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# Equivalences of Riemann Integral Based on *p*-Norm

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**Abstract:** In the usual Riemann integral setting, the Riemann norm or a mesh is adopted for Riemann sums. In this article, we use the $p$-norm to define the $p$-integral and show the equivalences between the Riemann integral and the $p$-integral. The $p$-norm provides an alternative approach to define the Riemann integral. Based on this norm, we also derive some other equivalences of the Riemann integral and the $p$-integral.

**Keywords:** $p$-integral; Riemann integral; Darboux integral

**MSC:** 28A10

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## 1. Introduction

In this article, only the bounded functions defined on a closed interval $[a, b]$ are considered. A function $f$ is used to denote a bound function defined on $[a, b]$. To begin with, the usual settings of a Riemann integral are listed for comparison. Then, some notations for the preparation of our settings of the $p$-integral are introduced. Although the $p$-norm and the Riemann norm turn out to be equivalent, the $p$-norm has many merits in connecting functional analysis and integrals. Since the $p$-norm is massively used in functional analysis, by defining an alternative Riemann integral via this norm, one could further look at the typical the Riemann integral from a new aspect. This might further extend the Riemann integral to other territories. Defining the $p$-norm from the beginning and deriving all the equivalences directly gives us some insightful knowledge between all these equivalences.

### *1.1. Background*

Let $P = \{P(0), P(1), ..., P(n)\}$ denote a partition of $[a, b]$, in which $P(0) = a < P(1) < P(2) < \cdots < P(n) = b$. Let $|P|$ denote the length of the partition, in this case $n$. Let $Tag(P)$ denote the set of all the tags of $P$. The usual setting of the Riemann integral is based on the concept of meshes [1]. Firstly, one defines a Riemann sum corresponding to a partition $P$ of $[a, b]$ and its tags $\zeta^P$ as follows [2,3]:

$$\sigma(f, P, \zeta^P) = \sum_{j=1}^{|P|} f(\zeta_j^P) \cdot \Delta_j^P,$$

where $\Delta_j^P = P(j) - P(j-1)$ and $\zeta_j^P \in [P(j-1), P(j)]$. Then, one defines the mesh of the partition $P$,

$$\lambda(P) = max\{\Delta_j^P : j \in \{1, 2, ..., |P|\}.$$

Moreover, if there exists $I \in \mathbb{R}$ such that $\forall \epsilon > 0\ \exists \delta_\epsilon > 0$ such that for all partition $P$ of $[a, b]$

$$\lambda(P) < \delta_\epsilon \to \forall \zeta^P \in Tag(P) |I - \sigma(f, P, \zeta^P)| < \epsilon,$$

then we say $f$ is Riemann integrable [4] and the integral of $f$ is defined as $\int_a^b f(x)dx = I$ or

$$\int_a^b f(x)dx = \lim_{\lambda(P)\to 0} f(\zeta_j^P) \cdot \Delta_j^P.$$

*1.2. Notations*

In this article, we use the $p-$norm (where $p > 1$) instead of a mesh (or the Riemann norm) to derive some equivalences of the Riemann integral. Unlike the usual setting that combines partitions and tags, they are reintroduced via two individual parts: partition vectors and their corresponding product (tag) spaces. For a (partition) vector $\vec{v} = (v_0, v_1, ..., v_n) \in \mathbb{R}^{n+1}$, we use $|\vec{v}|$ to denote the length of $\vec{v}$, i.e., $n$ in this case and $\vec{v}_j$ to denote the $j$th element of $\vec{v}$, i.e., $v_j$ in this case. To unify the presentation of the Riemann integral and the $p$-integral, some notations are defined beforehand:

- $FPV^{(n)}[a,b]$ denotes the set of all the finite partition vectors of $[a,b]$ whose length is $n$, i.e., $FPV^{(n)}[a,b] = \{\vec{B} \in \mathbb{R}^{n+1} : B_0 = a < B_1 < \ldots < B_j < B_{j+1} < \ldots < B_n = b\}$. For example, $\vec{B} = (1, 1.2, 1.5, 2.2, 2.6, 3) \in FPV^{(5)}[1,3]$, in which $B_0 = 1, B_1 = 1.2, ..., B_5 = 3$ and $|\vec{B}| = 5$.
- $FPV[a,b] = \bigcup_{n=1}^{\infty} FPV^{(n)}[a,b]$, i.e., the set of all the finite partitions of $[a,b]$. Observe that $FPV[a,b] \subseteq \bigcup_{n=1}^{\infty} \mathbb{R}^{n+1}$.
- $\prod[\vec{B}] = \prod_{j=0}^{|\vec{B}|-1} [B_j, B_{j+1}]$. For example, if $\vec{B} = (1, 1.2, 1.5, 2.2, 2.6, 3) \in FPV^{(5)}[1,3]$, then $(1.1, 1.4, 1.5, 2.4, 2.9) \in \prod[\vec{B}]$. $\prod[\vec{B}]$ represents the space where the tags are located, given a partition $\vec{B}$ of $[a,b]$.
- $\prod FPV[a,b] = \prod_{j=1}^{\infty} FPV[a,b]$. $\prod FPV[a,b]$ represents an ordered sequence of finite partition vectors.
- (Riemann norm) $\langle\langle\vec{B}\rangle\rangle := max\{B_{j+1} - B_j : 0 \le j \le |\vec{B}| - 1\}$. This is exactly the usual definition of a mesh.
- (Riemann sum) $A_f(\vec{B}, \vec{H}) := \sum_{j=0}^{|\vec{B}|-1} (B_{j+1} - B_j) \cdot f(H_j)$, where $\vec{H}_j \in [B_j, B_{j+1}]$, where $\vec{H}$ represents a sequence of tags, i.e., $\vec{H} \in \prod[\vec{B}]$.
- ($p$-norm) $\|\vec{B}\|_p = \sqrt[p]{\sum_{j=0}^{|\vec{B}|-1} |B_{j+1} - B_j|^p}$, where $p > 1$. For example, if $\vec{B} = (1, 2, 4, 7, 10)$, then $\|\vec{B}\|_3 = \sqrt[3]{(1 + 2^3 + 3^3 + 3^3)} = \sqrt[3]{63}$. By exploiting Minkowski inequality, one could easily verify $\|\cdot\|_p$ is a norm.
- For any $\delta > 0$, define $FPV(\delta) = \{\vec{B} \in FPV[a,b] : \langle\langle\vec{B}\rangle\rangle < \delta\}$.
- For any $\delta > 0$, define $FPV_p(\delta) = \{\vec{B} \in FPV[a,b] : \|\vec{B}\|_p < \delta\}$.

