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Article

Convergence in Fuzzy Semi-Metric Spaces

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Abstract: The convergence using the fuzzy semi-metric and dual fuzzy semi-metric is studied in this paper. The infimum type of dual fuzzy semi-metric and the supremum type of dual fuzzy semi-metric are proposed in this paper. Based on these two types of dual fuzzy semi-metrics, the different types of triangle inequalities can be obtained. We also study the convergence of these two types of dual fuzzy semi-metrics.

Keywords: dual fuzzy semi-metric; fuzzy semi-metric space; metric convergence; triangle inequality; triangular norm

MSC: 54E35; 54H25

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1. Introduction

Let $*$ be a t-norm, and let M be a mapping defined on $X \times X \times [0, \infty)$ into $[0, 1]$. The three-tuple $(X, M, *)$ is called a fuzzy metric space if and only if some required conditions are satisfied. For different researchers, the required conditions are slightly different. However, the following symmetric condition and triangle inequality will be included

- (Symmetric condition) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t \geq 0$;
- (Triangle inequality) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t \geq 0$.

George and Veeramani [1,2] studied some properties of fuzzy metric spaces. Gregori and Romaguera [3–5] also extended this to study the properties of fuzzy metric spaces and fuzzy quasi-metric spaces. The Hausdorff topology induced by the fuzzy metric space was studied in Wu [6]. Gregori and Romaguera [4] proposed the fuzzy quasi-metric spaces in which the symmetric condition was not assumed. Wu [7] studied the so-called fuzzy semi-metric space without assuming the symmetric condition in which four forms of triangle inequalities are considered. The common coincidence points and common fixed points in fuzzy semi-metric spaces were also studied in Wu [8].

Since the symmetric condition is not satisfied in the fuzzy semi-metric space, three kinds of limit concepts will be considered in this paper. Based on these limit concepts, we shall study the metric convergence in this paper. On the other hand, we also propose the concepts of the dual fuzzy semi-metric. We shall separately study the infimum type of dual fuzzy semi-metric and the supremum type of dual fuzzy semi-metric. Under these settings, we shall also investigate the convergence of the dual fuzzy semi-metric. The potential application for using the convergence of dual fuzzy semi-metric is to study the fixed point theorems in fuzzy semi-metric space by considering the Cauchy sequences, which will be the future research.

This paper is organized as follows. In Section 2, the basic properties of fuzzy semi-metric space are presented that will be used for the further discussion. In Section 3, since the symmetric condition is not necessarily satisfied, we study the metric convergence based on the different concepts of limits. In Section 4, we propose the concept of the infimum type of dual fuzzy semi-metric and study its convergent properties. Four different types of triangle inequalities can be obtained for this kind of dual fuzzy semi-metric. In Section 5, we also propose the concept of the supremum type of dual fuzzy semi-metric and study its convergent properties and triangle inequalities.

2. Fuzzy Semi-Metric Space

In the sequel, we shall define the concept of fuzzy semi-metric space without considering the symmetric condition. Due to lacking symmetry, the concept of the triangle inequality should be carefully interpreted. First of all, the so-called fuzzy semi-metric space is defined below.

Definition 1. *Let X be a nonempty universal set, and let M be a mapping defined on $X \times X \times [0, \infty)$ into $[0, 1]$. Then, (X, M) is called a fuzzy semi-metric space if and only if the following conditions are satisfied*

- *for any $x, y \in X$, $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;*
- *$M(x, y, 0) = 0$ for all $x, y \in X$ with $x \neq y$.*

We say that M satisfies the symmetric condition if and only if $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$. We say that M satisfies the strongly symmetric condition if and only if $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t \geq 0$.

From the first condition, we see that the value of $M(x, x, 0)$ is free and $M(x, x, t) = 1$ for all $t > 0$. Since the value $M(x, y, t)$ is interpreted as the membership degree of the distance that is less than t between x and y, the value $M(x, x, t) = 1$ for all $t > 0$ means that the distance that is less than $t > 0$ between x and x is always true. Regarding the second condition, the value $M(x, y, 0) = 0$ for $x \neq y$ can be similarly realized such that the distance that is less than zero between two distinct elements x and y is impossible.

In order to consider the triangle inequalities, we need to introduce the concept of the t-norm. We say that the function $* : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm if and only if the following conditions are satisfied

- (Boundary condition) $a * 1 = a$.
- (Commutativity) $a * b = b * a$.
- (Increasing property) If $b < c$, then $a * b \leq a * c$.
- (Associativity) $(a * b) * c = a * (b * c)$.

Since the symmetric condition is not assumed to be true in fuzzy semi-metric space, four kinds of triangle inequalities proposed by Wu [7] are shown below.

Definition 2. *Let X be a nonempty universal set; let $*$ be a t-norm; and let M be a mapping defined on $X \times X \times [0, \infty)$ into $[0, 1]$.*

- *We say that M satisfies the $\bowtie$-triangle inequality if and only if the following inequality is satisfied*

  $$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

  We say that M satisfies the strict $\bowtie$-triangle inequality if and only if the inequality "$\leq$" is replaced by the strict inequality "$<$".
- *We say that M satisfies the $\triangleright$-triangle inequality if and only if the following inequality is satisfied*

  $$M(x, y, t) * M(z, y, s) \leq M(x, z, t + s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

  We say that M satisfies the strict $\triangleright$-triangle inequality if and only if the inequality "$\leq$" is replaced by the strict inequality "$<$".
- *We say that M satisfies the $\triangleleft$-triangle inequality if and only if the following inequality is satisfied*

  $$M(y, x, t) * M(y, z, s) \leq M(x, z, t + s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

We say that M satisfies the strict $\triangleleft$*-triangle inequality if and only if the inequality "*$\leq$*" is replaced by the strict inequality "*$<$*".*

- *We say that M satisfies the* $\diamond$*-triangle inequality if and only if the following inequality is satisfied*

$$M(y,x,t) * M(z,y,s) \leq M(x,z,t+s) \text{ for all } x,y,z \in X \text{ and } s,t > 0.$$

We say that M satisfies the strict $\diamond$*-triangle inequality if and only if the inequality "*$\leq$*" is replaced by the strict inequality "*$<$*".*

We say that M satisfies the strong $\circ$*-triangle inequality for* $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$ *when* $s,t > 0$ *is replaced by* $s,t \geq 0$*. The concept of the strong strict* $\circ$*-triangle inequality for* $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$ *can be similarly defined.*

The following observations will be used in the further study.

Remark 1. *Let* (X, M) *be a fuzzy semi-metric space.*

- *Suppose that M satisfies the* $\bowtie$*-triangle inequality. Then*

$$M(a,b,t_1) * M(b,c,t_2) * M(c,d,t_3) \leq M(a,c,t_1+t_2) * M(c,d,t_3) \leq M(a,d,t_1+t_2+t_3).$$

In general, we have

$$M\left(x_1, x_2, t_1\right) * M\left(x_2, x_3, t_2\right) * \cdots * M\left(x_p, x_{p+1}, t_p\right) \leq M\left(x_1, x_{p+1}, t_1+t_2+\cdots+t_p\right). \tag{1}$$

- *Suppose that M satisfies the* $\triangleright$*-triangle inequality. Since*

$$M(a,b,t_1) * M(c,b,t_2) \leq \min\left\{M(a,c,t_1+t_2), M(c,a,t_1+t_2)\right\},$$

this implies

$$M(a,b,t_1) * M(c,b,t_2) * M(d,c,t_3) \leq \min\left\{M(a,d,t_1+t_2+t_3), M(d,a,t_1+t_2+t_3)\right\}. \tag{2}$$

In general, we have

$$\begin{aligned} &M\left(x_1, x_2, t_1\right) * M\left(x_3, x_2, t_2\right) * M\left(x_4, x_3, t_3\right) * \cdots * M\left(x_{p+1}, x_p, t_p\right) \\ &\quad \leq \min\left\{M\left(x_1, x_{p+1}, t_1+t_2+\cdots+t_p\right), M\left(x_{p+1}, x_1, t_1+t_2+\cdots+t_p\right)\right\}. \end{aligned}$$

- *Suppose that M satisfies the* $\triangleleft$*-triangle inequality. Since*

$$M(b,a,t_1) * M(b,c,t_2) \leq \min\left\{M(a,c,t_1+t_2), M(c,a,t_1+t_2)\right\},$$

this implies

$$M(b,a,t_1) * M(b,c,t_2) * M(c,d,t_3) \leq \min\left\{M(a,d,t_1+t_2+t_3), M(d,a,t_1+t_2+t_3)\right\}. \tag{3}$$

In general, we have

$$\begin{aligned} &M\left(x_2, x_1, t_1\right) * M\left(x_2, x_3, t_2\right) * M\left(x_3, x_4, t_3\right) * \cdots * M\left(x_p, x_{p+1}\right) \\ &\quad \leq \min\left\{M\left(x_1, x_{p+1}, t_1+t_2+\cdots+t_p\right), M\left(x_{p+1}, x_1, t_1+t_2+\cdots+t_p\right)\right\}. \end{aligned}$$

- *Suppose that M satisfies the* $\diamond$*-triangle inequality. Then*

$$\begin{aligned} M(a,b,t_1) * M(b,c,t_2) * M(d,c,t_3) &= M(b,c,t_1) * M(a,b,t_2) * M(d,c,t_3) \\ &\leq M(c,a,t_1+t_2) * M(d,c,t_3) \leq M(a,d,t_1+t_2+t_3) \end{aligned} \tag{4}$$

and

$$
\begin{aligned}
M(b,a,t_1)*M(c,b,t_2)*M(c,d,t_3) &\le M(a,c,t_1+t_2)*M(c,d,t_3)\\
&= M(c,d,t_3)*M(a,c,t_1+t_2) \le M(d,a,t_1+t_2+t_3).
\end{aligned} \tag{5}
$$

From (4), we also have

$$
\begin{aligned}
M(a,b,t_1)*M(c,b,t_2)*M(d,c,t_3) &= M(d,c,t_3)*M(c,b,t_2)*M(a,b,t_1)\\
&\le M(d,a,t_1+t_2+t_3),
\end{aligned} \tag{6}
$$

which implies

$$
M(b,a,t_1)*M(b,c,t_2)*M(c,d,t_3) \ge M(a,d,t_1+t_2+t_3) \tag{7}
$$

by referring to (5). In general, we have the following cases.

