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Positive Implicative Ideals of BCK-Algebras Based on Intuitionistic Falling Shadows

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Abstract: The concepts of a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal and a positive implicative falling intuitionistic fuzzy ideal are introduced, and several properties are investigated. Characterizations of a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal are obtained, and relations between a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal and an intuitionistic fuzzy ideal are discussed. Conditions for an intuitionistic fuzzy ideal to be a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal are provided, and relations between a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal, a falling intuitionistic fuzzy ideal and a positive implicative falling intuitionistic fuzzy ideal are considered. Conditions for a falling intuitionistic fuzzy ideal to be positive implicative are given.

Keywords: intuitionistic random set; intuitionistic falling shadow; (positive implicative) $(\in, \in)$-intuitionistic fuzzy ideal; (positive implicative) falling intuitionistic fuzzy ideal

1. Introduction

Zadeh introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. Recently, Zhang [1] studied the fuzzy set theory of anti-grouped filters and normal filters in pseudo-BCI-algebras. Atanassov [2] introduced the degree of non-membership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Goodman [3] dealt with the equivalence of a fuzzy set and a class of random sets in the study of a unified treatment of uncertainty modeling by means of combining probability and fuzzy set theory. Wang and Sanchez [4,5] introduced the theory of falling shadows, which directly relates probability concepts with the membership function of fuzzy sets. In fact, the concept of random sets was firstly introduced by Kendall [6] and Matheron [7]. Using the theory of falling shadows, Tan et al. [8] constructed a theoretical approach to define fuzzy set operations based on the theory of falling shadows. Furthermore, Tan et al. [9] discussed a theoretical approach to define a fuzzy inference relation and showed that the formulae of the fuzzy inference relation given by Łukasiewicz, Zadeh and the probability formula are consequences of their definition under three different correlations of the propositions. The theory of falling shadows was applied to d-algebras [10], Tarski algebras [11], BCK/BCI-algebras [12–15], lattice implication algebras [16], EQ-algebras [17], MV-algebras [18], near-rings [19], BL-algebras [20], R_0-algebras [21] and vector spaces [22]. Using the notion of intuitionistic random set and intuitionistic falling shadow, which was introduced by Jun et al. [23], the concepts of the falling intuitionistic subalgebra and falling intuitionistic ideal in BCK/BCI-algebras were introduced, and related properties were investigated in [23]. Jun et al. [23] discussed relations between the falling intuitionistic ideal and falling intuitionistic subalgebra and established a characterization of the falling intuitionistic ideal.

In this paper, we introduce the concepts of a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal and a positive implicative falling intuitionistic fuzzy ideal, and we investigate several properties. We obtain characterizations of a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal and discuss relations between a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal and an intuitionistic fuzzy ideal. We provide conditions for an intuitionistic fuzzy ideal to be a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal and consider relations between a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal, a falling intuitionistic fuzzy ideal and a positive implicative falling intuitionistic fuzzy ideal. We give conditions for a falling intuitionistic fuzzy ideal to be positive implicative. Based on our results, we will try to find a way to solve nonlinear models such as the paper [24]. Furthermore, as future research topics, we will apply the generalizations of these results to other algebraic systems such as pseudo-BCI-algebras or neutrosophic triplet groups, etc; see [25–27].

2. Preliminaries

A BCK/BCI-algebra, which is an important class of logical algebras, was introduced by K. Iséki (see [28,29]).

A BCI-algebra is defined to be the structure $(X, *, 0)$, which satisfies the following conditions (see [30]):

(I) $(\forall x, y, z \in X)\ ((x * y) * (x * z) \leq z * y)$,
(II) $(\forall x, y \in X)\ ((x * (x * y) \leq y)$,
(III) $(\forall x \in X)\ (x \leq x)$,
(IV) $(\forall x, y \in X)\ (x \leq y,\ y \leq x \ \Rightarrow\ x = y)$ where $x \leq y$ means $x * y = 0$ for all $x, y \in X$. A BCI-algebra X with the following identity:
(V) $(\forall x \in X)\ (0 \leq x)$, is called a BCK-algebra. Every BCK/BCI-algebra X has the following conditions (see [30]).

$$(\forall x \in X)\ (x * 0 = x)\,, \tag{1}$$

$$(\forall x, y, z \in X)\ (x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2}$$

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)\,, \tag{3}$$

$$(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)\,. \tag{4}$$

We say that a BCK-algebra X is positive implicative (see [30]) if it satisfies the following condition.

$$(\forall x, y, z \in X)\ ((x * z) * (y * z) = (x * y) * z)\,. \tag{5}$$

By a subalgebra of a BCK/BCI-algebra X, we mean a nonempty subset S of X such that $x * y \in S$ for all $x, y \in S$ (see [30]).

By an ideal of a BCK/BCI-algebra X, we mean a subset I of X such that:

$$0 \in I, \tag{6}$$

$$(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \ \Rightarrow\ x \in I)\,. \tag{7}$$

By a positive implicative ideal of a BCK-algebra X (see [30]), we mean a subset I of X satisfying (6) and:

$$(\forall x, y, z \in X)(((x * y) * z \in I,\ y * z \in I \ \Rightarrow\ x * z \in I)\,. \tag{8}$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [30]).

An intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a BCK/BCI-algebra X is called an intuitionistic fuzzy subalgebra of X (see [31]) if it satisfies:

$$(\forall x, y \in X)\left(\begin{array}{l} h_\alpha(x*y) \geq \min\{h_\alpha(x), h_\alpha(y)\} \\ h_\beta(x*y) \leq \max\{h_\beta(x), h_\beta(y)\} \end{array}\right). \tag{9}$$

An intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a BCK/BCI-algebra X is called an intuitionistic fuzzy ideal of X (see [31]) if it satisfies:

$$(\forall x \in X)\left(h_\alpha(0) \geq h_\alpha(x), h_\beta(0) \leq h_\beta(x) \right). \tag{10}$$

$$(\forall x, y \in X)\left(\begin{array}{l} h_\alpha(x) \geq \min\{h_\alpha(x*y), h_\alpha(y)\} \\ h_\beta(x) \leq \max\{h_\beta(x*y), h_\beta(y)\} \end{array}\right). \tag{11}$$

For any $\alpha, \beta \in [0,1]$ and an intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a BCK/BCI-algebra X, consider the following sets:

$$U_\in(h;\alpha) = \{x \in X \mid h_\alpha(x) \geq \alpha\}$$

and:

$$L_\in(h;\beta) = \{x \in X \mid h_\beta(x) \leq \beta\}.$$

We say $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are intuitionistic $\in$-subsets.