$FPV(\delta)$ collects all the finite partition vectors whose Riemann norms are less than $\delta$, while $FPV_p(\delta)$ collects all the finite partition vectors whose $p$-norms are less than $\delta$. One could easily verify that $FPV_p(\delta) \subsetneq FPV(\delta)$. The equality is obvious not true. For example, if $\vec{B} = (0, 0.2, 1, 1.5, 2)$ and $\delta = 1$, then $\vec{B} \in FPV(1)$, but $\vec{B} \notin FPV_p(1)$. We use the notation $\vec{\mathbb{C}} = (\vec{\mathbb{C}}_0, \vec{\mathbb{C}}_1, \vec{\mathbb{C}}_2, \ldots, \vec{\mathbb{C}}_n, \ldots) \in \prod FPV[a,b]$ to denote a sequence of partition vectors in $FPV[a,b]$, where each $\vec{\mathbb{C}}_j \in FPV[a,b]$ and denotes the $(j+1)$th partition vectors in $\vec{\mathbb{C}}$.

**Definition 1.** $[0] = \{\vec{\mathbb{C}} \in \prod FPV[a,b] : \lim_{j\to\infty} \langle\langle\vec{\mathbb{C}}_j\rangle\rangle = 0\}$.

**Definition 2.** $[0]_p = \{\vec{\mathbb{C}} \in \prod FPV[a,b] : \lim_{j\to\infty} \|\vec{\mathbb{C}}_j\|_p = 0\}$.

For example, if $\vec{\mathbb{C}} = (\vec{\mathbb{C}}_0, \vec{\mathbb{C}}_1, \vec{\mathbb{C}}_2, \ldots, \vec{\mathbb{C}}_n, \ldots) = ((a,b), (a, a + \frac{1}{2}(b-a), b), (a, a + \frac{1}{3}(b-a), a + \frac{2}{3}(b-a), b), \ldots, (a, a + \frac{1}{n}(b-a), a + \frac{2}{n}(b-a), \ldots, a + \frac{n-1}{n}(b-a), b), \ldots)$, then one has

$$\lim_{j\to\infty} \langle\langle \vec{\mathbb{C}}_j \rangle\rangle = \lim_{j\to\infty} \frac{b-a}{j+1} = 0,$$

and

$$\lim_{j\to\infty} \|\vec{\mathbb{C}}_j\|_p = \lim_{j\to\infty} \sqrt[p]{j \cdot \left(\frac{b-a}{j}\right)^p} = 0,$$

i.e., $\vec{\mathbb{C}} \in [0]$ and $\vec{\mathbb{C}} \in [0]_p$. $[0]$ is the set of all the sequences of partitioned vectors whose Riemann norms converge to 0, while $[0]_p$ is the set of all the sequences of partition vectors whose $p$-norms converge to 0. Since Riemann norm and $p$-norm is equivalent, $[0] = [0]_p$.

## 2. Definitions

To compare the differences between our settings and the usual Riemann integral and to facilitate our introduction of the $p$-integral, we rephrase some terminologies regarding the Riemann integral and have the following definitions.

**Definition 3.** *(Riemann integrable) If there exists $s \in \mathbb{R}$ such that $\forall \epsilon > 0\ \exists \delta_\epsilon > 0$ such that $\forall \vec{B} \in FPV(\delta_\epsilon)$ and $\forall \vec{H} \in \prod[\vec{B}]$, $|A_f(\vec{B}, \vec{H}) - s| < \epsilon$, we say $f$ is Riemann-integrable, denoted by $(R) \int_a^b f = s$. Furthermore, we use $R[a,b]$ to denote the set of all the Riemann-integrable functions defined on $[a,b]$.*

Let us define a partial ordering $\preceq$ on $FPV[a,b]$. For any $\vec{P}, \vec{Q} \in FPV[a,b]$, $\vec{P} \preceq \vec{Q}$ if and only if $\forall j \in \{1,2,...,|\vec{Q}|\}\ \exists k \in \{1,2,...,|\vec{P}|\}$ such that $\vec{Q}(k) = \vec{P}(j)$. If $\vec{P} \preceq \vec{Q}$, we say $P$ is a refinement of $\vec{Q}$. Then, we define the set of all the refinements of $\vec{Q}$ by $REF(\vec{Q}) = \{\vec{P} \in FPV[a,b] : \vec{P} \preceq \vec{Q}\}$. One could easily check that $\forall \delta > 0$ and $\forall \vec{B} \in FPV(\delta)[REF(\vec{B}) \subseteq FPV(\delta)]$.