(a) *If p is even, then*

$$
\begin{aligned}
&M(x_1,x_2,t_1)*M(x_2,x_3,t_2)*M(x_4,x_3,t_3)*M(x_4,x_5,t_4)*M(x_6,x_5,t_5)\\
&\quad *M(x_6,x_7,t_6)*\cdots*M\left(x_p,x_{p+1},t_p\right) \le M\left(x_{p+1},x_1,t_1+t_2+\cdots+t_p\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&M(x_2,x_1,t_1)*M(x_3,x_2,t_2)*M(x_3,x_4,t_3)*M(x_5,x_4,t_4)*M(x_5,x_6,t_5)\\
&\quad *M(x_7,x_6,t_6)*\cdots*M\left(x_p,x_{p+1},t_p\right) \le M\left(x_1,x_{p+1},t_1+t_2+\cdots+t_p\right).
\end{aligned}
$$

(b) *If p is odd, then*

$$
\begin{aligned}
&M(x_1,x_2,t_1)*M(x_2,x_3,t_2)*M(x_4,x_3,t_3)*M(x_4,x_5,t_4)*M(x_6,x_5,t_5)\\
&\quad *M(x_6,x_7,t_6)*\cdots*M\left(x_p,x_{p+1},t_p\right) \le M\left(x_1,x_{p+1},t_1+t_2+\cdots+t_p\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&M(x_2,x_1,t_1)*M(x_3,x_2,t_2)*M(x_3,x_4,t_3)*M(x_5,x_4,t_4)*M(x_5,x_6,t_5)\\
&\quad *M(x_7,x_6,t_6)*\cdots*M\left(x_{p+1},x_p\,t_p\right) \le M\left(x_{p+1},x_1,t_1+t_2+\cdots+t_p\right).
\end{aligned}
$$

Definition 3. *Let (X,M) be a fuzzy semi-metric space.*

- *We say that M is nondecreasing if and only if, given any fixed $x,y \in X$, $M(x,y,t_1) \ge M(x,y,t_2)$ for $t_1 > t_2 > 0$.*
- *We say that M is increasing if and only if, given any fixed $x,y \in X$, $M(x,y,t_1) > M(x,y,t_2)$ for $t_1 > t_2 > 0$.*
- *We say that M is symmetrically nondecreasing if and only if, given any fixed $x,y \in X$, $M(x,y,t_1) \ge M(y,x,t_2)$ for $t_1 > t_2 > 0$.*
- *We say that M is symmetrically increasing if and only if, given any fixed $x,y \in X$, $M(x,y,t_1) > M(y,x,t_2)$ for $t_1 > t_2 > 0$.*

Remark 2. *We want to claim $M(x,y,t_1) \ge M(x,y,0)$ for $t_1 > 0$. If $x \ne y$, then it is obvious since $M(x,y,0) = 0$. If $x = y$, then $M(x,y,t_1) = 1 \ge M(x,y,0)$. We can similarly obtain $M(x,y,t_1) \ge M(y,x,0)$.*

The following results are modified from Wu [7] by using a similar argument, which will be used in the further discussion.

Proposition 1. *Let (X,M) be a fuzzy semi-metric space. Then, we have the following properties.*

(i) *If M satisfies the $\bowtie$-triangle inequality, then M is nondecreasing. If M satisfies the strict $\bowtie$-triangle inequality, then M is increasing.*
(ii) *If M satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality, then M is both nondecreasing and symmetrically nondecreasing. If M satisfies the strict $\triangleright$-triangle inequality or the strict $\triangleleft$-triangle inequality, then M is both increasing and symmetrically increasing.*
(iii) *If M satisfies the $\diamond$-triangle inequality, then M is symmetrically nondecreasing. If M satisfies the strict $\diamond$-triangle inequality, then M is symmetrically increasing.*

Definition 4. *Let (X, M) be a fuzzy semi-metric space.*

- *We say that M is left-continuous with respect to the distance at $t_0 > 0$ if and only if, for any fixed $x, y \in X$, given any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < t_0 - t < \delta$ implies $|M(x, y, t) - M(x, y, t_0)| < \epsilon$; that is, the mapping $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is left-continuous at t_0. We say that M is left-continuous with respect to the distance on $(0, \infty)$ if and only if the mapping $M(x, y, \cdot)$ is left-continuous on $(0, \infty)$ for any fixed $x, y \in X$.*
- *We say that M is right-continuous with respect to the distance at $t_0 \geq 0$ if and only if, for any fixed $x, y \in X$, given any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < t - t_0 < \delta$ implies $|M(x, y, t) - M(x, y, t_0)| < \epsilon$; that is, the mapping $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is right-continuous at t_0. We say that M is right-continuous with respect to the distance on $[0, \infty)$ if and only if the mapping $M(x, y, \cdot)$ is left-continuous on $[0, \infty)$ for any fixed $x, y \in X$.*
- *We say that M is continuous with respect to the distance at $t_0 \geq 0$ if and only if, for any fixed $x, y \in X$, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies $|M(x, y, t) - M(x, y, t_0)| < \epsilon$; that is, the mapping $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous at t_0. We say that M is continuous with respect to the distance on $[0, \infty)$ if and only if the mapping $M(x, y, \cdot)$ is continuous on $[0, \infty)$ for any fixed $x, y \in X$.*
- *We say that M is symmetrically left-continuous with respect to the distance at $t_0 > 0$ if and only if, for any fixed $x, y \in X$, given any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < t_0 - t < \delta$ implies $|M(x, y, t) - M(y, x, t_0)| < \epsilon$. We say that M is symmetrically left-continuous with respect to the distance on $(0, \infty)$ if and only if it is symmetrically left-continuous with respect to the distance at each $t > 0$.*
- *We say that M is symmetrically right-continuous with respect to the distance at $t_0 \geq 0$ if and only if, for any fixed $x, y \in X$, given any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < t - t_0 < \delta$ implies $|M(x, y, t) - M(y, x, t_0)| < \epsilon$. We say that M is symmetrically right-continuous with respect to the distance on $[0, \infty)$ if and only if it is symmetrically right-continuous with respect to the distance at each $t \geq 0$.*
- *We say that M is symmetrically continuous with respect to the distance at $t_0 \geq 0$ if and only if, for any fixed $x, y \in X$, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies $|M(x, y, t) - M(y, x, t_0)| < \epsilon$. We say that M is symmetrically continuous with respect to the distance on $[0, \infty)$ if and only if it is symmetrically continuous with respect to the distance at each $t \geq 0$.*

Example 1. *Let X be a universal set. We consider a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions*

- *$d(x, y) \geq 0$ for any $x, y \in X$;*
- *$d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$;*
- *$d(x, y) + d(y, z) \geq d(x, z)$ for any $x, y, z \in X$.*

We do not assume $d(x, y) = d(y, x)$. For example, we can take $X = [0, 1]$ and define

$$d(x,y) = \begin{cases} y - x & \text{if } y \geq x \\ 1 & \text{otherwise} \end{cases}$$

Then, $d(x,y) \neq d(y,x)$ and the above three conditions are satisfied. Now, we take t-norm $*$ as the product $a * b = ab$ and define

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+d(x,y)} & \text{if } t > 0 \\ 1 & \text{if } t = 0 \text{ and } d(x,y) = 0 \\ 0 & \text{if } t = 0 \text{ and } d(x,y) > 0 \end{cases}$$

Then, $(X,M,*)$ is a fuzzy semi-metric space satisfying the $\bowtie$-triangle inequality. For any fixed $x,y \in X$, it is obvious that $M(x,y,t)$ is continuous on $(0,\infty)$. Therefore, M is continuous with respect to the distance on $(0,\infty)$.

The following results are from Wu [7]. Especially, Part (ii) is modified from Wu [7] by using the similar argument.

Proposition 2. *Let (X,M) be a fuzzy semi-metric space such that the $\circ$-triangle inequality is satisfied for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then, we have the following properties.*

(i) *Suppose that M is left-continuous or symmetrically left-continuous with respect to the distance at $t > 0$. Then, $M(x,y,t) = M(y,x,t)$. In other words, if M is left-continuous or symmetrically left-continuous with respect to the distance on $(0,\infty)$, then M satisfies the symmetric condition.*

(ii) *Suppose that M is right-continuous or symmetrically right-continuous with respect to the distance at $t > 0$. Then, $M(x,y,t) = M(y,x,t)$. In other words, if M is right-continuous or symmetrically right-continuous with respect to the distance on $(0,\infty)$, then M satisfies the symmetric condition.*

From Proposition 2, if M is left-continuous or symmetrically left-continuous with respect to the distance on $(0,\infty)$, or right-continuous and or symmetrically right-continuous with respect to the distance on $(0,\infty]$, then we can just consider the $\bowtie$-triangle inequality. The following results are modified from Wu [7] by using a similar argument.

Proposition 3. *Let (X,M) be a fuzzy semi-metric space.*

(i) *Suppose that M is left-continuous or symmetrically left-continuous with respect to the distance on $(0,\infty)$. If $M(x,x,0) = 1$ or $M(x,x,0) = 0$ for any $x \in X$, then M satisfies the $\circ$-triangle inequality if and only if M satisfies the strong $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$.*

(ii) *Suppose that M is right-continuous or symmetrically right-continuous with respect to the distance on $[0,\infty)$. Then, M satisfies the $\circ$-triangle inequality if and only if M satisfies the strong $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$.*

3. Metric Convergence

Since the symmetric condition is not satisfied in the fuzzy semi-metric space, three kinds of limit concepts will also be considered in this paper by referring to Wu [7].

Let (X,d) be a metric space. If the sequence $\{x_n\}_{n=1}^{\infty}$ in (X,d) converges to x, i.e., $d(x_n,x) \to 0$ as $n \to \infty$, then it is denoted by $x_n \xrightarrow{d} x$ as $n \to \infty$. In this case, we also say that x is a d-limit of the sequence $\{x_n\}_{n=1}^{\infty}$. From Wu [7], the limits based on the fuzzy semi-metric M are given below.

Definition 5. *Let (X,M) be a fuzzy semi-metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.*

- *We write $x_n \xrightarrow{M^{\triangleright}} x$ as $n \to \infty$ if and only if*

$$\lim_{n\to\infty} M(x_n,x,t) = 1 \text{ for all } t > 0.$$

In this case, we call x a $M^{\triangleright}$-limit of the sequence $\{x_n\}_{n=1}^{\infty}$.

- *We write* $x_n \xrightarrow{M^{\triangleleft}} x$ *as* $n \to \infty$ *if and only if*

$$\lim_{n\to\infty} M(x, x_n, t) = 1 \text{ for all } t > 0.$$

In this case, we call x *a* $M^{\triangleleft}$*-limit of the sequence* $\{x_n\}_{n=1}^{\infty}$.

- *We write* $x_n \xrightarrow{M} x$ *as* $n \to \infty$ *if and only if*

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(x, x_n, t) = 1 \text{ for all } t > 0.$$

In this case, we call x *an* M*-limit of the sequence* $\{x_n\}_{n=1}^{\infty}$.

The uniqueness of limits obtained from Wu [7] are shown below.

Proposition 4. *Let* (X, M) *be a fuzzy semi-metric space along with a t-norm* $*$ *that is left-continuous at one with respect to the first or second component, and let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* X.