3. Positive Implicative $(\in, \in)$ -Intuitionistic Fuzzy Ideals

Definition 1. *An intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a BCK-algebra X is called a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X if it satisfies the condition (10) and:*

$$\begin{array}{l} (x*y)*z \in U_\in(h;\alpha_x),\ y*z \in U_\in(h;\alpha_y) \ \Rightarrow\ x*z \in U_\in(h;\min\{\alpha_x,\alpha_y\}) \\ (x*y)*z \in L_\in(h;\beta_x),\ y*z \in L_\in(h;\beta_y) \ \Rightarrow\ x*z \in L_\in(h;\max\{\beta_x,\beta_y\}) \end{array} \tag{12}$$

for all $x, y, z \in X$, $(\alpha_x, \beta_x) \in [0,1] \times [0,1]$ and $(\alpha_y, \beta_y) \in [0,1] \times [0,1]$.

Example 1. *Consider a set $X = \{0,1,2,3,4\}$ with the binary operation *, which is given in Table 1.*

Table 1. Cayley table for the binary operation "*".

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	4	4	4	0

Then, $(X; *, 0)$ is a BCK-algebra (see [30]). Let $h = (h_\alpha, h_\beta)$ be an intuitionistic fuzzy set in X defined by Table 2.

Table 2. Tabular representation of $h = (h_\alpha, h_\beta)$.

X	$h_\alpha(x)$	$h_\beta(x)$
0	0.8	0.1
1	0.7	0.4
2	0.6	0.4
3	0.4	0.6
4	0.2	0.9

Routine calculations show that $h = (h_\alpha, h_\beta)$ is a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X.

Theorem 1. *For an intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a BCK-algebra X, the following are equivalent.*

(1) *The non-empty $\in$-subsets $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are positive implicative ideals of X for all $\alpha, \beta \in [0,1]$.*
(2) *$h = (h_\alpha, h_\beta)$ satisfies the condition (10) and:*

$$(\forall x, y, z \in X)\left(\begin{array}{l} h_\alpha(x*z) \geq \min\{h_\alpha((x*y)*z), h_\alpha(y*z)\} \\ h_\beta(x*z) \leq \max\{h_\beta((x*y)*z) \vee h_\beta(y*z)\} \end{array}\right) \tag{13}$$

Proof. Assume that the non-empty $\in$-subsets $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are positive implicative ideals of X for all $\alpha, \beta \in [0,1]$. If $h_\alpha(0) < h_\alpha(a)$ for some $a \in X$, then $a \in U_\in(h;h_\alpha(a))$ and $0 \notin U_\in(h;h_\alpha(a))$. This is a contradiction, and so, $h_\alpha(0) \geq h_\alpha(x)$ for all $x \in X$. Suppose that $h_\beta(0) > h_\beta(a)$ for some $a \in X$. Then, $a \in L_\in(h;h_\beta(a))$ and $0 \notin L_\in(h;h_\beta(a))$. This is a contradiction, and thus, $h_\beta(0) \leq h_\beta(x)$ for all $x \in X$. Therefore (10) is valid. Assume that there exist $a, b, c \in X$ such that:

$$h_\alpha(a*c) < \min\{h_\alpha((a*b)*c), h_\alpha(b*c)\}.$$

Taking $\alpha := \min\{h_\alpha((a*b)*c), h_\alpha(b*c)\}$ implies that $(a*b)*c \in U_\in(h;\alpha)$ and $b*c \in U_\in(h;\alpha)$, but $a*c \notin U_\in(h;\alpha)$, which is a contradiction. Hence:

$$h_\alpha(x*z) \geq \min\{h_\alpha((x*y)*z), h_\alpha(y*z)\}$$

for all $x, y, z \in X$. Now, suppose there are $x, y, z \in X$ such that:

$$h_\beta(x*z) > \max\{h_\beta((x*y)*z), h_\beta(y*z)\} := \beta.$$

Then, $(x*y)*z \in L_\in(h;\beta)$ and $y*z \in L_\in(h;\beta)$, but $x*z \notin L_\in(h;\beta)$, a contradiction. Thus:

$$h_\beta(x*z) \leq \max\{h_\beta((x*y)*z), h_\beta(y*z)\}$$

for all $x, y, z \in X$.

Conversely, let $h = (h_\alpha, h_\beta)$ be an intuitionistic fuzzy set in X satisfying two conditions (10) and (13). Assume that $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are nonempty for $\alpha, \beta \in [0,1]$. Let $x \in U_\in(h;\alpha)$ and $u \in L_\in(h;\beta)$ for $\alpha, \beta \in [0,1]$. Then, $h_\alpha(0) \geq h_\alpha(x) \geq \alpha$ and $h_\beta(0) \leq h_\beta(u) \leq \beta$ by (10). It follows that $0 \in U_\in(h;\alpha)$ and $0 \in L_\in(h;\beta)$. Let $a, b, c \in X$ be such that $(a*b)*c \in U_\in(h;\alpha)$ and $b*c \in U_\in(h;\alpha)$ for $\alpha \in [0,1]$. Then:

$$h_\alpha(a*c) \geq \min\{h_\alpha((a*b)*c), h_\alpha(b*c)\} \geq \alpha$$

by (13), and so, $a*c \in U_\in(h;\alpha)$. Suppose that: $(x*y)*z \in L_\in(h;\beta)$ and $y*z \in L_\in(h;\beta)$ for all $x, y, z \in X$ and $\beta \in [0,1]$. Then, $h_\beta((x*y)*z) \leq \beta$ and $h_\beta(y*z) \leq \beta$, which imply from the condition (13) that:

$$h_\beta(x*z) \leq \max\{h_\beta((x*y)*z), h_\beta(y*z)\} \leq \beta.$$

Hence, $x*z \in L_\in(h;\beta)$. Therefore, the non-empty $\in$-subsets $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are positive implicative ideals of X for all $\alpha, \beta \in [0,1]$. □