**Definition 4.** *If there exists $s \in \mathbb{R}$ such that $\forall \epsilon' > 0\ \exists \vec{B}^{\epsilon'} \in FPV[a,b]\ \forall \vec{B} \in REF(\vec{B}^{\epsilon'})\ \forall \vec{H} \in \prod[\vec{B}] |A_f(\vec{B}, \vec{H}) - s| < \epsilon'$, we say $f$ is refinement-integrable, denoted by $(REF) \int_a^b f = s$. Furthermore, we use $REF[a,b]$ to denote the set of all the refinement-integrable functions defined on $[a,b]$.*

Let us define the upper Darboux sum of $f$ with respect to $\vec{B}$ as follows:

$$\bar{A}_f(\vec{B}) = \sum_{j=0}^{|\vec{B}|-1} [B_j - B_{j-1}] \cdot f^*([B_{j-1}, B_j]),$$

where $f^*([B_{j-1}, B_j]) = sup\{f(x) : x \in [B_{j-1}, B_j]\}$ and the lower Darboux sum of $f$ with respect to $\vec{B}$ as follows:

$$\underline{A}_f(\vec{B}) = \sum_{j=0}^{|\vec{B}|-1} [B_j - B_{j-1}] \cdot f_*([B_{j-1}, B_j]),$$

where $f_*([B_{j-1}, B_j]) = inf\{f(x) : x \in [B_{j-1}, B_j]\}$. Since $f$ is a bounded function, these definitions are all well-defined.

**Definition 5.** *(Darboux Cauchy integrable [5,6]) If*

$$\forall \hat{\epsilon} > 0\ \exists \vec{B}^{\hat{\epsilon}} \in FPV[a,b] \text{ such that } \bar{A}_f(\vec{B}^{\hat{\epsilon}}) - \underline{A}_f(\vec{B}^{\hat{\epsilon}}) < \hat{\epsilon},$$

*we say $f$ is Darboux Cauchy integrable and use $DC[a,b]$ to denote the set of all the Darboux Cauchy integrable functions defined on $[a,b]$.*

Define $\underline{\int}_a^b f = sup\{\bar{A}_f(\vec{B}) : \vec{B} \in FPV[a,b]\}$ and $\bar{\int}_a^b f = inf\{\bar{A}^f(\vec{B}) : \vec{B} \in FPV[a,b]\}$. Since $\forall \vec{B} \in FPV[a,b][\underline{A}_f(\vec{B}) \leq \bar{A}_f(\vec{B})]$, one has $\underline{\int}_a^b f \leq \bar{\int}_a^b f$.

**Definition 6.** *(Darboux integrable [7], p. 120) If there exists $s \in \mathbb{R}$ such that $s = \underline{\int}_a^b f = \bar{\int}_a^b f$, we say $f$ is Darboux integrable, denoted by $(DI) \int_a^b f = s$. Furthermore, we use $DI[a,b]$ to denote the set of all the Darboux integrable functions defined on $[a,b]$.*

Based on $p$-norm, we have the following new definitions.

**Definition 7.** *($p$-integrable) If there exists $s \in \mathbb{R}$ such that $\forall \tilde{\epsilon} > 0\ \exists \tilde{\delta}_{\tilde{\epsilon}} > 0$ such that $\forall \vec{B} \in FPV_p(\tilde{\delta}_{\tilde{\epsilon}})\ \forall \vec{H} \in \prod[\vec{B}]|A_f(\vec{B},\vec{H}) - s| < \tilde{\epsilon}$, we say $f$ is $p$-integrable, denoted by $(p) \int_a^b f = s$. Furthermore, we use $p[a,b]$ to denote the set of all the $p$-integrable functions defined on $[a,b]$.*

**Example 1.** *In this example, we demonstrate that if $f, g \in p[a,b]$, then $\int_a^b f(x)x + \int_a^b g(x)dx = \int_a^b (f + g)(x)dx$. Suppose $\int_a^b f(x)dx = c$ and $\int_a^b g(x)dx = d$. Let $\tilde{\epsilon} > 0$ be arbitrary. By $f, g \in p[a,b]$, one has there exist $c, d$ and there exist $\tilde{\delta}^f_{\frac{\tilde{\epsilon}}{2}}$ , $\tilde{\delta}^g_{\frac{\tilde{\epsilon}}{2}}$ such that*

$$\forall \vec{B} \in FPV_p(\tilde{\delta}^f_{\frac{\tilde{\epsilon}}{2}})\ \forall \vec{H} \in \prod[\vec{B}]|A_f(\vec{B},\vec{H}) - c| < \frac{\tilde{\epsilon}}{2},$$

*and*

$$\forall \vec{B} \in FPV_p(\tilde{\delta}^g_{\frac{\tilde{\epsilon}}{2}})\ \forall \vec{H} \in \prod[\vec{B}]|A_f(\vec{B},\vec{H}) - c| < \frac{\tilde{\epsilon}}{2}.$$

*Let $\tilde{\delta}_{\tilde{\epsilon}} = min\{\tilde{\delta}^f_{\frac{\tilde{\epsilon}}{2}}, \tilde{\delta}^g_{\frac{\tilde{\epsilon}}{2}}\}$. Let $\vec{B} \in FPV_p(\tilde{\delta}_{\tilde{\epsilon}})$ and let $\vec{H} \in \prod[\vec{B}]$ be arbitrary. By the property that $FPV_p(\tilde{\delta}_{\tilde{\epsilon}}) \subseteq FPV_p(\tilde{\delta}^f_{\frac{\tilde{\epsilon}}{2}}), FPV_p(\tilde{\delta}^g_{\frac{\tilde{\epsilon}}{2}})$, one has*

$$\left|\sum_{j=0}^{|\vec{B}|-1}(B_{j+1} - B_j)\cdot f(H_j) - c\right| < \frac{\tilde{\tilde{\epsilon}}}{2},$$

$$\left|\sum_{j=0}^{|\vec{B}|-1}(B_{j+1} - B_j)\cdot g(H_j) - d\right| < \frac{\tilde{\tilde{\epsilon}}}{2},$$

