their properties (automatically inherited from the original aggregation function $\mathbf{A}$ acting on $\mathbb{I}$), as it was done in, e.g., [178].

Finally, let us stress that we are not against reasonable generalizations of fuzzy sets in the sense of [1], in particular if they proved to be useful in certain applications.

However, as one of the referees for this paper noted, "the crucial point is: not to introduce the same under different name" and "not to re-prove the same [...] facts". Therefore we have underlined that it is superfluous to (re-)prove "new" results for isomorphic lattices when the corresponding results are already known for at least one of the (already existing) isomorphic lattices. Also, we will continue to argue against "new" fantasy names for known mathematical objects and against the (ab)use of established (historical) mathematical notions in an improper context.

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