Theorem 2. *Let $h = (h_\alpha, h_\beta)$ be an intuitionistic fuzzy set in a BCK-algebra X. Then, $h = (h_\alpha, h_\beta)$ is a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X if and only if the non-empty intuitionistic $\in$-subsets $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are positive implicative ideals of X for all $\alpha, \beta \in [0,1]$.*

Proof. Let $h = (h_\alpha, h_\beta)$ be a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X, and assume that $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are nonempty for $\alpha, \beta \in [0,1]$. Then, there exist $x, z \in X$ such that $x \in U_\in(h;\alpha)$ and $z \in L_\in(h;\beta)$. It follows from (10) that $h_\alpha(0) \geq h_\alpha(x) \geq \alpha$ and $h_\beta(0) \leq h_\beta(x) \leq \beta$. Hence, $0 \in U_\in(h;\alpha)$ and $0 \in L_\in(h;\beta)$. Let $x, y, z, u, v, w \in X$ be such that $(x * y) * z \in U_\in(h;\alpha)$, $y * z \in U_\in(h;\alpha)$, $(u * v) * w \in L_\in(h;\beta)$ and $v * w \in L_\in(h;\beta)$. Then, $x * z \in U_\in(h; \min\{\alpha, \alpha\}) = U_\in(h;\alpha)$ and $u * w \in L_\in(h; \max\{\beta, \beta\}) = L_\in(h;\beta)$ by (12). Hence, the non-empty intuitionistic $\in$-subsets $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are positive implicative ideals of X for all $\alpha, \beta \in [0,1]$.

Conversely, let $h = (h_\alpha, h_\beta)$ be an intuitionistic fuzzy set in X for which $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are nonempty and are positive implicative ideals of X for all $\alpha, \beta \in [0,1]$. Obviously, (10) is valid. Let $x, y, z \in X$ and $(\alpha_x, \alpha_y) \in [0,1] \times [0,1]$ be such that $(x * y) * z \in U_\in(h;\alpha_x)$ and $y * z \in U_\in(h;\alpha_y)$. Then, $(x * y) * z \in U_\in(h;\alpha)$ and $y * z \in U_\in(h;\alpha)$ where $\alpha = \min\{\alpha_x, \alpha_y\}$. Since $U_\in(h;\alpha)$ is a positive implicative ideal of X, it follows that $x * z \in U_\in(h;\alpha) = U_\in(h; \min\{\alpha_x, \alpha_y\})$. Suppose that $(x * y) * z \in L_\in(h;\beta_x)$ and $y * z \in L_\in(h;\beta_y)$ for all $x, y, z \in X$ and $(\beta_x, \beta_y) \in [0,1] \times [0,1]$. Then, $(x * y) * z \in L_\in(h;\beta)$ and $y * z \in L_\in(h;\beta)$ where $\beta = \max\{\beta_x, \beta_y\}$. Hence, $x * z \in L_\in(h;\beta) = L_\in(h; \max\{\beta_x, \beta_y\})$ since $L_\in(h;\beta)$ is a positive implicative ideal of X. Therefore, $h = (h_\alpha, h_\beta)$ is a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X. □

Corollary 1. *Let $h = (h_\alpha, h_\beta)$ be an intuitionistic fuzzy set in a BCK-algebra X. Then, $h = (h_\alpha, h_\beta)$ is a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X if and only if it satisfies two conditions* (10) *and* (13).

Theorem 3. *Every positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of a BCK-algebra X is an intuitionistic fuzzy ideal of X.*

Proof. It is clear by taking $z = 0$ in (13) and using (1). □

Lemma 1 ([31]). *Every intuitionistic fuzzy ideal $h = (h_\alpha, h_\beta)$ of a BCK/BCI-algebra X satisfies the following assertion.*

$$(\forall x, y \in X) \left(x \leq y \Rightarrow \begin{cases} h_\alpha(x) \geq h_\alpha(y) \\ h_\beta(x) \leq h_\beta(y) \end{cases} \right). \tag{14}$$

Proposition 1. *Every positive implicative $(\in, \in)$-intuitionistic fuzzy ideal $h = (h_\alpha, h_\beta)$ of a BCK-algebra X satisfies the following assertions.*

$$(\forall x, y \in X) \begin{pmatrix} h_\alpha(x * y) \geq h_\alpha((x * y) * y) \\ h_\beta(x * y) \leq h_\beta((x * y) * y) \end{pmatrix}, \tag{15}$$

$$(\forall x, y \in X) \begin{pmatrix} h_\alpha((x * z) * (y * z)) \geq h_\alpha((x * y) * z) \\ h_\beta((x * z) * (y * z)) \leq h_\beta((x * y) * z) \end{pmatrix}, \tag{16}$$

and:

$$(\forall x, y \in X) \begin{pmatrix} h_\alpha(x * y) \geq \min\{h_\alpha(((x * y) * y) * z), h_\alpha(z)\} \\ h_\beta(x * y) \leq \max\{h_\beta(((x * y) * y) * z), h_\beta(z)\} \end{pmatrix}. \tag{17}$$

Proof. Let $h = (h_\alpha, h_\beta)$ be a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of a BCK-algebra X. Then, $h = (h_\alpha, h_\beta)$ is an intuitionistic fuzzy ideal of a BCK-algebra X (see Theorem 3). Since $x * x = 0$ for all $x \in X$, putting $z = y$ in (13) and using (10) induce (15). Since:

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in X$, we have:

$$\begin{aligned} h_\alpha((x*z)*(y*z)) &= h_\alpha((x*(y*z))*z) \\ &\geq h_\alpha(((x*(y*z))*z)*z) \\ &\geq h_\alpha((x*y)*z) \end{aligned}$$

and:

$$\begin{aligned} h_\beta((x*z)*(y*z)) &= h_\beta((x*(y*z))*z) \\ &\leq h_\beta(((x*(y*z))*z)*z) \\ &\leq h_\beta((x*y)*z) \end{aligned}$$

by (3), (15) and Lemma 1. Thus, (16) is valid. Note that:

$$(x*y)*z = ((x*z)*y)*(y*y)$$

for all $x, y \in X$. It follows from (11), (16) and (3) that:

$$\begin{aligned} h_\alpha(x*y) &\geq \min\{h_\alpha((x*y)*z), h_\alpha(z)\} \\ &= \min\{h_\alpha(((x*z)*y)*(y*y)), h_\alpha(z)\} \\ &\geq \min\{h_\alpha(((x*z)*y)*y), h_\alpha(z)\} \\ &= \min\{h_\alpha(((x*y)*y)*z), h_\alpha(z)\} \end{aligned}$$

and:

$$\begin{aligned} h_\beta(x*y) &\leq \max\{h_\beta((x*y)*z), h_\beta(z)\} \\ &= \max\{h_\beta(((x*z)*y)*(y*y)), h_\beta(z)\} \\ &\leq \max\{h_\beta(((x*z)*y)*y), h_\beta(z)\} \\ &= \max\{h_\beta(((x*y)*y)*z), h_\beta(z)\} \end{aligned}$$

for all $x, y, z \in X$. Therefore, (17) is valid. □

The converse of Theorem 3 is not true as seen in the following example.

Example 2. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation *, which is given in Table 3 (see [30]).*

Table 3. Cayley table for the binary operation "*".

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Let $h = (h_\alpha, h_\beta)$ be an intuitionistic fuzzy set in X defined by Table 4.

Table 4. Tabular representation of $h = (h_\alpha, h_\beta)$.

X	$h_\alpha(x)$	$h_\beta(x)$
0	0.7	0.3
1	0.4	0.5
2	0.5	0.4
3	0.4	0.5
4	0.1	0.6

Routine calculations show that $h = (h_\alpha, h_\beta)$ is an $(\in, \in)$-intuitionistic fuzzy ideal of X, and intuitionistic $\in$-subsets are given by:

$$U_\in(h;\alpha) = \begin{cases} \emptyset & \text{if } \alpha \in (0.7, 1], \\ \{0\} & \text{if } \alpha \in (0.5, 0.7], \\ \{0,2\} & \text{if } \alpha \in (0.4, 0.5], \\ \{0,1,2,3\} & \text{if } \alpha \in (0.1, 0.4], \\ X & \text{if } \alpha \in (0, 0.1], \end{cases}$$

and:

$$L_\in(h;\beta) = \begin{cases} X & \text{if } \beta \in [0.6, 1), \\ \{0,1,2,3\} & \text{if } \beta \in [0.5, 0.6), \\ \{0,2\} & \text{if } \beta \in [0.4, 0.5), \\ \{0\} & \text{if } \beta \in [0.3, 0.4), \\ \emptyset & \text{if } \beta \in [0, 0.3). \end{cases}$$

If $\alpha \in (0.4, 0.5]$ and $\beta \in [0.4, 0.5)$, then $U_\in(h;\alpha)$ and $L_\in(h;\beta)$ are not positive implicative ideals of X. Thus, $h = (h_\alpha, h_\beta)$ is not a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X by Theorems 1 and 2.

We provide conditions for an intuitionistic fuzzy ideal to be a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal.

Theorem 4. *Given an intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a BCK-algebra X, the following assertions are equivalent.*

(1) $h = (h_\alpha, h_\beta)$ *is a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X.*
(2) $h = (h_\alpha, h_\beta)$ *is an intuitionistic fuzzy ideal of X that satisfies the condition (15).*
(3) $h = (h_\alpha, h_\beta)$ *is an intuitionistic fuzzy ideal of X that satisfies the condition (16).*
(4) $h = (h_\alpha, h_\beta)$ *satisfies two conditions (10) and (17).*

Proof. Assume that $h = (h_\alpha, h_\beta)$ is a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X. Then, $h = (h_\alpha, h_\beta)$ is an intuitionistic fuzzy ideal of X by Theorem 3. If we put $z = y$ in (13) and use (10), then we get the condition (15). Suppose that $h = (h_\alpha, h_\beta)$ is an intuitionistic fuzzy ideal of X satisfying the condition (15). Note that:

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in X$. It follows from (3), (15) and Lemma 1 that:

$$\begin{aligned} h_\alpha((x * z) * (y * z)) &= h_\alpha((x * (y * z)) * z) \\ &\geq h_\alpha(((x * (y * z)) * z) * z) \\ &\geq h_\alpha((x * y) * z) \end{aligned}$$

and:

$$\begin{aligned} h_\beta((x*z)*(y*z)) &= h_\beta((x*(y*z))*z) \\ &\leq h_\beta(((x*(y*z))*z)*z) \\ &\leq h_\beta((x*y)*z). \end{aligned}$$

Hence, (16) is valid. Assume that $h = (h_\alpha, h_\beta)$ is an intuitionistic fuzzy ideal of X satisfying the condition (16). It is clear that $h = (h_\alpha, h_\beta)$ satisfies the condition (10). Using (11), (III), (3) and (16), we have:

$$\begin{aligned} h_\alpha(x*y) &\geq \min\{h_\alpha((x*y)*z), h_\alpha(z)\} \\ &= \min\{h_\alpha(((x*z)*y)*(y*y)), h_\alpha(z)\} \\ &\geq \min\{h_\alpha(((x*z)*y)*y), h_\alpha(z)\} \\ &= \min\{h_\alpha(((x*y)*y)*z), h_\alpha(z)\} \end{aligned}$$

and:

$$\begin{aligned} h_\beta(x*y) &\leq \max\{L_\in((x*y)*z), h_\beta(z)\} \\ &= \max\{h_\beta(((x*z)*y)*(y*y)), h_\beta(z)\} \\ &\leq \max\{h_\beta(((x*z)*y)*y), h_\beta(z)\} \\ &= \max\{h_\beta(((x*y)*y)*z), h_\beta(z)\} \end{aligned}$$

for all $x, y, z \in X$. Thus, (17) is valid. Finally, suppose that $h = (h_\alpha, h_\beta)$ satisfies two conditions (10) and (17). Using (1) and (17), we get:

$$\begin{aligned} h_\alpha(x) &= h_\alpha(x*0) \\ &\geq \min\{h_\alpha(((x*0)*0)*y), h_\alpha(y)\} \\ &= \min\{h_\alpha(x*y), h_\alpha(y)\} \end{aligned}$$

and:

$$\begin{aligned} h_\beta(x) &= h_\beta(x*0) \\ &\leq \max\{h_\beta(((x*0)*0)*y), h_\beta(y)\} \\ &= \max\{h_\beta(x*y), h_\beta(y)\} \end{aligned}$$

for all $x, y \in X$. Hence, $h = (h_\alpha, h_\beta)$ is an intuitionistic fuzzy ideal of X. Since:

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$$

for all $x, y, z \in X$, it follows from (17) and (14) that:

$$\begin{aligned} h_\alpha(x*z) &\geq \min\{h_\alpha(((x*z)*z)*(y*z)), h_\alpha(y*z)\} \\ &\geq \min\{h_\alpha((x*y)*z), h_\alpha(y*z)\} \end{aligned}$$

and:

$$\begin{aligned} h_\beta(x*z) &\leq \max\{h_\beta(((x*z)*z)*(y*z)), h_\beta(y*z)\} \\ &\leq \max\{h_\beta((x*y)*z), h_\beta(y*z)\} \end{aligned}$$

for all $x, y, z \in X$. Therefore, $h = (h_\alpha, h_\beta)$ is a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of X. $\square$

4. Positive Implicative Falling Intuitionistic Ideals

Given an element x of a BCK/BCI-algebra X and $D \in 2^X$, let:

$$\bar{x} := \{C \in 2^X \mid x \in C\}, \tag{18}$$

and:

$$\bar{D} := \{\bar{x} \mid x \in D\}. \tag{19}$$

A pair $(2^X, \mathcal{B})$ is called a hyper-measurable structure on X if $\mathcal{B}$ is a σ-field in 2^X and $\bar{X} \subseteq \mathcal{B}$.

Let $(\mho, \mathcal{A}, P)$ be a probability space and $(2^X, \mathcal{B})$ a hyper-measurable structure on X. An intuitionistic random set on X is defined to be a couple $\psi := (\psi_\alpha, \psi_\beta)$ in which ψ_α and ψ_β are mappings from $\mho$ to 2^X which are $\mathcal{A}$-$\mathcal{B}$ measurables, that is,

$$(\forall C \in \mathcal{B}) \begin{pmatrix} \psi_\alpha^{-1}(C) = \{\varepsilon_\alpha \in \mho \mid \psi_\alpha(\varepsilon_\alpha) \in C\} \in \mathcal{A} \\ \psi_\beta^{-1}(C) = \{\varepsilon_\beta \in \mho \mid \psi_\beta(\varepsilon_\beta) \in C\} \in \mathcal{A} \end{pmatrix}. \tag{20}$$

Given an intuitionistic random set $\psi := (\psi_\alpha, \psi_\beta)$ on X, consider functions:

$$\begin{aligned} &\tilde{F}_\alpha : X \to [0,1],\ x_\alpha \mapsto P(\varepsilon_\alpha \mid x_\alpha \in \psi_\alpha(\varepsilon_\alpha)), \\ &\tilde{F}_\beta : X \to [0,1],\ x_\beta \mapsto 1 - P(\varepsilon_\beta \mid x_\beta \in \psi_\beta(\varepsilon_\beta)). \end{aligned}$$

Then, $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ is an intuitionistic fuzzy set on X, and we call it the intuitionistic falling shadow of the intuitionistic random set $\psi := (\psi_\alpha, \psi_\beta)$, and $\psi := (\psi_\alpha, \psi_\beta)$ is called an intuitionistic cloud of $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$.

For example, consider a probability space $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ be an intuitionistic fuzzy set in X. Then, a couple $\psi := (\psi_\alpha, \psi_\beta)$ in which:

$$\begin{aligned} &\psi_\alpha : [0,1] \to 2^X, \alpha \mapsto U_\in(\tilde{F}; \alpha), \\ &\psi_\beta : [0,1] \to 2^X, \beta \mapsto L_\in(\tilde{F}; \beta) \end{aligned}$$

is an intuitionistic random set and $\psi := (\psi_\alpha, \psi_\beta)$ is an intuitionistic cloud of $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$. We will call $\psi := (\psi_\alpha, \psi_\beta)$ defined above the intuitionistic cut-cloud of $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$.

Definition 2. *Consider a probability space $(\mho, \mathcal{A}, P)$, and let $\psi := (\psi_\alpha, \psi_\beta)$ be an intuitionistic fuzzy random set on a BCK-algebra X. If $\psi_\alpha(\varepsilon_\alpha)$ and $\psi_\beta(\varepsilon_\beta)$ are positive implicative ideals of X for all ε_α, $\varepsilon_\beta \in \mho$, then the intuitionistic falling shadow $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ of the intuitionistic random set $\psi := (\psi_\alpha, \psi_\beta)$ on X, that is,*

$$\begin{aligned} &\tilde{F}_\alpha(x_\alpha) = P(\varepsilon_\alpha \mid x_\alpha \in \psi_\alpha(\varepsilon_\alpha)), \\ &\tilde{F}_\beta(x_\beta) = 1 - P(\varepsilon_\beta \mid x_\beta \in \psi_\beta(\varepsilon_\beta)) \end{aligned} \tag{21}$$

is called a positive implicative falling intuitionistic fuzzy ideal of X.

Example 3. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation *, which is given in Table 5 (see [30]).*

Table 5. Cayley table for the binary operation "*".

*	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Let $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, and let $\psi := (\psi_\alpha, \psi_\beta)$ be an intuitionistic fuzzy random set on X, which is given as follows:

$$\psi_\alpha : [0,1] \to 2^X, \quad x \mapsto \begin{cases} \{0,3\} & \text{if } x \in [0, 0.25), \\ \{0,1\} & \text{if } x \in [0.25, 0.55), \\ \{0,1,2\} & \text{if } x \in [0.55, 0.85), \\ \{0,1,3\} & \text{if } x \in [0.85, 0.95), \\ X & \text{if } x \in [0.95, 1], \end{cases}$$

and:

$$\psi_\beta : [0,1] \to 2^X, \quad x \mapsto \begin{cases} \{0\} & \text{if } x \in (0.9, 1], \\ \{0,3\} & \text{if } x \in (0.7, 0.9], \\ \{0,1,2\} & \text{if } x \in (0.5, 0.7], \\ \{0,1,2,3\} & \text{if } x \in (0.3, 0.5], \\ \{0,1,2,4\} & \text{if } x \in [0, 0.3]. \end{cases}$$