*i.e.,*

$$\left|\sum_{j=0}^{|\vec{B}|-1}(B_{j+1} - B_j)(f+g)(H_j) - (c+d)\right| \leq \left|\sum_{j=0}^{|\vec{B}|-1}(B_{j+1} - B_j)\cdot f(H_j) - c\right| + \left|\sum_{j=0}^{|\vec{B}|-1}(B_{j+1} - B_j)\cdot g(H_j) - d\right| < \tilde{\epsilon},$$

*i.e., $f + g \in p[a,b]$ and*

$$\int_a^b (f+g)(x)dx = c + d = \int_a^b f(x)x + \int_a^b g(x)dx.$$

Since the main purpose of this article is to show all sorts of equivalences of integrals and since the Riemann integral and the $p$-norm integral are equivalent, we do not focus too much on how to reinvent the wheel. The interested readers could simply use the $p$-norm integral to construct a whole parallel results with respect to the Riemann integral.

**Definition 8.** *(Discrete Darboux Cauchy integrable) If there exists* $s \in \mathbb{R}$ *such that* $\forall \ddot{\epsilon} > 0\, \forall \vec{\mathbb{C}} \in [0]_p\, \exists \ddot{N}_{\ddot{\epsilon},\vec{\mathbb{C}}}$ *such that* $\forall n \geq \ddot{N}_{\ddot{\epsilon},\vec{\mathbb{C}}} [\bar{A}_f(\vec{\mathbb{C}}_n) - \underline{A}_f(\vec{\mathbb{C}}_n) < \ddot{\epsilon}]$*, we say* $f$ *is discrete Darboux Cauchy integrable, denoted by* $(DDC_p) \int_a^b f = s$*. Furthermore, we use* $DDC_p[a,b]$ *to denote the set of all the discrete Darboux Cauchy integrable functions defined on* $[a,b]$*.*

**Definition 9.** *(Discrete* $p-$*integrable) If there exists* $s \in \mathbb{R}$ *such that* $\forall \check{\epsilon} > 0\, \forall \vec{\mathbb{C}} \in [0]_p\, \exists \check{N}_{\check{\epsilon},\vec{\mathbb{C}}} \in \mathbb{N}$ *such that* $\forall m \geq \check{N}_{\check{\epsilon},\vec{\mathbb{C}}}\, \forall \vec{H} \in \prod[\vec{\mathbb{C}}_m] |A_f(\vec{\mathbb{C}}_m, \vec{H}) - s| < \check{\epsilon}$*, we say* $f$ *is discrete* $p$*-integrable, denoted by* $(Dp) \int_a^b f = s$*. Furthermore, we use* $Dp[a,b]$ *to denote the set of all the discrete* $p$*-integrable functions defined on* $[a,b]$*.*

**Definition 10.** *(Ranged Darboux Cauchy integrable) If there exists* $s \in \mathbb{R}$ *such that* $\forall \bar{\epsilon} > 0\, \exists \bar{\delta}_{\bar{\epsilon}} > 0$ *such that* $\forall \vec{B} \in FPV_p(\bar{\delta}_{\bar{\epsilon}})[\bar{A}_f(\vec{B}) - \underline{A}_f(\vec{B}) < \bar{\epsilon}]$*, we say* $f$ *is ranged Darboux Cauchy integrable, denoted by* $(RDC_p) \int_a^b f = s$*. Furthermore, we use* $RDC_p[a,b]$ *to denote the set of all the ranged Darboux Cauchy integrable functions defined on* $[a,b]$*.*

## 3. Theorems

**Claim 1.** $R[a,b] \subseteq REF[a,b]$.

**Proof.** Let $f \in R[a,b]$ be arbitrary. We show $f \in REF[a,b]$. Let $\epsilon' > 0$ be arbitrary. By Definition 3, it follows

$$\exists \delta_{\epsilon'} > 0 \text{ such that } \forall \vec{B} \in FPV(\delta_{\epsilon'})\, \forall \vec{H} \in \prod[\vec{B}]\; |A_f(\vec{B}, \vec{H}) - s| < \epsilon', \tag{1}$$

for some $s \in \mathbb{R}$. Choose one $\vec{B}^{\epsilon'} \in FPV(\delta_{\epsilon'})$. Then, $REF(\vec{B}^{\epsilon'}) \subseteq FPV(\delta_{\epsilon'})$ and thus, by (1),

$$\forall \vec{B} \in REF(\vec{B}^{\epsilon'})\, \forall \vec{H} \in \prod[\vec{B}]\; |A_f(\vec{B}, \vec{H}) - s| < \epsilon'.$$

By Definition 4, this completes our proof. □

**Claim 2.** $R[a,b] = DC[a,b] = DI[a,b]$.

**Proof.** The proofs could be found in ([4], Theorem 6.6) and ([7], Theorems 3.2.6 and 3.2.7). □

**Lemma 1.** $R[a,b] = REF[a,b]$.

**Proof.** Due to Claims 1 and 2, it suffices to show $REF[a,b] \subseteq DC[a,b]$. Let $f \in REF[a,b]$ be arbitrary. We show $f \in DC[a,b]$. Let $\hat{\epsilon} > 0$ be arbitrary. Then, by Definition 4

$$\exists \vec{B}^{\frac{\hat{\epsilon}}{4}} \in FPV[a,b]\, \forall \vec{B} \in REF(\vec{B}^{\frac{\hat{\epsilon}}{4}})\, \forall \vec{H} \in \prod[\vec{B}]\; |A_f(\vec{B}, \vec{H}) - s| < \frac{\hat{\epsilon}}{4} \tag{2}$$

for some $s \in \mathbb{R}$. Furthermore, by the definitions of upper and lower Darboux sums of $f$ with respect to $\vec{B}^{\frac{\hat{\epsilon}}{4}}$, one has $\exists \vec{H}^{\frac{\hat{\epsilon}}{4}}, \vec{K}^{\frac{\hat{\epsilon}}{4}} \in \prod[\vec{B}^{\frac{\hat{\epsilon}}{4}}]$ such that

$$\begin{cases} \bar{A}_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}) - \frac{\hat{\epsilon}}{4} < A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{H}^{\frac{\hat{\epsilon}}{4}}) \\ A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{K}^{\frac{\hat{\epsilon}}{4}}) < \underline{A}_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}) + \frac{\hat{\epsilon}}{4}. \end{cases} \tag{3}$$