(i) *Suppose that* M *satisfies the* $\bowtie$*-triangle inequality or the* $\diamond$*-triangle inequality. Then, we have the following properties.*
    - *If* $x_n \xrightarrow{M^{\triangleleft}} x$ *and* $x_n \xrightarrow{M^{\triangleright}} y$, *then* $x = y$.
    - *If* $x_n \xrightarrow{M^{\triangleright}} x$ *and* $x_n \xrightarrow{M^{\triangleleft}} y$, *then* $x = y$.

(ii) *Suppose that* M *satisfies the* $\triangleleft$*-triangle inequality. If* $x_n \xrightarrow{M^{\triangleright}} x$ *and* $x_n \xrightarrow{M^{\triangleright}} y$, *then* $x = y$. *In other words, the* $M^{\triangleright}$*-limit is unique.*

(iii) *Suppose that* M *satisfies the* $\triangleright$*-triangle inequality. If* $x_n \xrightarrow{M^{\triangleleft}} x$ *and* $x_n \xrightarrow{M^{\triangleleft}} y$, *then* $x = y$. *In other words, the* $M^{\triangleleft}$*-limit is unique.*

Example 2. *Continued from Example 1, we see that* $x_n \xrightarrow{M^{\triangleright}} x$ *as* $n \to \infty$ *if and only if*

$$1 = \lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} \frac{t}{t + d(x_n, x)} = \frac{t}{t + \lim_{n\to\infty} d(x_n, x)} \text{ for all } t > 0.$$

Therefore, we obtain

$$x_n \xrightarrow{M^{\triangleright}} x \text{ if and only if } \lim_{n\to\infty} d(x_n, x) = 0$$

and

$$x_n \xrightarrow{M^{\triangleleft}} x \text{ if and only if } \lim_{n\to\infty} d(x, x_n) = 0.$$

Suppose that $x_n \xrightarrow{M^{\triangleleft}} x$ *and* $x_n \xrightarrow{M^{\triangleright}} y$. *Then, we have*

$$\lim_{n\to\infty} d(x, x_n) = 0 = \lim_{n\to\infty} d(x_n, y).$$

Since

$$d(x, y) \leq d(x, x_n) + d(x_n, y),$$

we obtain

$$0 \leq d(x, y) \leq \lim_{n\to\infty} d(x, x_n) + \lim_{n\to\infty} d(x_n, y) = 0,$$

which implies $x = y$. *This verifies Part (i) of Proposition 4.*

In the sequel, we are going to present the different kinds of convergence of real-valued sequence $\{M(x_n, y_n, t_n)\}_{n=1}^{\infty}$, where $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ is a sequence in the product set $X \times X \times (0, \infty)$. We first provide some useful lemmas.

Lemma 1. *Let $*$ be a t-norm. We have the following properties.*

(i) *Given any fixed $a, b \in (0,1]$, suppose that the t-norm $*$ is left-continuous at a and b with respect to the first or second component. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences in $[0,1]$ such that $a_n \to a-$ and $b_n \to b-$ as $n \to \infty$, then $a_n * b_n \to a * b$ as $n \to \infty$.*

(ii) *Given any fixed $a, b \in [0,1)$, suppose that the t-norm $*$ is right-continuous at a and b with respect to the first or second component. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences in $[0,1]$ such that $a_n \to a+$ and $b_n \to b+$ as $n \to \infty$, then $a_n * b_n \to a * b$ as $n \to \infty$.*

Proof. To prove Part (i), we note that there exist two increasing sequences $\{p_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ such that $p_n \uparrow a$ and $r_n \uparrow b$ satisfying $p_n \leq a_n$ and $r_n \leq b_n$. According to the concept of commutativity of the t-norm, we see that the t-norm is left-continuous with respect to each component. Given any $\epsilon > 0$, using the left-continuity of the t-norm at b with respect to the second component, there exists $n_0 \in \mathbb{N}$ such that

$$a * b - \frac{\epsilon}{2} < a * r_{n_0}.$$

Furthermore, using the left-continuity of the t-norm at a with respect to the first component, there exists $n_1 \in \mathbb{N}$ such that

$$a * r_{n_0} - \frac{\epsilon}{2} < p_{n_1} * r_{n_0}.$$

Using the increasing property of the t-norm, for $m \geq n_1$ and $n \geq n_0$, we have

$$a * r_{n_0} - \frac{\epsilon}{2} < p_{n_1} * r_{n_0} \leq p_m * r_n \leq a_m * b_n.$$

Since $a_n \to a-$ and $b_n \to b-$, we see that $a_n \leq a$ and $b_n \leq b$ for all n. By taking $n_2 = \max\{n_0, n_1\}$, for $n \geq n_2$, we obtain

$$a * b - \epsilon < a * r_{n_0} - \frac{\epsilon}{2} < a_n * b_n \leq a * b < a * b + \epsilon,$$

which says that $|a * b - a_n * b_n| < \epsilon$. This shows the desired convergence. Part (ii) can be similarly proven, and the proof is complete. □

Lemma 2. *Let $*$ be a t-norm. Given any two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in $[0,1]$, the following statements hold true.*

(i) *We have*

$$\sup_n (a_n * b_n) \leq \left(\sup_n a_n\right) * \left(\sup_n b_n\right) \text{ and } \inf_n (a_n * b_n) \geq \left(\inf_n a_n\right) * \left(\inf_n b_n\right).$$

(ii) *Suppose that the t-norm $*$ is left-continuous with respect to the first or second component.*

- *We have*
$$\sup_n (a_n * b) = \left(\sup_n a_n\right) * b \text{ and } \sup_n (a * b_n) = a * \left(\sup_n b_n\right).$$
- *If, for any $m, n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $a_m * b_n \leq a_{n_0} * b_{n_0}$, then*
$$\sup_n (a_n * b_n) = \left(\sup_n a_n\right) * \left(\sup_n b_n\right).$$
- *We have*
$$\limsup_{n\to\infty} (a_n * b) \geq \left(\limsup_{n\to\infty} a_n\right) * b.$$

(iii) *Suppose that the t-norm $*$ is right-continuous with respect to the first or second component.*

- *We have*
$$\inf_n (a_n * b) = \left(\inf_n a_n\right) * b \text{ and } \inf_n (a * b_n) = a * \left(\inf_n b_n\right).$$
- *If, for any $m, n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $a_m * b_n \geq a_{n_0} * b_{n_0}$, then*
$$\inf_n (a_n * b_n) = \left(\inf_n a_n\right) * \left(\inf_n b_n\right).$$

(iv) *If*
$$\sup_n (a_n * b_n) \geq \left(\sup_n a_n\right) * \left(\sup_n b_n\right)$$
then
$$\limsup_{n\to\infty} (a_n * b_n) \geq \left(\limsup_{n\to\infty} a_n\right) * \left(\limsup_{n\to\infty} b_n\right).$$

(v) *If*
$$\inf_n (a_n * b_n) \leq \left(\inf_n a_n\right) * \left(\inf_n b_n\right),$$
then
$$\liminf_{n\to\infty} (a_n * b_n) \leq \left(\liminf_{n\to\infty} a_n\right) * \left(\liminf_{n\to\infty} b_n\right).$$

(vi) *Suppose that the t-norm $*$ is left-continuous with respect to the first or second component. Let $\{c_n\}_{n=1}$ be another increasing sequence in $[0,1]$. If $a_n \to 1$ and $b_n \to 1$ as $n \to \infty$, then*
$$\sup_n (a_n * c_n * b_n) = \sup_n c_n = \lim_{n\to\infty} c_n$$
and
$$\limsup_{n\to\infty} (a_n * c_n * b_n) = \lim_{n\to\infty} c_n.$$

Proof. To prove Part (i), by the increasing property of the t-norm, we immediately have
$$a_k * b_k \leq \left(\sup_n a_n\right) * \left(\sup_n b_n\right) \text{ and } a_k * b_k \geq \left(\inf_n a_n\right) * \left(\inf_n b_n\right)$$
for all k, which proves the desired inequalities.

To prove Part (ii), given any $\epsilon > 0$, there exists $m, n \in \mathbb{N}$ such that
$$\sup_k a_k - \epsilon < a_m \text{ and } \sup_k b_k - \epsilon < b_n.$$

The increasing property of the t-norm says that
$$\left(\sup_k a_k - \epsilon\right) * \left(\sup_k b_k - \epsilon\right) \leq a_m * b_n \leq a_{n_0} * b_{n_0} \leq \sup_n (a_n * b_n).$$

Since the t-norm is left-continuous with respect to each component, by taking $\epsilon \to 0$, we obtain the desired inequality. Now, according to the inequalities in Part (i), we have
$$\begin{aligned}\limsup_{n\to\infty} (a_n * b) = \inf_{k\geq 1} \sup_{n\geq k} (a_n * b) &= \inf_{k\geq 1} \left[\left(\sup_{n\geq k} a_n\right) * b\right] \\ &\geq \left(\inf_{k\geq 1} \sup_{n\geq k} a_n\right) * b = \left(\limsup_{n\to\infty} a_n\right) * b.\end{aligned}$$

To prove Part (iii), given any $\epsilon > 0$, there exists $m, n \in \mathbb{N}$ such that

$$\inf_k a_k + \epsilon > a_m \text{ and } \inf_k b_k + \epsilon > b_n.$$

The increasing property of the t-norm says that

$$\left(\inf_k a_k + \epsilon\right) * \left(\inf_k b_k + \epsilon\right) \geq a_m * b_n \geq a_{n_0} * b_{n_0} \geq \inf_n (a_n * b_n).$$

Since the t-norm is right-continuous with respect to each component, by taking $\epsilon \to 0$, we obtain the desired inequality.