Then, $\psi_\alpha(t)$ and $\psi_\beta(t)$ are positive implicative ideals of X for all $t \in [0,1]$. Hence, the intuitionistic fuzzy falling shadow $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ of $\psi := (\psi_\alpha, \psi_\beta)$ is a positive implicative falling intuitionistic fuzzy ideal of X, and it is given as follows:

$$\tilde{F}_\alpha(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.75 & \text{if } x = 1, \\ 0.35 & \text{if } x = 2, \\ 0.4 & \text{if } x = 3, \\ 0.05 & \text{if } x = 4, \end{cases}$$

and:

$$\tilde{F}_\beta(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.7 & \text{if } x \in \{1,2\}, \\ 0.4 & \text{if } x = 3, \\ 0.3 & \text{if } x = 4. \end{cases}$$

Given a probability space $(\mho, \mathcal{A}, P)$, let $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ be an intuitionistic fuzzy falling shadow of an intuitionistic fuzzy random set $\psi := (\psi_\alpha, \psi_\beta)$. For $x \in X$, let:

$$\mho(x; \psi_\alpha) := \{\varepsilon_\alpha \in \mho \mid x \in \psi_\alpha(\varepsilon_\alpha)\},$$
$$\mho(x; \psi_\beta) := \{\varepsilon_\beta \in \mho \mid x \in \psi_\beta(\varepsilon_\beta)\}.$$

Then, $\mho(x; \psi_\alpha), \mho(x; \psi_\beta) \in \mathcal{A}$ (see [23]).

Proposition 2. *Let $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ be an intuitionistic fuzzy falling shadow of the intuitionistic fuzzy random set $\psi := (\psi_\alpha, \psi_\beta)$ on a BCK-algebra X. If $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ is a positive implicative falling intuitionistic fuzzy ideal of X, then:*

$$(\forall x, y, z \in X) \begin{pmatrix} \mho((x*y)*z;\psi_\alpha) \cap \mho(y*z;\psi_\alpha) \subseteq \mho(x*z;\psi_\alpha) \\ \mho((x*y)*z;\psi_\beta) \cap \mho(y*z;\psi_\beta) \subseteq \mho(x*z;\psi_\beta) \end{pmatrix}, \tag{22}$$

$$(\forall x, y, z \in X) \begin{pmatrix} \mho(x*z;\psi_\alpha) \subseteq \mho((x*y)*z;\psi_\alpha) \\ \mho(x*z;\psi_\beta) \subseteq \mho((x*y)*z;\psi_\beta) \end{pmatrix}. \tag{23}$$

Proof. Let $\varepsilon_\alpha \in \mho((x*y)*z;\psi_\alpha) \cap \mho(y*z;\psi_\alpha)$ and $\varepsilon_\beta \in \mho((x*y)*z;\psi_\beta) \cap \mho(y*z;\psi_\beta)$ for all $x, y, z \in X$. Then, $(x*y)*z \in \psi_\alpha(\varepsilon_\alpha)$, $y*z \in \psi_\alpha(\varepsilon_\alpha)$, $(x*y)*z \in \psi_\beta(\varepsilon_\beta)$ and $y*z \in \psi_\beta(\varepsilon_\beta)$. Since $\psi_\alpha(\varepsilon_\alpha)$ and $\psi_\beta(\varepsilon_\beta)$ are positive implicative ideals of X, it follows from (8) that $x*z \in \psi_\alpha(\varepsilon_\alpha) \cap \psi_\beta(\varepsilon_\beta)$ and so that $\varepsilon_\alpha \in \mho(x*z;\psi_\alpha)$ and $\varepsilon_\beta \in \mho(x*z;\psi_\beta)$. Hence, (22) is valid. Now, let $x, y, z \in X$ be such that $\varepsilon_\alpha \in \mho(x*z;\psi_\alpha)$ and $\varepsilon_\beta \in \mho(x*z;\psi_\beta)$. Then, $x*z \in \psi_\alpha(\varepsilon_\alpha) \cap \psi_\beta(\varepsilon_\beta)$. Note that:

$$\begin{aligned} &((x*y)*z)*(x*z) = ((x*y)*(x*z))*z \\ &\leq (z*y)*z = (z*z)*y \\ &= 0*y = 0, \end{aligned}$$

which yields $((x*y)*z)*(x*z) = 0 \in \psi_\alpha(\varepsilon_\alpha) \cap \psi_\beta(\varepsilon_\beta)$. Since $\psi_\alpha(\varepsilon_\alpha)$ and $\psi_\beta(\varepsilon_\beta)$ are positive implicative ideals and hence ideals of X, it follows that $(x*y)*z \in \psi_\alpha(\varepsilon_\alpha) \cap \psi_\beta(\varepsilon_\beta)$. Hence, $\varepsilon_\alpha \in \mho((x*y)*z;\psi_\alpha)$ and $\varepsilon_\beta \in \mho((x*y)*z;\psi_\beta)$. Therefore, (23) is valid. □

For a probability space $(\mho, \mathcal{A}, P)$, consider:

$$\mathcal{F}(X) := \{h \mid h : \mho \to X \text{ is a mapping}\}. \tag{24}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \varepsilon \in \mho)\ ((f \circledast g)(\varepsilon) = f(\varepsilon) * g(\varepsilon)) \tag{25}$$

for all $f, g \in \mathcal{F}(X)$. Then, $(\mathcal{F}(X); \circledast, \theta)$ is a BCK/BCI-algebra (see [13]) where θ is given as follows:

$$\theta : \mho \to X,\ \varepsilon \mapsto 0.$$

For any subset A of X and $g_\alpha, g_\beta \in \mathcal{F}(X)$, consider the following sets and mappings:

$$\begin{aligned} A_\alpha^g &:= \{\varepsilon_\alpha \in \mho \mid g_\alpha(\varepsilon_\alpha) \in A\}, \\ A_\beta^g &:= \{\varepsilon_\beta \in \mho \mid g_\beta(\varepsilon_\beta) \in A\} \end{aligned}$$

and:

$$\begin{aligned} \psi_\alpha &: \mho \to \mathcal{P}(\mathcal{F}(X)),\ \varepsilon_\alpha \mapsto \{g_\alpha \in \mathcal{F}(X) \mid g_\alpha(\varepsilon_\alpha) \in A\}, \\ \psi_\beta &: \mho \to \mathcal{P}(\mathcal{F}(X)),\ \varepsilon_\beta \mapsto \{g_\beta \in \mathcal{F}(X) \mid g_\beta(\varepsilon_\beta) \in A\}. \end{aligned}$$

Then, $A_\alpha^g, A_\beta^g \in \mathcal{A}$ (see [23]).