By Equation (3), it follows directly that

$$\begin{aligned}\bar{A}_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}) - \underline{A}_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}) - \frac{\hat{\epsilon}}{2} &< A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{H}^{\frac{\hat{\epsilon}}{4}}) - A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{K}^{\frac{\hat{\epsilon}}{4}}) \\ &\leq |A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{H}^{\frac{\hat{\epsilon}}{4}}) - A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{K}^{\frac{\hat{\epsilon}}{4}})| \\ &= |(A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{H}^{\frac{\hat{\epsilon}}{4}}) - s) + (-A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{K}^{\frac{\hat{\epsilon}}{4}}) + s)| \\ &\leq |(A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{H}^{\frac{\hat{\epsilon}}{4}}) - s)| + |A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{K}^{\frac{\hat{\epsilon}}{4}}) - s|.\end{aligned} \tag{4}$$

By Equations (2) and (4), one has $\bar{A}_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}) - \underline{A}_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}) < \hat{\epsilon}$. Hence, by Definition 5, $f \in DC[a,b]$. $\square$

**Claim 3.** $\|\vec{K}\|_p < \gamma \Rightarrow \langle\langle\vec{K}\rangle\rangle < \gamma, \forall \vec{K} \in FPV[a,b]$ .

**Proof.** By definition, $\|\vec{K}\|_p = \sqrt[p]{\sum_{i=0}^{|\vec{K}|-1} |\vec{K}_{i+1} - \vec{K}_i|^p} < \gamma$, i.e.,

$$\forall i \in \{0, 1, 2, ..., |\vec{K} - 1|\}[|\vec{K}_{i+1} - \vec{K}_i| < \gamma],$$

i.e., by definition $\langle\langle\vec{K}\rangle\rangle < \gamma$. $\square$

By this, we know $FPV_p(\delta) \subsetneq FPV(\delta)$, since the equality is obviously false.

**Lemma 2.** $R[a,b] \subseteq p[a,b]$.

**Proof.** Let $f \in R[a,b]$ be arbitrary. We show $f \in p[a,b]$. Let $\tilde{\epsilon} > 0$ be arbitrary. Then, by Definition 3

$$\exists \delta_{\tilde{\epsilon}} > 0 \text{ such that } \forall \vec{B} \in FPV(\delta_{\tilde{\epsilon}})\ \forall \vec{H} \in \prod[\vec{B}]\ |A_f(\vec{B}, \vec{H}) - s| < \tilde{\epsilon}, \tag{5}$$

for some $s \in \mathbb{R}$. Let us take $\tilde{\delta}_{\tilde{\epsilon}} = \delta_{\tilde{\epsilon}}$. Let $\vec{B} \in FPV_p(\tilde{\delta}_{\tilde{\epsilon}}) = FPV_p(\delta_{\tilde{\epsilon}})$ be arbitrary. Then, by definition, one has $\|\vec{B}\|_p = \sqrt[p]{\sum_{j=0}^{|\vec{B}|-1} |\vec{B}_{j+1} - \vec{B}_j|^p} < \tilde{\delta}_{\tilde{\epsilon}}$, i.e., by Claim 3, $\langle\langle\vec{B}\rangle\rangle < \tilde{\delta}_{\tilde{\epsilon}}$, i.e., $FPV_p(\tilde{\delta}_{\tilde{\epsilon}}) \subsetneq FPV(\tilde{\delta}_{\tilde{\epsilon}}) = FPV(\delta_{\tilde{\epsilon}})$, then the result $f \in p[a,b]$ follows immediately from Equation (5) and Definition 7. $\square$

Since each $FPV_p(\delta)$ is a proper subset of $FPV(\delta)$, it appears that $R[a,b] = p[a,b]$ is doubtful. However, the following lemma proves otherwise.

**Lemma 3.** $p[a,b] \subseteq REF[a,b]$.

**Proof.** Let $f \in p[a,b]$ be arbitrary. We show $f \in REF[a,b]$. Let $\epsilon' > 0$ be arbitrary. Then, by Definition 7

$$\exists \tilde{\delta}_{\epsilon'} > 0 \text{ such that } \forall \vec{B} \in FPV_p(\tilde{\delta}_{\epsilon'})\ \forall \vec{H} \in \prod[\vec{B}]\ |A_f(\vec{B}, \vec{H}) - s| < \epsilon', \tag{6}$$

for some $s \in \mathbb{R}$. Let us arbitrarily choose one $\vec{B}^{\epsilon'} \in FPV_p(\tilde{\delta}_{\epsilon'})$. Let $\vec{B} \in REF(\vec{B}^{\epsilon'})$ be arbitrary. Then,

$$\|\vec{B}\|_p = \sqrt[p]{\sum_{i=0}^{|\vec{B}|-1} |\vec{B}_{i+1} - \vec{B}_i|^p} \leq \|\vec{B}^{\epsilon'}\|_p = \sqrt[p]{\sum_{j=0}^{|\vec{B}^{\epsilon'}|-1} |\vec{B}^{\epsilon'}_{j+1} - \vec{B}^{\epsilon'}_j|^p} < \tilde{\delta}_{\epsilon'},$$

i.e., $REF(\vec{B}^{\epsilon'}) \subseteq FPV_p(\tilde{\delta}_{\epsilon'})$. Then, the result $f \in REF[a,b]$ follows immediately from Equation (6). $\square$