To prove Part (iv), according to the inequalities in Part (i), we have

$$\begin{aligned}\limsup_{n\to\infty} (a_n * b_n) &= \inf_{k\geq 1} \sup_{n\geq k} (a_n * b_n) \geq \inf_{k\geq 1} \left[\left(\sup_{n\geq k} a_n\right) * \left(\sup_{n\geq k} b_n\right)\right] \\ &\geq \left(\inf_{k\geq 1} \sup_{n\geq k} a_n\right) * \left(\inf_{k\geq 1} \sup_{n\geq k} b_n\right) = \left(\limsup_{n\to\infty} a_n\right) * \left(\limsup_{n\to\infty} b_n\right).\end{aligned}$$

To prove Part (v), we also have

$$\begin{aligned}\liminf_{n\to\infty} (a_n * b_n) &= \sup_{k\geq 1} \inf_{n\geq k} (a_n * b_n) \leq \sup_{k\geq 1} \left[\left(\inf_{n\geq k} a_n\right) * \left(\inf_{n\geq k} b_n\right)\right] \\ &\leq \left(\sup_{k\geq 1} \inf_{n\geq k} a_n\right) * \left(\sup_{k\geq 1} \inf_{n\geq k} b_n\right) = \left(\liminf_{n\to\infty} a_n\right) * \left(\liminf_{n\to\infty} b_n\right).\end{aligned}$$

To prove Part (vi), since $0 \leq a_n \leq 1$ and $0 \leq b_n \leq 1$ for all n and $a_n \to 1$ and $b_n \to 1$ as $n \to \infty$, we see that

$$\sup_n a_n = \lim_{n\to\infty} a_n = 1 \text{ and } \sup_n b_n = \lim_{n\to\infty} b_n = 1$$

Using Part (i), we obtain

$$\sup_n (a_n * c_n * b_n) =\leq \left(\sup_n a_n\right) * \left(\sup_n c_n\right) * \left(\sup_n b_n\right) = 1 * \left(\sup_n c_n\right) * 1 = \sup_n c_n = \lim_{n\to\infty} c_n.$$

Given any $\epsilon > 0$, there exists $m, n, r \in \mathbb{N}$ such that

$$\sup_k a_k - \epsilon < a_m, \quad \sup_k b_k - \epsilon < b_n \text{ and } \sup_k c_k - \epsilon < c_r.$$

Since $a_n \to 1$, $b_n \to 1$ and $\{c_n\}_{n=1}$ is an increasing sequence, there exists $n_0 \in \mathbb{N}$ such that $a_m \leq a_{n_0}$, $b_n \leq b_{n_0}$ and $c_r \leq c_{n_0}$. The increasing property of the t-norm says that

$$\begin{aligned}(1-\epsilon) * \left(\sup_k c_k - \epsilon\right) * (1-\epsilon) &= \left(\sup_k a_k - \epsilon\right) * \left(\sup_k c_k - \epsilon\right) * \left(\sup_k b_k - \epsilon\right) \\ &\leq a_m * c_r * b_n \leq a_{n_0} * c_{n_0} * b_{n_0} \leq \sup_n (a_n * c_n * b_n).\end{aligned}$$

Since the t-norm is left-continuous with respect to each component, by taking $\epsilon \to 0$, we obtain

$$1 * \left(\sup_n c_n\right) * 1 \geq \sup_n (a_n * c_n * b_n).$$

Then, we obtain the desired equality. On the other hand, we have

$$\limsup_{n\to\infty} (a_n * c_n * b_n) = \inf_{k\geq 1} \sup_{n\geq k} (a_n * c_n * b_n) = \inf_{k\geq 1} \sup_{n\geq k} c_n = \limsup_{n\to\infty} c_n = \lim_{n\to\infty} c_n.$$

This completes the proof. □

Proposition 5. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component, and let $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X \times (0, \infty)$. Given $t^\circ > 0$, we have the following properties.*

(i) Suppose that M satisfies the $\bowtie$-triangle inequality, and that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleright} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$.

- *If M is left-continuous with respect to the distance at t°, then*

$$\liminf_{n\to\infty} M(x_n, y_n, t_n) \geq M(x^\circ, y^\circ, t^\circ).$$

- *If M is symmetrically left-continuous with respect to the distance at t°, then*

$$\liminf_{n\to\infty} M(x_n, y_n, t_n) \geq M(y^\circ, x^\circ, t^\circ).$$

(ii) Suppose that M satisfies the $\triangleright$-triangle inequality and that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleright} x^\circ$ and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$. If M is left-continuous or symmetrically left-continuous with respect to the distance at t°, then

$$\liminf_{n\to\infty} M(x_n, y_n, t_n) \geq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

(iii) Suppose that M satisfies the $\triangleleft$-triangle inequality and that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$. If M is left-continuous or symmetrically left-continuous with respect to the distance at t°, then

$$\liminf_{n\to\infty} M(x_n, y_n, t_n) \geq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

(iv) Suppose that M satisfies the $\diamond$-triangle inequality, and that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleright} x^\circ$ and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$ simultaneously, or $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$ simultaneously. If M is left-continuous or symmetrically left-continuous with respect to the distance at t°, then

$$\liminf_{n\to\infty} M(x_n, y_n, t_n) \geq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

Proof. To prove Part (i), given any sufficiently small $\epsilon > 0$ with $\epsilon < t^\circ/2$, there exists $n_0 \in \mathbb{N}$ such that $t^\circ - t_n < \epsilon$ for all $n \geq n_0$. Using Part (i) of Proposition 1 and (1) in the first observation of Remark 1, we have

$$M(x_n, y_n, t_n) \geq M(x_n, y_n, t^\circ - \epsilon) \geq M\left(x_n, x^\circ, \frac{\epsilon}{2}\right) * M(x^\circ, y^\circ, t^\circ - 2\epsilon) * M\left(y^\circ, y_n, \frac{\epsilon}{2}\right). \quad (8)$$

Since $x_n \xrightarrow{M^\triangleright} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$, we have

$$\lim_{n\to\infty} M\left(x_n, x^\circ, \frac{\epsilon}{2}\right) = 1 \text{ and } \lim_{n\to\infty} M\left(y^\circ, y_n, \frac{\epsilon}{2}\right) = 1, \quad (9)$$

which also say that the sequences $\{M(x_n, x^\circ, \frac{\epsilon}{2})\}_{n=1}^{\infty}$ and $\{M(y^\circ, y_n, \frac{\epsilon}{2})\}_{n=1}^{\infty}$ converge to one from the left, since each element of the sequences is less than one. The existence of the limit in the right-hand side of (8) is guaranteed by applying Part (i) of Lemma 1. Then, we obtain

$$\liminf_{n\to\infty} M\left(x_n, y_n, t_n\right) \geq 1 * M\left(x^\circ, y^\circ, t^\circ - 2\epsilon\right) * 1 = M\left(x^\circ, y^\circ, t^\circ - 2\epsilon\right).$$

If M is left-continuous with respect to the distance at t°, then

$$\liminf_{n\to\infty} M\left(x_n, y_n, t_n\right) \geq M(x^\circ, y^\circ, t^\circ)$$

by taking $\epsilon \to 0$. Furthermore, if M is symmetrically left-continuous with respect to the distance at t°, then

$$\liminf_{n\to\infty} M\left(x_n, y_n, t_n\right) \geq M(y^\circ, x^\circ, t^\circ).$$

To prove Part (ii), using Part (ii) of Proposition 1 and (2) in the second observation of Remark 1, we have

$$M\left(x_n, y_n, t_n\right) \geq M\left(x_n, y_n, t^\circ - \epsilon\right) \geq M\left(x_n, x^\circ, \frac{\epsilon}{2}\right) * M\left(y^\circ, x^\circ, t^\circ - 2\epsilon\right) * M\left(y_n, y^\circ, \frac{\epsilon}{2}\right)$$

and

$$M\left(x_n, y_n, t_n\right) \geq M\left(y_n, x_n, t^\circ - \epsilon\right) \geq M\left(y_n, y^\circ, \frac{\epsilon}{2}\right) * M\left(x^\circ, y^\circ, t^\circ - 2\epsilon\right) * M\left(x_n, x^\circ, \frac{\epsilon}{2}\right).$$

From Part (i) of Proposition 2, we also have $M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ)$. Therefore, we can similarly obtain

$$\liminf_{n\to\infty} M\left(x_n, y_n, t_n\right) \geq M(y^\circ, x^\circ, t^\circ) = M(x^\circ, y^\circ, t^\circ).$$

To prove Part (iii), using Part (ii) of Proposition 1 and (3) in the third observation of Remark 1, we have

$$M\left(x_n, y_n, t_n\right) \geq M\left(x_n, y_n, t^\circ - \epsilon\right) \geq M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) * M\left(x^\circ, y^\circ, t^\circ - 2\epsilon\right) * M\left(y^\circ, y_n, \frac{\epsilon}{2}\right)$$

and

$$M\left(x_n, y_n, t_n\right) \geq M\left(y_n, x_n, t^\circ - \epsilon\right) \geq M\left(y^\circ, y_n, \frac{\epsilon}{2}\right) * M\left(y^\circ, x^\circ, t^\circ - 2\epsilon\right) * M\left(x^\circ, x_n, \frac{\epsilon}{2}\right).$$

From Part (i) of Proposition 2, we also have $M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ)$. Therefore, we can similarly obtain the desired results.

To prove Part (iv), using Part (iii) of Proposition 1 and (4) and (8) in the fourth observation of Remark 1, we have

$$M\left(x_n, y_n, t_n\right) \geq M\left(y_n, x_n, t^\circ - \epsilon\right) \geq M\left(y_n, y^\circ, \frac{\epsilon}{2}\right) * M\left(y^\circ, x^\circ, t^\circ - 2\epsilon\right) * M\left(x_n, x^\circ, \frac{\epsilon}{2}\right)$$

and

$$M\left(x_n, y_n, t_n\right) \geq M\left(y_n, x_n, t^\circ - \epsilon\right) \geq M\left(y^\circ, y_n, \frac{\epsilon}{2}\right) * M\left(y^\circ, x^\circ, t^\circ - 2\epsilon\right) * M\left(x^\circ, x_n, \frac{\epsilon}{2}\right).$$

From Part (i) of Proposition 2, we also have $M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ)$. Therefore, we can similarly obtain the desired results, and the proof is complete. □

In Proposition 5, we remark that $M(x_n, y_n, t_n) \neq M(y_n, x_n, t_n)$ in general for $n \in \mathbb{N}$.

Proposition 6. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component, and let $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X \times (0, \infty)$. Assume that the following inequality is satisfied*

$$\sup_n (a_n * b_n) \geq \left(\sup_n a_n\right) * \left(\sup_n b_n\right) \tag{10}$$

for any sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in $[0,1]$. Given $t^\circ > 0$, we have the following properties.