Theorem 5. *If K is a positive implicative ideal of a BCK-algebra X, then:*

$$\begin{aligned} \psi_\alpha(\varepsilon_\alpha) &= \{g_\alpha \in \mathcal{F}(X) \mid g_\alpha(\varepsilon_\alpha) \in K\}, \\ \psi_\beta(\varepsilon_\beta) &= \{g_\beta \in \mathcal{F}(X) \mid g_\beta(\varepsilon_\beta) \in K\} \end{aligned}$$

are positive implicative ideals of $\mathcal{F}(X)$.

Proof. Assume that K is a positive implicative ideal of a BCK-algebra X. Since $\theta(\varepsilon_\alpha) = 0 \in K$ and $\theta(\varepsilon_\beta) = 0 \in K$ for all $\varepsilon_\alpha, \varepsilon_\beta \in \mho$, we have:

$$\theta \in \psi_\alpha(\varepsilon_\alpha) \cap \psi_\beta(\varepsilon_\beta).$$

Let $f_\alpha, g_\alpha, h_\alpha \in \mathcal{F}(X)$ be such that $(f_\alpha \circledast g_\alpha) \circledast h_\alpha \in \psi_\alpha(\varepsilon_\alpha)$ and $g_\alpha \circledast h_\alpha \in \psi_\alpha(\varepsilon_\alpha)$. Then:

$$(f_\alpha(\varepsilon_\alpha) * g_\alpha(\varepsilon_\alpha)) * h_\alpha(\varepsilon_\alpha) = ((f_\alpha \circledast g_\alpha) \circledast h_\alpha)(\varepsilon_\alpha) \in K$$

and $g_\alpha(\varepsilon_\alpha) * h_\alpha(\varepsilon_\alpha) \in K$. Since K is a positive implicative ideal of X, it follows from (8) that:

$$(f_\alpha \circledast h_\alpha)(\varepsilon_\alpha) = f_\alpha(\varepsilon_\alpha) * h_\alpha(\varepsilon_\alpha) \in K,$$

that is, $f_\alpha \circledast h_\alpha \in \psi_\alpha(\varepsilon_\alpha)$. Hence, $\psi_\alpha(\varepsilon_\alpha)$ is a positive implicative ideal of $\mathcal{F}(X)$. Now, let f_β, g_β, $h_\beta \in \mathcal{F}(X)$ be such that $(f_\beta \circledast g_\beta) \circledast h_\beta \in \psi_\beta(\varepsilon_\beta)$ and $g_\beta \circledast h_\beta \in \psi_\beta(\varepsilon_\beta)$. Then:

$$(f_\beta(\varepsilon_\beta) * g_\beta(\varepsilon_\beta)) * h_\beta(\varepsilon_\beta) = ((f_\beta \circledast g_\beta) \circledast h_\beta)(\varepsilon_\beta) \in K$$

and $g_\beta(\varepsilon_\beta) * h_\beta(\varepsilon_\beta) \in K$. Then:

$$(f_\beta \circledast h_\beta)(\varepsilon_\beta) = f_\beta(\varepsilon_\beta) * h_\beta(\varepsilon_\beta) \in K,$$

and so $f_\beta \circledast h_\beta \in \psi_\beta(\varepsilon_\beta)$. Hence, $\psi_\beta(\varepsilon_\beta)$ is a positive implicative ideal of $\mathcal{F}(X)$. This completes the proof. □

Theorem 6. *If we consider a probability space $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, then every positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of a BCK-algebra is a positive implicative falling intuitionistic fuzzy ideal.*

Proof. Let $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ be a positive implicative $(\in, \in)$-intuitionistic fuzzy ideal of a BCK-algebra X. Then, $U_\in(\tilde{F}; \alpha)$ and $L_\in(\tilde{F}; \beta)$ are positive implicative ideals of X for all $\alpha, \beta \in (0,1]$ and $\beta \in [0,1)$ by Theorem 2. Hence, a couple $\psi := (\psi_\alpha, \psi_\beta)$ in which:

$$\psi_\alpha : [0,1] \to 2^X, \ \alpha \mapsto U_\in(\tilde{F}; \alpha),$$
$$\psi_\beta : [0,1] \to 2^X, \ \beta \mapsto L_\in(\tilde{F}; \beta)$$

is an intuitionistic fuzzy cut-cloud of $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$, and so, $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ is a positive implicative falling intuitionistic fuzzy ideal of X. □

The converse of Theorem 6 is not true, as seen in the following example.

Example 4. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation *, which is given in Table 6 (see [30]).*

Table 6. Cayley table for the binary operation "*".

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Let $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$*, and let* $\psi := (\psi_\alpha, \psi_\beta)$ *be an intuitionistic fuzzy random set on* X*, which is given as follows:*

$$\psi_\alpha : [0,1] \to 2^X, \ \ x \mapsto \begin{cases} \{0,1\} & \text{if } x \in [0, 0.2), \\ \{0,2\} & \text{if } x \in [0.2, 0.55), \\ \{0,2,4\} & \text{if } x \in [0.55, 0.75), \\ \{0,1,2,3\} & \text{if } x \in [0.75, 1], \end{cases}$$

and:

$$\psi_\beta : [0,1] \to 2^X, \ \ x \mapsto \begin{cases} \{0\} & \text{if } x \in (0.77, 1], \\ \{0,1\} & \text{if } x \in (0.66, 0.77], \\ \{0,2\} & \text{if } x \in (0.48, 0.66], \\ \{0,2,4\} & \text{if } x \in (0.23, 0.48], \\ \{0,1,2,3\} & \text{if } x \in [0, 0.23]. \end{cases}$$