**Theorem 1.** $p[a,b] = R[a,b]$.

**Proof.** From Lemmas 1–3, one has

$$R[a,b] \subseteq p[a,b] \subseteq REF[a,b] = R[a,b]$$

and thus the result follows immediately. □

**Example 2.** *Take $[a,b] = [0,1]$ and $f(x) = x$ for example. Then, $f \in R[a,b]$. Here, based on the definition of p-norm integral, we give a detailed proof showing $f \in p[a,b]$. Let $\tilde{\epsilon} > 0$ be arbitrary. Let $N_{\tilde{\epsilon}} \in \mathbb{N}$ be the natural number satisfying $\frac{1}{N_{\tilde{\epsilon}}} \leq \tilde{\epsilon} < \frac{1}{N_{\tilde{\epsilon}}-1}$ for all $N_{\tilde{\epsilon}} \geq 2$. Observe that when $\tilde{\epsilon}$ decreases, $N_{\tilde{\epsilon}}$ increases. Now, take $\tilde{\delta}_{\tilde{\epsilon}} = \frac{1}{N_{\tilde{\epsilon}}}$. Let $\vec{B} \in FPV_p(\frac{1}{N_{\tilde{\epsilon}}})$ and $\vec{H} \in \prod[\vec{B}]$ be arbitrary. Then,*

$$\sum_{j=0}^{|\vec{B}|-1} (\vec{B}_{j+1} - \vec{B}_j) \cdot \vec{B}_j \leq A_f(\vec{B}, \vec{H}) = \sum_{j=0}^{|\vec{B}|-1} (\vec{B}_{j+1} - \vec{B}_j) \cdot f(\vec{H}_j) \leq \sum_{j=0}^{|\vec{B}|-1} (\vec{B}_{j+1} - \vec{B}_j) \cdot \vec{B}_{j+1}.$$

*Since $\vec{B} \in FPV_p(\frac{1}{N_{\tilde{\epsilon}}})$, one has*

$$(\sum_{j=0}^{|\vec{B}|-1} \vec{B}_j^p)^{\frac{1}{p}} < \frac{1}{N_{\tilde{\epsilon}}},$$

*i.e., $(\sum_{j=0}^{|\vec{B}|-1} \vec{B}_j^p) < (\frac{1}{N_{\tilde{\epsilon}}})^p$, i.e., $\vec{B}_j < \frac{1}{N_{\tilde{\epsilon}}}$ for all $0 \leq j \leq |\vec{B}| - 1$. Hence, one has*

$$\frac{1}{N_{\tilde{\epsilon}}} \cdot [\frac{1}{N_{\tilde{\epsilon}}} + 2\frac{1}{N_{\tilde{\epsilon}}} + ... + (N_{\tilde{\epsilon}} - 1) \cdot \frac{1}{N_{\tilde{\epsilon}}}] \leq \sum_{j=0}^{|\vec{B}|-1} (\vec{B}_{j+1} - \vec{B}_j) \cdot \vec{B}_j,$$

$$\sum_{j=0}^{|\vec{B}|-1} (\vec{B}_{j+1} - \vec{B}_j) \cdot \vec{B}_{j+1} \leq \frac{1}{N_{\tilde{\epsilon}}} \cdot [\frac{1}{N_{\tilde{\epsilon}}} + 2\frac{1}{N_{\tilde{\epsilon}}} + ... + N_{\tilde{\epsilon}} \cdot \frac{1}{N_{\tilde{\epsilon}}}].$$

*Hence, one has*

$$(\frac{1}{N_{\tilde{\epsilon}}})^2 \cdot [\frac{(N_{\tilde{\epsilon}} - 1)N_{\tilde{\epsilon}}}{2}] \leq A_f(\vec{B}, \vec{H}) \leq (\frac{1}{N_{\tilde{\epsilon}}})^2 \cdot [(\frac{(N_{\tilde{\epsilon}})(N_{\tilde{\epsilon}} + 1)}{2}],$$

*i.e.,*

$$\frac{1}{2} - \frac{1}{N_{\tilde{\epsilon}}} \leq A_f(\vec{B}, \vec{H}) \leq \frac{1}{2} + \frac{1}{N_{\tilde{\epsilon}}}.$$

*Since $\lim_{\tilde{\epsilon} \to 0} \frac{1}{N_{\tilde{\epsilon}}} = 0$, one has $f \in p[a,b]$.*