(i) *Suppose that M satisfies the $\bowtie$-triangle inequality and that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$.*

- *If M is right-continuous with respect to the distance at t°, then*

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ).$$

- *If M is symmetrically right-continuous with respect to the distance at t°, then*

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(y^\circ, x^\circ, t^\circ).$$

(ii) *Suppose that M satisfies the $\triangleright$-triangle inequality and that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$. If M is right-continuous or symmetrically right-continuous with respect to the distance at t°, then*

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

(iii) *Suppose that M satisfies the $\triangleleft$-triangle inequality and that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleright} x^\circ$ and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$. If M is right-continuous or symmetrically right-continuous with respect to the distance at t°, then*

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

(iv) *Suppose that M satisfies the $\diamond$-triangle inequality, and that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleright} x^\circ$ and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$ simultaneously, or $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$ simultaneously. If M is right-continuous or symmetrically right-continuous with respect to the distance at t°, then*

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

Proof. To prove Part (i), given any fixed $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $t_n - t^\circ < \epsilon$ for all $n \geq n_0$. Using Part (i) of Proposition 1 and the first observation of Remark 1, we have

$$M(x^\circ, y^\circ, t^\circ + 2\epsilon) \geq M(x^\circ, y^\circ, t_n + \epsilon) \geq M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) * M(x_n, y_n, t_n) * M\left(y_n, y^\circ, \frac{\epsilon}{2}\right)$$

for all $n \geq n_0$. To prove the first case of Part (i), we have

$$\begin{aligned}
M(x^\circ, y^\circ, t^\circ + 2\epsilon) &\geq \limsup_{n\to\infty} \left[M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) * M(x_n, y_n, t_n) * M\left(y_n, y^\circ, \frac{\epsilon}{2}\right) \right] \\
&\geq \left(\limsup_{n\to\infty} M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) \right) * \left(\limsup_{n\to\infty} M(x_n, y_n, t_n) \right) * \left(\limsup_{n\to\infty} M\left(y_n, y^\circ, \frac{\epsilon}{2}\right) \right) \\
&\quad \text{(by Part (iv) of Lemma 2)} \\
&= 1 * \left(\limsup_{n\to\infty} M(x_n, y_n, t_n) \right) * 1 \text{ (since } x_n \overset{M^\triangleleft}{\longrightarrow} x^\circ \text{ and } y_n \overset{M^\triangleright}{\longrightarrow} y^\circ\text{)} \\
&= \limsup_{n\to\infty} M(x_n, y_n, t_n).
\end{aligned} \tag{11}$$

If M is right-continuous with respect to the distance at t°, then

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ)$$

by taking $\epsilon \to 0$. Furthermore, if M is symmetrically right-continuous with respect to the distance at t°, then

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(y^\circ, x^\circ, t^\circ).$$

To prove Part (ii), using Part (ii) of Proposition 1 and (2) in the second observation of Remark 1, we have

$$M(y^\circ, x^\circ, t^\circ + 2\epsilon) \geq M(y^\circ, x^\circ, t_n + \epsilon) \geq M\left(y^\circ, y_n, \frac{\epsilon}{2}\right) * M(x_n, y_n, t_n) * M\left(x^\circ, x_n, \frac{\epsilon}{2}\right)$$

and

$$M(x^\circ, y^\circ, t^\circ + 2\epsilon) \geq M(y^\circ, x^\circ, t_n + \epsilon) \geq M\left(y^\circ, y_n, \frac{\epsilon}{2}\right) * M(x_n, y_n, t_n) * M\left(x^\circ, x_n, \frac{\epsilon}{2}\right).$$

From Part (ii) of Proposition 2, we also have $M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ)$. Therefore, we can similarly obtain

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

To prove Part (iii), using Part (ii) of Proposition 1 and (3) in the third observation of Remark 1, we have

$$M(x^\circ, y^\circ, t^\circ + 2\epsilon) \geq M(x^\circ, y^\circ, t_n + \epsilon) \geq M\left(x_n, x^\circ, \frac{\epsilon}{2}\right) * M(x_n, y_n, t_n) * M\left(y_n, y^\circ, \frac{\epsilon}{2}\right)$$

and

$$M(y^\circ, x^\circ, t^\circ + 2\epsilon) \geq M(x^\circ, y^\circ, t_n + \epsilon) \geq M\left(x_n, x^\circ, \frac{\epsilon}{2}\right) * M(x_n, y_n, t_n) * M\left(y_n, y^\circ, \frac{\epsilon}{2}\right).$$

From Part (ii) of Proposition 2, we also have $M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ)$. Therefore, we can similarly obtain

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

To prove Part (iv), using Part (iii) of Proposition 1 and (4) and (8) in the fourth observation of Remark 1, we have

$$M(y^\circ, x^\circ, t^\circ + 2\epsilon) \geq M(x^\circ, y^\circ, t_n + \epsilon) \geq M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) * M(x_n, y_n, t_n) * M\left(y^\circ, y_n, \frac{\epsilon}{2}\right)$$

and

$$M\left(y^{\circ},x^{\circ},t^{\circ}+2\epsilon\right)\geq M\left(x^{\circ},y^{\circ},t_{n}+\epsilon\right)\geq M\left(x_{n},x^{\circ},\frac{\epsilon}{2}\right)*M\left(x_{n},y_{n},t_{n}\right)*M\left(y_{n},y^{\circ},\frac{\epsilon}{2}\right).$$

From Part (ii) of Proposition 2, we also have $M(x^{\circ},y^{\circ},t^{\circ})=M(y^{\circ},x^{\circ},t^{\circ})$. Therefore, we can similarly obtain

$$\limsup_{n\to\infty}M\left(x_{n},y_{n},t_{n}\right)\leq M(y^{\circ},x^{\circ},t^{\circ})=M(x^{\circ},y^{\circ},t^{\circ}).$$

Therefore, we can similarly obtain the desired results, and the proof is complete. □

Proposition 7. *Let (X,M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component, and let $\{(x_n,y_n,t_n)\}_{n=1}^{\infty}$ be a sequence in $X\times X\times(0,\infty)$. Assume that the following inequality is satisfied*

$$\sup_{n}\left(a_{n}*b_{n}\right)\geq\left(\sup_{n}a_{n}\right)*\left(\sup_{n}b_{n}\right)$$

for any sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in $[0,1]$. Given $t^{\circ}>0$, we have the following properties.

(i) Suppose that M satisfies the $\bowtie$-triangle inequality, and that $t_n\to t^{\circ}$, $x_n\xrightarrow{M}x^{\circ}$ and $y_n\xrightarrow{M}y^{\circ}$ as $n\to\infty$. Then, the following statements hold true.

- *If M is continuous with respect to the distance at t°, then*

$$\lim_{n\to\infty}M(x_{n},y_{n},t_{n})=M(x^{\circ},y^{\circ},t^{\circ}).$$

- *If M is symmetrically continuous with respect to the distance at t°, then*

$$\lim_{n\to\infty}M(x_{n},y_{n},t_{n})=M(y^{\circ},x^{\circ},t^{\circ}).$$

(ii) Suppose that M satisfies the $\circ$-triangle inequality for $\circ\in\{\triangleright,\triangleleft\}$ and that $t_n\to t^{\circ}$, $x_n\xrightarrow{M}x^{\circ}$ and $y_n\xrightarrow{M}y^{\circ}$ as $n\to\infty$. If M is continuous or symmetrically continuous with respect to the distance at t°, then

$$\lim_{n\to\infty}M(x_{n},y_{n},t_{n})=M(x^{\circ},y^{\circ},t^{\circ})=M(y^{\circ},x^{\circ},t^{\circ}).$$

(iii) Suppose that M satisfies the $\diamond$-triangle inequality and that $t_n\to t^{\circ}$ as $n\to\infty$, $x_n\xrightarrow{M^{\triangleright}}x^{\circ}$ and $y_n\xrightarrow{M^{\triangleright}}y^{\circ}$ as $n\to\infty$ simultaneously, or $x_n\xrightarrow{M^{\triangleleft}}x^{\circ}$ and $y_n\xrightarrow{M^{\triangleleft}}y^{\circ}$ as $n\to\infty$ simultaneously. If M is continuous or symmetrically continuous with respect to the distance at t°, then

$$\lim_{n\to\infty}M(x_{n},y_{n},t_{n})=M(y^{\circ},x^{\circ},t^{\circ})=M(x^{\circ},y^{\circ},t^{\circ}).$$

Proof. From Propositions 5 and 6, we have

$$M(x^{\circ},y^{\circ},t^{\circ})\leq\liminf_{n\to\infty}M\left(x_{n},y_{n},t_{n}\right)\leq\limsup_{n\to\infty}M\left(x_{n},y_{n},t_{n}\right)\leq M(x^{\circ},y^{\circ},t^{\circ})$$

and

$$M(y^{\circ},x^{\circ},t^{\circ})\leq\liminf_{n\to\infty}M\left(y_{n},x_{n},t_{n}\right)\leq\limsup_{n\to\infty}M\left(y_{n},x_{n},t_{n}\right)\leq M(y^{\circ},x^{\circ},t^{\circ}).$$

This completes the proof. □

Proposition 8. *Let (X,M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component, and let $\{(x_n,y_n,t_n)\}_{n=1}^{\infty}$ be a sequence in $X\times X\times(0,\infty)$. Given $t^{\circ}>0$, we have the following properties.*

(i) *Suppose that M satisfies the $\bowtie$-triangle inequality, that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$, and that $\{M(x_n, y_n, t_n)\}_{n=1}^\infty$ is an increasing sequence.*

- *If M is right-continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ).$$

- *If M is symmetrically right-continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) \leq M(y^\circ, x^\circ, t^\circ).$$

(ii) *Suppose that M satisfies the $\triangleright$-triangle inequality, that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$, and that $\{M(x_n, y_n, t_n)\}_{n=1}^\infty$ is an increasing sequence. If M is right-continuous or symmetrically right-continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

(iii) *Suppose that M satisfies the $\triangleleft$-triangle inequality, that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleright} x^\circ$ and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$, and that $\{M(x_n, y_n, t_n)\}_{n=1}^\infty$ is an increasing sequence. If M is right-continuous or symmetrically right-continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

(iv) *Suppose that M satisfies the $\diamond$-triangle inequality, that $t_n \to t^\circ$, $x_n \xrightarrow{M^\triangleright} x^\circ$ and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$ simultaneously, or $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$ simultaneously, and that $\{M(x_n, y_n, t_n)\}_{n=1}^\infty$ is an increasing sequence. If M is right-continuous or symmetrically right-continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

Proof. To prove Part (i), given any fixed $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $t_n - t^\circ < \epsilon$ for all $n \geq n_0$. Using Part (i) of Proposition 1 and the first observation of Remark 1, we have

$$M(x^\circ, y^\circ, t^\circ + 2\epsilon) \geq M(x^\circ, y^\circ, t_n + \epsilon) \geq M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) * M(x_n, y_n, t_n) * M\left(y_n, y^\circ, \frac{\epsilon}{2}\right)$$

for all $n \geq n_0$. To prove the first case of Part (i), since

$$\lim_{n\to\infty} M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) = 1 \text{ and } \limsup_{n\to\infty} M\left(y_n, y^\circ, \frac{\epsilon}{2}\right) = 1,$$

using Part (vi) of Lemma 2, we have

$$\begin{aligned} M(x^\circ, y^\circ, t^\circ + 2\epsilon) &\geq \limsup_{n\to\infty} \left[M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) * M(x_n, y_n, t_n) * M\left(y_n, y^\circ, \frac{\epsilon}{2}\right)\right] \\ &= \lim_{n\to\infty} M(x_n, y_n, t_n) = \sup_n M(x_n, y_n, t_n). \end{aligned}$$

If M is right-continuous with respect to the distance at t°, then

$$\lim_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ)$$

by taking $\epsilon \to 0$. Furthermore, if M is symmetrically right-continuous with respect to the distance at t°, then

$$\lim_{n\to\infty} M\left(x_n, y_n, t_n\right) \leq M(y^\circ, x^\circ, t^\circ).$$

The remaining proof follows from the arguments of the proof of Proposition 6. This completes the proof. □

Proposition 9. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component, and let $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X \times (0, \infty)$. Given $t^\circ > 0$, we have the following properties.*

(i) Suppose that M satisfies the $\bowtie$-triangle inequality, that $t_n \to t^\circ$, $x_n \xrightarrow{M} x^\circ$ and $y_n \xrightarrow{M} y^\circ$ as $n \to \infty$, and that $\{M(x_n, y_n, t_n)\}_{n=1}^{\infty}$ is an increasing sequence. Then, the following statements hold true.