Then, $\psi_\alpha(t)$ *and* $\psi_\beta(t)$ *are positive implicative ideals of* X *for all* $t \in [0,1]$*. Hence, the intuitionistic fuzzy falling shadow* $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ *of* $\psi := (\psi_\alpha, \psi_\beta)$ *is a positive implicative falling intuitionistic fuzzy ideal of* X*, and it is given as follows:*

$$\tilde{F}_\alpha(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.45 & \text{if } x = 1, \\ 0.8 & \text{if } x = 2, \\ 0.25 & \text{if } x = 3, \\ 0.2 & \text{if } x = 4, \end{cases}$$

and:

$$\tilde{F}_\beta(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.66 & \text{if } x = 1, \\ 0.34 & \text{if } x = 2, \\ 0.77 & \text{if } x = 3, \\ 0.75 & \text{if } x = 4. \end{cases}$$

However, $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ *is not a positive implicative* $(\in, \in)$*-intuitionistic fuzzy ideal of* X *since:*

$$\tilde{F}_\alpha(3 * 4) = \tilde{F}_\alpha(3) = 0.25 < 0.8 = \tilde{F}_\alpha((3 * 2) * 4) \wedge \tilde{F}_\alpha(2 * 4)$$

and/or:

$$\tilde{F}_\beta(3 * 4) = \tilde{F}_\beta(3) = 0.77 > 0.66 = \tilde{F}_\beta((3 * 1) * 4) \vee \tilde{F}_\beta(1 * 4).$$

We provide relations between a falling intuitionistic fuzzy ideal and a positive implicative falling intuitionistic fuzzy ideal.

Theorem 7. *Let* $(\mho, \mathcal{A}, P)$ *be a probability space, and let* $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ *be an intuitionistic fuzzy falling shadow of an intuitionistic fuzzy random set* $\psi := (\psi_\alpha, \psi_\beta)$ *on a BCK-algebra* X*. If* $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ *is a positive implicative falling intuitionistic fuzzy ideal of* X*, then it is a falling intuitionistic fuzzy ideal of* X*.*

Proof. Let $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ be a positive implicative falling intuitionistic fuzzy ideal of a BCK-algebra X. Then, $\psi_\alpha(\varepsilon_\alpha)$ and $\psi_\beta(\varepsilon_\beta)$ are positive implicative ideals of X, and so, $\psi_\alpha(\varepsilon_\alpha)$ and $\psi_\beta(\varepsilon_\beta)$ are ideals of X for all $\varepsilon_\alpha, \varepsilon_\beta \in \mho$. Therefore, $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ is a falling intuitionistic fuzzy ideal of X. □

The converse of Theorem 7 is false, as seen in the following example.

Example 5. *Consider a BCK-algebra $X = \{0,1,2,3,4\}$ with the binary operation *, which is given in Table 7 (see [30]).*

Table 7. Cayley table for the binary operation "*".

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	3	2	0	3
4	4	4	4	4	0

Let $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, and let $\psi := (\psi_\alpha, \psi_\beta)$ be an intuitionistic fuzzy random set on X, which is given as follows:

$$\psi_\alpha : [0,1] \to 2^X, \;\; x \mapsto \begin{cases} \{0,4\} & \text{if } x \in [0, 0.37), \\ \{0,1,2,3\} & \text{if } x \in [0.37, 0.67), \\ \{0,1,4\} & \text{if } x \in [0.67, 1], \end{cases}$$

and:

$$\psi_\beta : [0,1] \to 2^X, \;\; x \mapsto \begin{cases} \{0\} & \text{if } x \in (0.74, 1], \\ \{0,1\} & \text{if } x \in (0.66, 0.74], \\ \{0,4\} & \text{if } x \in (0.48, 0.66], \\ \{0,1,2,3\} & \text{if } x \in [0, 0.48]. \end{cases}$$

Then, $\psi_\alpha(t)$ and $\psi_\beta(t)$ are ideals of X for all $t \in [0,1]$. Hence, the intuitionistic fuzzy falling shadow $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ of $\psi := (\psi_\alpha, \psi_\beta)$ is a falling intuitionistic fuzzy ideal of X. However, it is not a positive implicative falling intuitionistic fuzzy ideal of X because if $\alpha \in [0.67, 1]$, $\beta \in [0, 0.45)$ and $\beta \in (0.66, 0.74]$, then $\psi_\alpha(\alpha) = \{0,1,4\}$ and $\psi_\beta(\beta) = \{0,1\}$ are not positive implicative ideals of X, respectively.

Since every ideal is positive implicative in a positive implicative BCK-algebra, we have the following theorem.

Theorem 8. *Let $(\mho, \mathcal{A}, P)$ be a probability space, and let $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ be an intuitionistic fuzzy falling shadow of an intuitionistic fuzzy random set $\psi := (\psi_\alpha, \psi_\beta)$ on a positive implicative BCK-algebra. If $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ is a falling intuitionistic fuzzy ideal of X, then it is a positive implicative falling intuitionistic fuzzy ideal of X.*

Corollary 2. *Let $(\mho, \mathcal{A}, P)$ be a probability space. For any BCK-algebra X that satisfies one of the following assertions*

$$\begin{aligned} &(\forall x, y \in X)(x * y = (x * y) * y), \\ &(\forall x, y \in X)((x * (x * y)) * (y * x) = x * (x * (y * (y * x)))), \\ &(\forall x, y \in X)(x * y = (x * y) * (x * (x * y))), \\ &(\forall x, y \in X)(x * (x * y) = (x * (x * y)) * (x * y)), \\ &(\forall x, y \in X)((x * (x * y)) * (y * x) = (y * (y * x)) * (x * y)), \end{aligned}$$

let $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ be an intuitionistic fuzzy falling shadow of an intuitionistic fuzzy random set $\psi := (\psi_\alpha, \psi_\beta)$ on X. If $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ is a falling intuitionistic fuzzy ideal of X, then it is a positive implicative falling intuitionistic fuzzy ideal of X.

5. Conclusions

In this paper, some new notions of fuzzy ideals in BCK-algebras are proposed, and the relationships between these new fuzzy ideals are investigated. The results of this paper are of positive significance for the further study of the structure of BCK-algebras. As future research topics, the generalizations of these results to other algebraic systems (pseudo-BCI algebras or neutrosophic triplet groups; see [25–27]) are meaningful.

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