**Lemma 4.** *$Dp[a,b] \subseteq DC[a,b]$.*

**Proof.** Let $f \in Dp[a,b]$ be arbitrary, we show $f \in DC[a,b]$. Let $\vec{\mathbb{C}} \in [0]_p$ be arbitrary. Let $\hat{\epsilon} > 0$ be arbitrary. Then, by Definition 9, $\exists \check{N}_{\frac{\hat{\epsilon}}{4}, \vec{\mathbb{C}}} \in \mathbb{N}$ such that

$$\forall m \geq \check{N}_{\frac{\hat{\epsilon}}{4}, \vec{\mathbb{C}}} \; \forall \vec{H} \in \prod[\vec{\mathbb{C}}_m] \; |A_f(\vec{\mathbb{C}}_m, \vec{H}) - s| < \frac{\hat{\epsilon}}{4}, \tag{7}$$

for some $s \in \mathbb{R}$. Take $\vec{B}^{\hat{\epsilon}} = \vec{\mathbb{C}}_{\check{N}_{\frac{\hat{\epsilon}}{4}, \vec{\mathbb{C}}}}$. Then, by the definitions of upper and lower Darboux integral, one has $\exists \vec{H}^{\frac{\hat{\epsilon}}{4}}, \vec{K}^{\frac{\hat{\epsilon}}{4}} \in \prod[\vec{B}^{\frac{\hat{\epsilon}}{4}}]$ such that

$$\begin{cases} \bar{A}_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}) - \frac{\hat{\epsilon}}{4} < A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{H}^{\frac{\hat{\epsilon}}{4}}) \\ A_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}, \vec{K}^{\frac{\hat{\epsilon}}{4}}) < \underline{A}_f(\vec{B}^{\frac{\hat{\epsilon}}{4}}) + \frac{\hat{\epsilon}}{4}. \end{cases}$$

Then, by the identical proof of Equation (4) in Lemma 1, the result follows. □

**Lemma 5.** $R[a,b] \subseteq Dp[a,b]$.

**Proof.** Let $f \in R[a,b]$ be arbitrary. We show $f \in D_p[a,b]$. Let $\check{\epsilon} > 0, \vec{\mathbb{C}} \in [0]_p$ be arbitrary. By Definition 3

$$\exists \delta_{\check{\epsilon}} > 0 \, \forall \vec{B} \in FPV(\delta_{\check{\epsilon}}) \, \forall \vec{H} \in \prod[\vec{B}] \, |A_f(\vec{B}, \vec{H}) - s| < \check{\epsilon}, \tag{8}$$

for some $s \in \mathbb{R}$. By $\vec{\mathbb{C}} \in [0]_p$, it follows

$$\forall \gamma > 0 \, \exists N_\gamma \in \mathbb{N} \text{ such that } \forall k \geq N_\gamma [\|\vec{\mathbb{C}}_k\|_p < \gamma].$$

Now, take $\gamma = \delta_{\check{\epsilon}}$, one has $\exists N_{\delta_{\check{\epsilon}}} \in \mathbb{N}$ such that $\forall k \geq N_{\delta_{\check{\epsilon}}} \|\vec{\mathbb{C}}_k\|_p < \delta_{\check{\epsilon}}$, i.e., by Claim 3, $\langle\langle \vec{\mathbb{C}}_k \rangle\rangle < \delta_{\check{\epsilon}}$, i.e., $FPV_p(\delta_{\check{\epsilon}}) \subseteq FPV(\delta_{\check{\epsilon}})$. Take $\check{N}_{\check{\epsilon},\vec{\mathbb{C}}} = N_{\delta_{\check{\epsilon}}}$. Let $m \geq \check{N}_{\check{\epsilon},\vec{\mathbb{C}}} = N_{\delta_{\check{\epsilon}}}$, $\vec{H} \in \prod[\vec{\mathbb{C}}_m]$ be arbitrary. Then, the result $f \in Dp[a,b]$ follows immediately from Equation (8) and Definition 9. □

**Theorem 2.** $R[a,b] = Dp[a,b]$.

**Proof.** From Claim 2, and Lemmas 4 and 5, one has

$$R[a,b] \subseteq Dp[a,b] \subseteq DC[a,b] = R[a,b]$$

and thus the result follows immediately. □

**Lemma 6.** $DDC_p[a,b] \subseteq DC[a,b]$.

**Proof.** Let $f \in DDC_p[a,b]$ be arbitrary. We show $f \in DC[a,b]$. Let $\hat{\epsilon} > 0$ be arbitrary. Let $\vec{\mathbb{C}} \in [0]_p$ be arbitrary. By Definition 8,

$$\exists \ddot{N}_{\hat{\epsilon},\vec{\mathbb{C}}} \text{ such that } \forall n \geq \ddot{N}_{\hat{\epsilon},\vec{\mathbb{C}}} [\bar{A}_f(\vec{\mathbb{C}}^n) - \underline{A}_f(\vec{\mathbb{C}}^n) < \hat{\epsilon}].$$

Now, take $\vec{B}^{\hat{\epsilon}} = \vec{\mathbb{C}}_{\ddot{N}_{\hat{\epsilon},\vec{\mathbb{C}}}}$. Then, $\bar{A}_f(\vec{B}^{\hat{\epsilon}}) - \underline{A}_f(\vec{B}^{\hat{\epsilon}}) < \hat{\epsilon}$ and thus the result $f \in DC[a,b]$ follows immediately from Definition 5. □

**Lemma 7.** $Dp[a,b] \subseteq DDC_p[a,b]$.

**Proof.** Let $f \in D_p[a,b]$ be arbitrary. We show $f \in DDC_p[a,b]$. Let $\ddot{\epsilon} > 0$, $\vec{\mathbb{C}} \in [0]_p$ be arbitrary. By $f \in Dp[a,b]$, it follows

$$\exists N_{\frac{\ddot{\epsilon}}{4},\vec{\mathbb{C}}} \in \mathbb{N} \text{ such that } \forall m \geq N_{\frac{\ddot{\epsilon}}{4},\vec{\mathbb{C}}} \, \forall \vec{H} \in \prod[\vec{\mathbb{C}}_m] |A_f(\vec{\mathbb{C}}_m, \vec{H}) - s| < \frac{\ddot{\epsilon}}{4}, \tag{9}$$

for some $s \in \mathbb{R}$. Now, take $\ddot{N}_{\ddot{\epsilon},\vec{\mathbb{C}}} = N_{\frac{\ddot{\epsilon}}{4},\vec{\mathbb{C}}}$. Let $n \geq \ddot{N}_{\ddot{\epsilon},\vec{\mathbb{C}}}$ be arbitrary. By the definitions of upper and lower Darboux integral, $\exists \vec{H}, \vec{K} \in \prod[\vec{\mathbb{C}}_n]$ such that

$$\begin{cases} \bar{A}_f(\vec{\mathbb{C}}_n) - \frac{\ddot{\epsilon}}{4} < A_f(\vec{\mathbb{C}}_n, \vec{K}), \\ A_f(\vec{\mathbb{C}}_n, \vec{H}) < \underline{A}_f(\vec{\mathbb{C}}_n) + \frac{\ddot{\epsilon}}{4}. \end{cases}$$