- *If M is continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) = M(x^\circ, y^\circ, t^\circ).$$

- *If M is symmetrically continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) = M(y^\circ, x^\circ, t^\circ).$$

(ii) Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$, that $t_n \to t^\circ$, $x_n \xrightarrow{M} x^\circ$ and $y_n \xrightarrow{M} y^\circ$ as $n \to \infty$, and that $\{M(x_n, y_n, t_n)\}_{n=1}^{\infty}$ is an increasing sequence. If M is continuous or symmetrically continuous with respect to the distance at t°, then

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) = M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

(iii) Suppose that M satisfies the $\diamond$-triangle inequality, that $t_n \to t^\circ$ as $n \to \infty$, $x_n \xrightarrow{M^\triangleright} x^\circ$ and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$ simultaneously, or $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$ simultaneously and that $\{M(x_n, y_n, t_n)\}_{n=1}^{\infty}$ is an increasing sequence. If M is continuous or symmetrically continuous with respect to the distance at t°, then

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) = M(y^\circ, x^\circ, t^\circ) = M(x^\circ, y^\circ, t^\circ).$$

Proof. From Propositions 5 and 8, we have

$$M(x^\circ, y^\circ, t^\circ) \leq \liminf_{n\to\infty} M\left(x_n, y_n, t_n\right) \leq \lim_{n\to\infty} M\left(x_n, y_n, t_n\right) \leq M(x^\circ, y^\circ, t^\circ)$$

and

$$M(y^\circ, x^\circ, t^\circ) \leq \liminf_{n\to\infty} M\left(y_n, x_n, t_n\right) \leq \lim_{n\to\infty} M\left(y_n, x_n, t_n\right) \leq M(y^\circ, x^\circ, t^\circ).$$

This completes the proof. □

Example 3. *Continued from Example 2, suppose that $t_n \to t^\circ$, $x_n \xrightarrow{M} x^\circ$ and $y_n \xrightarrow{M} y^\circ$ as $n \to \infty$. Then, we have*

$$\lim_{n\to\infty} d(x_n, x^\circ) = 0 = \lim_{n\to\infty} d(x^\circ, x_n)$$

and

$$\lim_{n\to\infty} d(y_n, y^\circ) = 0 = \lim_{n\to\infty} d(y^\circ, y_n).$$

From the triangle inequality of d, we have

$$d(x_n, y_n) \leq d(x_n, x^\circ) + d(x^\circ, y^\circ) + d(y^\circ, y_n)$$

and

$$d(x^\circ, y^\circ) \leq d(x^\circ, x_n) + d(x_n, y_n) + d(y_n, y^\circ),$$

which imply

$$-d(x^\circ, x_n) - d(y_n, y^\circ) \leq d(x_n, y_n) - d(x^\circ, y^\circ) \leq d(x_n, x^\circ) + d(y^\circ, y_n).$$

Therefore, we obtain

$$|d(x_n, y_n) - d(x^\circ, y^\circ)| \leq d(x_n, x^\circ) + d(y^\circ, y_n) = d(x^\circ, x_n) + d(y_n, y^\circ),$$

which shows that

$$\lim_{n\to\infty} d(x_n, y_n) = d(x^\circ, y^\circ).$$

Finally, we obtain

$$\begin{aligned} \lim_{n\to\infty} M(x_n, y_n, t_n) &= \lim_{n\to\infty} \frac{t_n}{t_n + d(x_n, y_n)} = \frac{t^\circ}{t^\circ + \lim\limits_{n\to\infty} d(x_n, y_n)} \\ &= \frac{t^\circ}{t^\circ + d(x^\circ, y^\circ)} = M(x^\circ, y^\circ, t^\circ). \end{aligned}$$

The convergence of this example does not need any extra sufficient conditions.

Remark 3. *If the inequality (10) is satisfied, then, by Part (i) of Lemma 2,*

$$\sup_n (a_n * b_n) = \left(\sup_n a_n\right) * \left(\sup_n b_n\right)$$

According to Part (ii) of Lemma 2, the inequality (10) can be replaced by the following statement: given any sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, for any $m, n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $a_m * b_n \leq a_{n_0} * b_{n_0}$.

Definition 6. *Let (X, d) be a metric space, and let (X, M) be a fuzzy semi-metric space.*

- *Given any fixed $y \in X$ and $0 < t < \infty$, we consider the mapping $M(\cdot, y, t) : X \to [0, 1]$. We say that the mapping $M(\cdot, y, t)$ is $\triangleleft$-continuous at x with respect to d if and only if $x_n \xrightarrow{d} x$ as $n \to \infty$ implies $M(x_n, y, t) \to M(x, y, t)$ as $n \to \infty$.*
- *Given any fixed $x \in X$ and $0 < t < \infty$, we consider the mapping $M(x, \cdot, t) : X \to [0, 1]$. We say that the mapping $M(x, \cdot, t)$ is $\triangleright$-continuous at x with respect to d if and only if $y_n \xrightarrow{d} y$ as $n \to \infty$ implies $M(x, y_n, t) \to M(x, y, t)$ as $n \to \infty$.*

Proposition 10. *Let (X, d) be a metric space; let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component; and let $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X \times (0, \infty)$ such that $t_n \to t^\circ$, $x_n \xrightarrow{d} x^\circ$ and $y_n \xrightarrow{d} y^\circ$ as $n \to \infty$. Assume that the following conditions are satisfied*

- *given any fixed $t \in (0, \infty)$, the mapping $M(\cdot, x^\circ, t) : X \to [0, 1]$ is $\triangleleft$-continuous at x° with respect to d;*
- *given any fixed $t \in (0, \infty)$, the mapping $M(y^\circ, \cdot, t) : X \to [0, 1]$ is $\triangleright$-continuous at y° with respect to d.*

Given $t^\circ > 0$, we have the following properties.

(i) *Suppose that M satisfies the $\bowtie$-triangle inequality.*

- *If M is left-continuous with respect to the distance at t°, then*

$$\liminf_{n\to\infty} M(x_n, y_n, t_n) \geq M(x^\circ, y^\circ, t^\circ).$$

- *If M is symmetrically left-continuous with respect to the distance at t°, then*

$$\liminf_{n\to\infty} M(x_n, y_n, t_n) \geq M(y^\circ, x^\circ, t^\circ).$$

(ii) *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. If M is left-continuous or symmetrically left-continuous with respect to the distance at t°, then*

$$\liminf_{n\to\infty} M(x_n, y_n, t_n) \geq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

Proof. Since $x_n \xrightarrow{d} x^\circ$ and $y_n \xrightarrow{d} y^\circ$ as $n \to \infty$, according to the assumption for continuities, we have

$$\lim_{n\to\infty} M\left(x_n, x^\circ, \frac{\epsilon}{2}\right) = M\left(x^\circ, x^\circ, \frac{\epsilon}{2}\right) = 1 \text{ and } \lim_{n\to\infty} M\left(y^\circ, y_n, \frac{\epsilon}{2}\right) = M\left(y^\circ, y^\circ, \frac{\epsilon}{2}\right) = 1,$$

which correspond to (9). The arguments in the proof of Proposition 5 are still valid, and the proof is complete. □

Proposition 11. *Let (X, d) be a metric space; let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component; and let $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X \times (0, \infty)$ such that $t_n \to t^\circ$, $x_n \xrightarrow{d} x^\circ$ and $y_n \xrightarrow{d} y^\circ$ as $n \to \infty$. Assume that the following inequality is satisfied*

$$\sup_n (a_n * b_n) \geq \left(\sup_n a_n\right) * \left(\sup_n b_n\right)$$

for any sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in $[0,1]$ and that the following conditions are satisfied

- *Given any fixed $t \in (0, \infty)$, the mapping $M(x^\circ, \cdot, t) : X \to [0,1]$ is $\triangleright$-continuous at x° with respect to d;*
- *Given any fixed $t \in (0, \infty)$, the mapping $M(\cdot, y^\circ, t) : X \to [0,1]$ is $\triangleleft$-continuous at y° with respect to d.*

Given $t^\circ > 0$, we have the following properties.

(i) *Suppose that M satisfies the $\bowtie$-triangle inequality.*

- *If M is right-continuous with respect to the distance at t°, then*

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ).$$

- *If M is symmetrically right-continuous with respect to the distance at t°, then*

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(y^\circ, x^\circ, t^\circ).$$

(ii) *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. If M is right-continuous or symmetrically right-continuous with respect to the distance at t°, then*

$$\limsup_{n\to\infty} M(x_n, y_n, t_n) \leq M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

Proof. Since $x_n \xrightarrow{d} x^\circ$ and $y_n \xrightarrow{d} y^\circ$ as $n \to \infty$, according to the assumption for continuities, we have

$$\lim_{n\to\infty} M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) = M\left(x^\circ, x^\circ, \frac{\epsilon}{2}\right) = 1 \text{ and } \lim_{n\to\infty} M\left(y_n, y^\circ, \frac{\epsilon}{2}\right) = M\left(y^\circ, y^\circ, \frac{\epsilon}{2}\right) = 1,$$

which correspond to (11). The arguments in the proof of Proposition 6 are still valid, and the proof is complete. □

Proposition 12. *Let (X,d) be a metric space; let (X,M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component; and let $\{(x_n,y_n,t_n)\}_{n=1}^{\infty}$ be a sequence in $X\times X\times(0,\infty)$ such that $t_n\to t^\circ$, $x_n\xrightarrow{d}x^\circ$ and $y_n\xrightarrow{d}y^\circ$ as $n\to\infty$. Assume that the following inequality is satisfied*

$$\sup_n\,(a_n*b_n)\geq\left(\sup_n a_n\right)*\left(\sup_n b_n\right)$$

for any sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in $[0,1]$, and that the following conditions are satisfied

- *given any fixed $t\in(0,\infty)$, the mapping $M(\cdot,x^\circ,t):X\to[0,1]$ is $\triangleleft$-continuous at x° with respect to d and the mapping $M(x^\circ,\cdot,t):X\to[0,1]$ is $\triangleright$-continuous at x° with respect to d;*
- *given any fixed $t\in(0,\infty)$, the mapping $M(\cdot,y^\circ,t):X\to[0,1]$ is $\triangleleft$-continuous at y° with respect to d, and the mapping $M(y^\circ,\cdot,t):X\to[0,1]$ is $\triangleright$-continuous at y° with respect to d.*

Given $t^\circ>0$, we have the following properties.

(i) Suppose that M satisfies the $\bowtie$-triangle inequality.

- *If M is continuous with respect to the distance at t°, then*

$$\lim_{n\to\infty}M(x_n,y_n,t_n)=M(x^\circ,y^\circ,t^\circ).$$

- *If M is symmetrically continuous with respect to the distance at t°, then*

$$\lim_{n\to\infty}M(x_n,y_n,t_n)=M(y^\circ,x^\circ,t^\circ).$$

(ii) Suppose that M satisfies the $\circ$-triangle inequality for $\circ\in\{\triangleright,\triangleleft,\diamond\}$. If M is continuous or symmetrically continuous with respect to the distance at t°, then

$$\lim_{n\to\infty}M(x_n,y_n,t_n)=M(x^\circ,y^\circ,t^\circ)=M(y^\circ,x^\circ,t^\circ).$$

Proof. From Propositions 10 and 11, we have

$$M(x^\circ,y^\circ,t^\circ)\leq\liminf_{n\to\infty}M\left(x_n,y_n,t_n\right)\leq\limsup_{n\to\infty}M\left(x_n,y_n,t_n\right)\leq M(x^\circ,y^\circ,t^\circ)$$

and

$$M(y^\circ,x^\circ,t^\circ)\leq\liminf_{n\to\infty}M\left(y_n,x_n,t_n\right)\leq\limsup_{n\to\infty}M\left(y_n,x_n,t_n\right)\leq M(y^\circ,x^\circ,t^\circ).$$

This completes the proof. □

Proposition 13. *Let (X,d) be a metric space, and let (X,M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component. Let $\{(x_n,y_n,t_n)\}_{n=1}^{\infty}$ be a sequence in $X\times X\times(0,\infty)$ such that $t_n\to t^\circ$, $x_n\xrightarrow{d}x^\circ$ and $y_n\xrightarrow{d}y^\circ$ as $n\to\infty$ and that $\{M(x_n,y_n,t_n)\}_{n=1}^{\infty}$ is an increasing sequence. Assume that the following conditions are satisfied*

- *given any fixed $t\in(0,\infty)$, the mapping $M(x^\circ,\cdot,t):X\to[0,1]$ is $\triangleright$-continuous at x° with respect to d;*
- *given any fixed $t\in(0,\infty)$, the mapping $M(\cdot,y^\circ,t):X\to[0,1]$ is $\triangleleft$-continuous at y° with respect to d.*

Given $t^\circ>0$, we have the following properties.

(i) Suppose that M satisfies the $\bowtie$-triangle inequality.

- *If M is right-continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) \le M(x^\circ, y^\circ, t^\circ).$$

- *If M is symmetrically right-continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) \le M(y^\circ, x^\circ, t^\circ).$$

(ii) *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. If M is right-continuous or symmetrically right-continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) \le M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

Proof. Since $x_n \xrightarrow{d} x^\circ$ and $y_n \xrightarrow{d} y^\circ$ as $n \to \infty$, according to the assumption for continuities, we have

$$\lim_{n\to\infty} M\left(x^\circ, x_n, \frac{\epsilon}{2}\right) = M\left(x^\circ, x^\circ, \frac{\epsilon}{2}\right) = 1 \text{ and } \lim_{n\to\infty} M\left(y_n, y^\circ, \frac{\epsilon}{2}\right) = M\left(y^\circ, y^\circ, \frac{\epsilon}{2}\right) = 1,$$

which correspond to (11). The arguments in the proof of Proposition 8 are still valid, and the proof is complete. □

Proposition 14. *Let (X, d) be a metric space; let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component; and let $\{(x_n, y_n, t_n)\}_{n=1}^\infty$ be a sequence in $X \times X \times (0, \infty)$ such that $t_n \to t^\circ$, $x_n \xrightarrow{d} x^\circ$ and $y_n \xrightarrow{d} y^\circ$ as $n \to \infty$ and that $\{M(x_n, y_n, t_n)\}_{n=1}^\infty$ is an increasing sequence. Assume that the following conditions are satisfied*

- *Given any fixed $t \in (0, \infty)$, the mapping $M(\cdot, x^\circ, t) : X \to [0, 1]$ is $\triangleleft$-continuous at x° with respect to d, and the mapping $M(x^\circ, \cdot, t) : X \to [0, 1]$ is $\triangleright$-continuous at x° with respect to d;*
- *Given any fixed $t \in (0, \infty)$, the mapping $M(\cdot, y^\circ, t) : X \to [0, 1]$ is $\triangleleft$-continuous at y° with respect to d, and the mapping $M(y^\circ, \cdot, t) : X \to [0, 1]$ is $\triangleright$-continuous at y° with respect to d.*

Given $t^\circ > 0$, we have the following properties.

(i) *Suppose that M satisfies the $\bowtie$-triangle inequality.*

- *If M is continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) = M(x^\circ, y^\circ, t^\circ).$$

- *If M is symmetrically continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) = M(y^\circ, x^\circ, t^\circ).$$

(ii) *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. If M is continuous or symmetrically continuous with respect to the distance at t°, then*

$$\sup_n M(x_n, y_n, t_n) = \lim_{n\to\infty} M(x_n, y_n, t_n) = M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ).$$

Proof. From Propositions 10 and 13, we have

$$M(x^\circ, y^\circ, t^\circ) \le \liminf_{n\to\infty} M(x_n, y_n, t_n) \le \lim_{n\to\infty} M(x_n, y_n, t_n) \le M(x^\circ, y^\circ, t^\circ)$$

and

$$M(y^\circ, x^\circ, t^\circ) \le \liminf_{n\to\infty} M(y_n, x_n, t_n) \le \lim_{n\to\infty} M(y_n, x_n, t_n) \le M(y^\circ, x^\circ, t^\circ).$$

This completes the proof. □

4. Dual Metric Convergence Based on the Infimum

Recall that the fuzzy semi-metric M is a mapping from $X \times X \times [0, \infty)$ into $[0, 1]$. Now, we are going to consider the dual sense by considering the mapping from $(0, 1] \times X \times X$ into $[0, \infty)$, which will be named as a dual fuzzy semi-metric. The potential application of dual fuzzy semi-metric will be studying the fixed point theorems in fuzzy semi-metric space by referring to Wu [8]. The dual fuzzy semi-metric was called as the auxiliary function in Wu [8] for only considering the $\bowtie$-triangle inequality. In this paper, we shall extend to study the dual fuzzy semi-metric by considering the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$.

For any fixed $x, y \in X$, Proposition 1 says that the mapping $M(x, y, \cdot)$ is nondecreasing, where the value $M(x, y, t)$ is interpreted as the membership degree of the distance between x and y that is less than t. Therefore, when t is sufficiently large, it is reasonable to argue that the membership degree $M(x, y, t)$ will be close to one. Alternatively, when t is sufficiently small, the membership degree $M(x, y, t)$ will be close to zero. Therefore, we propose the following definition.

Definition 7. *Let (X, M) be a fuzzy semi-metric space.*

- *We say that M satisfies the canonical condition if and only if*

$$\lim_{t \to +\infty} M(x, y, t) = 1 \text{ for any fixed } x, y \in X.$$

- *We say that M satisfies the rational condition if and only if*

$$\lim_{t \to 0+} M(x, y, t) = 0 \text{ for any fixed } x, y \in X \text{ with } x \neq y.$$

Example 4. *Continued from Example 1, for any fixed $x, y \in X$, since $d(x, y) < +\infty$, we see that*

$$\lim_{t \to +\infty} M(x, y, t) = \lim_{t \to +\infty} \frac{t}{t + d(x, y)} = 1$$

and

$$\lim_{t \to 0+} M(x, y, t) = \lim_{t \to 0+} \frac{t}{t + d(x, y)} = 0 \text{ for } x \neq y.$$

Therefore, M satisfies both the canonical and rational conditions.

The concept of dual fuzzy semi-metric is defined below.

Definition 8. *Let (X, M) be a fuzzy semi-metric space such that M satisfies the canonical condition. Given any fixed $x, y \in X$ and any fixed $\lambda \in (0, 1]$, we define the set*

$$\Omega^{\downarrow}(\lambda, x, y) = \{t > 0 : M(x, y, t) \geq 1 - \lambda\}$$

and the function $\Gamma^{\downarrow}(\lambda, \cdot, \cdot) : X \times X \to [0, +\infty)$ by

$$\Gamma^{\downarrow}(\lambda, x, y) = \inf \Omega^{\downarrow}(\lambda, x, y) = \inf \{t > 0 : M(x, y, t) \geq 1 - \lambda\}.$$

The mapping $\Gamma^{\downarrow}$ from $(0, 1] \times X \times X$ into $[0, \infty)$ is also called the infimum type of dual fuzzy semi-metric.

Example 5. *Continued from Example 1, we have*

$$\Omega^{\downarrow}(\lambda, x, y) = \left\{t > 0 : \frac{t}{t + d(x, y)} \geq 1 - \lambda\right\} = \left\{t > 0 : t \geq \left(\frac{1}{\lambda} - 1\right) \cdot d(x, y)\right\}$$

and

$$\Gamma^{\downarrow}(\lambda, x, y) = \inf \Omega^{\downarrow}(\lambda, x, y) = \inf \left\{ t > 0 : t \geq \left(\frac{1}{\lambda} - 1 \right) \cdot d(x, y) \right\} = \left(\frac{1}{\lambda} - 1 \right) \cdot d(x, y).$$

Since the symmetric condition is not satisfied, it means that $\Omega^{\downarrow}(\lambda, x, y) \neq \Omega^{\downarrow}(\lambda, y, x)$ and $\Gamma^{\downarrow}(\lambda, x, y) \neq \Gamma^{\downarrow}(\lambda, y, x)$ in general. We also need to claim that $\Omega^{\downarrow}(\lambda, x, y) \neq \emptyset$. Suppose that $\Omega^{\downarrow}(\lambda, x, y) = \emptyset$. Then, we must have $M(x, y, t) < 1 - \lambda$ for all $t > 0$. Therefore, we obtain

$$\lim_{t \to +\infty} M(x, y, t) \leq 1 - \lambda < 1,$$

which contradicts the fact that M satisfies the canonical condition. This says that Definition 8 is well-defined and $\Gamma^{\downarrow}(\lambda, x, y) < \infty$.

Remark 4. *We have the following observations.*

- *For $\lambda \in (0, 1]$, we have*

$$\Gamma^{\downarrow}(1, x, y) = \inf \{t > 0 : M(x, y, t) \geq 0\} = \inf\{t > 0\} = 0 \tag{12}$$

  and

$$\Gamma^{\downarrow}(\lambda, x, x) = \inf \{t > 0 : M(x, x, t) \geq 1 - \lambda\} = \inf\{t > 0\} = 0$$

- *Given any fixed $x, y \in X$, if $\mu > \lambda$, then*

$$\Omega^{\downarrow}(\lambda, x, y) \subseteq \Omega^{\downarrow}(\mu, x, y) \text{ and } \Gamma^{\downarrow}(\mu, x, y) \leq \Gamma^{\downarrow}(\lambda, x, y).$$

- *Given any fixed $x, y \in X$, suppose that $M(y, x, t) \geq M(x, y, t)$ for all $t > 0$. Then*

$$\Omega^{\downarrow}(\lambda, x, y) \subseteq \Omega^{\downarrow}(\lambda, y, x) \text{ and } \Gamma^{\downarrow}(\lambda, y, x) \leq \Gamma^{\downarrow}(\lambda, x, y).$$

In the sequel, we are going to establish the triangle inequalities and the convergence for the infimum type of dual fuzzy semi-metric $\Gamma^{\downarrow}$. Therefore, we shall first provide some relationships between the fuzzy semi-metric M and the infimum type of dual fuzzy semi-metric $\Gamma^{\downarrow}$.