Thus,

$$\bar{A}_f(\vec{\mathbb{C}}_n) - \underline{A}_f(\vec{\mathbb{C}}_n) < A_f(\vec{\mathbb{C}}_n, \vec{K}) - A_f(\vec{\mathbb{C}}_n, \vec{H}) + \frac{\ddot{\epsilon}}{2}$$
$$< |A_f(\vec{\mathbb{C}}_n, \vec{K}) - s| + |A_f(\vec{\mathbb{C}}_n, \vec{H}) - s| + \frac{\ddot{\epsilon}}{2} < \frac{\ddot{\epsilon}}{4} + \frac{\ddot{\epsilon}}{4} + \frac{\ddot{\epsilon}}{2} = \ddot{\epsilon}.$$

Thus, the result $f \in DDC_p[a,b]$ follows immediately from Definition 8. □

**Theorem 3.** $DDC_p[a,b] = DC[a,b]$.

**Proof.** By Claim 2, Theorem 2, and Lemmas 6 and 7,

$$DC[a,b] = R[a,b] = Dp[a,b] \subseteq DDC_p[a,b] \subseteq DC[a,b]$$

and thus the result follows immediately. □

**Lemma 8.** $RDC_p[a,b] \subseteq DC[a,b]$.

**Proof.** Let $f \in RDC_p[a,b]$ be arbitrary. We show $f \in DC[a,b]$. Let $\hat{\epsilon} > 0$ be arbitrary. By Definition 10,

$$\exists \bar{\delta}_{\hat{\epsilon}} > 0 \text{ such that } \forall \vec{B} \in FPV_p(\bar{\delta}_{\hat{\epsilon}}) |\bar{A}_f(\vec{B}) - \underline{A}_f(\vec{B})| < \hat{\epsilon}.$$

Now, choose one $\vec{B}^{\hat{\epsilon}} \in FPV_p(\bar{\delta}_{\hat{\epsilon}}) \subseteq FPV[a,b]$, then the result $f \in DC[a,b]$ follows immediately via Definition 5. □

**Lemma 9.** $R[a,b] \subseteq RDC_p[a,b]$.

**Proof.** Let $f \in R[a,b]$ be arbitrary. We show $f \in RDC_p[a,b]$. Let $\bar{\epsilon} > 0$ be arbitrary. By Definition 3,

$$\exists \delta_{\frac{\bar{\epsilon}}{4}} > 0 \text{ such that } \forall \vec{B} \in FPV(\delta_{\frac{\bar{\epsilon}}{4}})\ \forall \vec{H} \in \prod[\vec{B}] |A_f(\vec{B}, \vec{H}) - s| < \frac{\bar{\epsilon}}{4}, \tag{10}$$

for some $s \in \mathbb{R}$. Now, take $\bar{\delta}_{\bar{\epsilon}} = \delta_{\frac{\bar{\epsilon}}{4}}$. Let $\vec{B} \in FPV_p(\bar{\delta}_{\bar{\epsilon}})$ be arbitrary. By Claim 3, $\vec{B} \in FPV_p(\bar{\delta}_{\bar{\epsilon}}) \subsetneq FPV(\bar{\delta}_{\bar{\epsilon}}) = FPV(\delta_{\frac{\bar{\epsilon}}{4}})$. By the definitions of upper and lower Darboux integral, one has $\exists \vec{H}, \vec{K} \in \prod[\vec{B}]$ such that

$$\begin{cases} A_f(\vec{B}, \vec{H}) < \underline{A}_f(\vec{B}) + \frac{\bar{\epsilon}}{4} \\ \bar{A}_f(\vec{B}) - \frac{\bar{\epsilon}}{4} < A_f(\vec{B}, \vec{K}). \end{cases}$$

Thus, by Equation (10) and the identical proof of Equation (4) in Lemma 1, one has

$$\bar{A}_f(\vec{B}) - \underline{A}_f(\vec{B}) < \bar{\epsilon}.$$

Thus the result $f \in DDC_p[a,b]$ follows immediately from Definition 10. □

**Theorem 4.** $RDC_p[a,b] = DC[a,b]$.

**Proof.** From Claim 2, and Lemmas 8 and 9, $DC[a,b] = R[a,b] \subseteq RDC_p[a,b] \subseteq DC[a,b]$, the result follows immediately. □

**Theorem 5.** $R[a,b] = REF[a,b] = DC[a,b] = DI[a,b] = p[a,b] = Dp[a,b] = DDC_p[a,b] = RDC_p[a,b]$.

**Proof.** By Lemma 1, Claim 2, and Theorems 1–4, the result follows immediately. □

## 4. Conclusions

In this article, we have shown how to give a new equivalent definition of the Riemann integral based on the $p$-norm. This norm could easily link functional analysis and the Riemann integral.

By directly defining an alternative definition based on this norm, one gains more insightful knowledge about all sorts of equivalences of the Riemann integral. Furthermore, we also derive some related equivalences based on this norm. The $p$-norm provides an alternative to the usual definition of the Riemann integral.

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