Proposition 15. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical and rational conditions. Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$. Then, given any fixed $x, y \in X$ with $x \neq y$, we have $\Gamma^{\downarrow}(\lambda, x, y) \neq 0$, i.e., $\Gamma^{\downarrow}(\lambda, x, y) > 0$ for $\lambda \in (0, 1)$.*

Proof. Assume that $\Gamma^{\downarrow}(\lambda, x, y) = 0$ for $\lambda \in (0, 1)$. By the concept of the infimum for $\Gamma^{\downarrow}(\lambda, x, y)$, given any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$M(x, y, t_\epsilon) \geq 1 - \lambda \text{ and } t_\epsilon < \Gamma^{\downarrow}(\lambda, x, y) + \epsilon = \epsilon.$$

We consider the following cases.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. By Parts (i) and (ii) of Proposition 1, we have

$$M(x, y, \epsilon) \geq M(x, y, t_\epsilon) \geq 1 - \lambda.$$

  Since M satisfies the rational condition, it follows that

$$0 = \lim_{\epsilon \to 0+} M(x, y, \epsilon) \geq 1 - \lambda,$$

  which contradicts $0 < \lambda < 1$.

- Suppose that M satisfies the $\diamond$-triangle inequality. By Part (iii) of Proposition 1, we have

$$M\,(y,x,\epsilon) \geq M(x,y,t_\epsilon) \geq 1-\lambda.$$

Since M satisfies the rational condition, it follows that

$$0 = \lim_{\epsilon\to 0+} M\,(y,x,\epsilon) \geq 1-\lambda,$$

which contradicts $0<\lambda<1$.

Therefore, we conclude that $\Gamma^{\downarrow}(\lambda,x,y) \neq 0$ for $\lambda \in (0,1)$. This completes the proof. □

Proposition 16. *Let (X,M) be a fuzzy semi-metric space such that M satisfies the canonical condition. Given any fixed $x,y \in X$ and any fixed $\lambda \in (0,1)$, we have the following properties.*

(i) *If $\epsilon > 0$ is sufficiently small such that $\epsilon < \Gamma^{\downarrow}(\lambda,x,y)$, then*

$$M\left(x,y,\Gamma^{\downarrow}(\lambda,x,y)-\epsilon\right) < 1-\lambda. \tag{13}$$

If we further assume that M is left-continuous with respect to the distance on $(0,\infty)$ and that $\Gamma^{\downarrow}(\lambda,x,y) > 0$, then

$$M\left(x,y,\Gamma^{\downarrow}(\lambda,x,y)\right) \leq 1-\lambda. \tag{14}$$

(ii) *For any $\epsilon > 0$, we have the following properties.*

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie,\triangleright,\triangleleft\}$. Then*

$$M\left(x,y,\Gamma^{\downarrow}(\lambda,x,y)+\epsilon\right) \geq 1-\lambda. \tag{15}$$

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright,\triangleleft,\diamond\}$. Then*

$$M\left(x,y,\Gamma^{\downarrow}(\lambda,y,x)+\epsilon\right) \geq 1-\lambda. \tag{16}$$

Proof. To prove Part (i), we first note that $\Gamma^{\downarrow}(\lambda,x,y)-\epsilon > 0$. Suppose that $M(x,y,\Gamma^{\downarrow}(\lambda,x,y)-\epsilon) \geq 1-\lambda$. By the definition of $\Gamma^{\downarrow}$, we have $\Gamma^{\downarrow}(\lambda,x,y) \leq \Gamma^{\downarrow}(\lambda,x,y)-\epsilon$, which is a contradiction. Therefore, we must have $M(x,y,\Gamma^{\downarrow}(\lambda,x,y)-\epsilon) < 1-\lambda$. Using the left-continuity of M and applying $\epsilon \to 0+$ to the inequality (13), we obtain (14).

To prove Part (ii), we consider the following cases.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie,\triangleright,\triangleleft\}$. By the concept of infimum for $\Gamma^{\downarrow}(\lambda,x,y)$, given any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $M(x,y,t_\epsilon) \geq 1-\lambda$ and $t_\epsilon < \Gamma^{\downarrow}(\lambda,x,y)+\epsilon$. By Parts (i) and (ii) of Proposition 1, we have

$$M\left(x,y,\Gamma^{\downarrow}(\lambda,x,y)+\epsilon\right) \geq M(x,y,t_\epsilon) \geq 1-\lambda.$$

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright,\triangleleft,\diamond\}$. By the concept of infimum for $\Gamma^{\downarrow}(\lambda,y,x)$, given any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $M(y,x,t_\epsilon) \geq 1-\lambda$ and $t_\epsilon < \Gamma^{\downarrow}(\lambda,y,x)+\epsilon$. By Parts (ii) and (iii) of Proposition 1, we have

$$M\left(x,y,\Gamma^{\downarrow}(\lambda,y,x)+\epsilon\right) \geq M(y,x,t_\epsilon) \geq 1-\lambda.$$

This completes the proof. □

Proposition 17. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition. Given any fixed $x, y \in X$ and any fixed $\lambda \in (0, 1)$, we have the following properties.*

(i) *If $t > \Gamma^{\downarrow}(\lambda, x, y)$, then the following statements hold true.*

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$. Then, $M(x, y, t) \geq 1 - \lambda$.*
- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then, $M(y, x, t) \geq 1 - \lambda$, i.e.,*

$$\min\{M(x, y, t), M(y, x, t)\} \geq 1 - \lambda.$$

(ii) *If $0 < t < \Gamma^{\downarrow}(\lambda, x, y)$, then the following statements hold true.*

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then, $M(x, y, t) < 1 - \lambda$.*
- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then, $M(y, x, t) < 1 - \lambda$.*

Proof. To prove the first case of Part (i), if $t > \Gamma^{\downarrow}(\lambda, x, y)$, there exists $\epsilon > 0$ such that $t \geq \Gamma^{\downarrow}(\lambda, x, y) + \epsilon$.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Using Parts (i) and (ii) of Proposition 1 and (15), we obtain

$$M(x, y, t) \geq M\left(x, y, \Gamma^{\downarrow}(\lambda, x, y) + \epsilon\right) \geq 1 - \lambda.$$

- Suppose that M satisfies the $\diamond$-triangle inequality. Using Part (iii) of Proposition 1 and (16), we obtain

$$M(x, y, t) \geq M\left(y, x, \Gamma^{\downarrow}(\lambda, x, y) + \epsilon\right) \geq 1 - \lambda.$$

The second case of Part (i) follows from the second case of Part (i) of Proposition 18 below by considering the contrapositive statement.

To prove Part (ii), if $t < \Gamma^{\downarrow}(\lambda, x, y)$, there exists a sufficiently small $\epsilon > 0$ such that $0 < t \leq \Gamma^{\downarrow}(\lambda, x, y) - \epsilon$.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Using Parts (i) and (ii) of Proposition 1 and (13), we obtain

$$M(x, y, t) \leq M\left(x, y, \Gamma^{\downarrow}(\lambda, x, y) - \epsilon\right) < 1 - \lambda.$$

- Suppose that M satisfies the $\diamond$-triangle inequality. Using Parts (ii) and (iii) of Proposition 1 and (13), we obtain

$$M(y, x, t) \leq M\left(x, y, \Gamma^{\downarrow}(\lambda, x, y) - \epsilon\right) < 1 - \lambda.$$

This completes the proof. □

Proposition 18. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition. Given any fixed $x, y \in X$ and any fixed $\lambda \in (0, 1)$, we have the following properties.*

(i) *If $M(x, y, t) < 1 - \lambda$, then the following statements hold true.*

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$. Then, $t \leq \Gamma^{\downarrow}(\lambda, x, y)$.*
- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then, $t \leq \Gamma^{\downarrow}(\lambda, y, x)$, i.e.,*

$$t \leq \max\{\Gamma^{\downarrow}(\lambda, x, y), \Gamma^{\downarrow}(\lambda, y, x)\}.$$

(ii) *If $M(x, y, t) = 1 - \lambda$ for $t > 0$, then the following statements hold true.*

- *Suppose that M satisfies the strict $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. If $\Gamma^{\downarrow}(\lambda, x, y) > 0$, then $t = \Gamma^{\downarrow}(\lambda, x, y)$.*

- *Suppose that M satisfies the strict $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. If $\Gamma^{\downarrow}(\lambda, y, x) > 0$, then $t = \Gamma^{\downarrow}(\lambda, y, x)$.*

(iii) If $M(x, y, t) \geq 1 - \lambda$, then the following statements hold true.

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then, $t \geq \Gamma^{\downarrow}(\lambda, x, y)$.*
- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then, $t \geq \Gamma^{\downarrow}(\lambda, y, x)$.*

Proof. The first case of Part (i) follows from Part (i) of Proposition 17 by considering the contrapositive statement. To prove the second case of Part (i), using the concept of the infimum for $\Gamma^{\downarrow}(\lambda, y, x)$, given any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $M(y, x, t_\epsilon) \geq 1 - \lambda$ and $t_\epsilon < \Gamma^{\downarrow}(\lambda, y, x) + \epsilon$. Using Parts (ii) and (iii) of Proposition 1, if $t > t_\epsilon$, then $M(x, y, t) \geq M(y, x, t_\epsilon)$, which contradicts $M(x, y, t) < 1 - \lambda$. It says that

$$t \leq t_\epsilon < \Gamma^{\downarrow}(\lambda, y, x) + \epsilon.$$

Since ϵ can be any positive real number, we must have $t \leq \Gamma^{\downarrow}(\lambda, y, x)$.

To prove Part (ii), we consider the following cases.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Using the concept of the infimum for $\Gamma^{\downarrow}(\lambda, x, y)$, given any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $M(x, y, t_\epsilon) \geq 1 - \lambda$ and $t_\epsilon < \Gamma^{\downarrow}(\lambda, x, y) + \epsilon$. Using Parts (i) and (ii) of Proposition 1, if $t > t_\epsilon$, then

$$M(x, y, t) > M(x, y, t_\epsilon) \geq 1 - \lambda,$$

which contradicts $M(x, y, t) = 1 - \lambda$. It says that

$$t \leq t_\epsilon < \Gamma^{\downarrow}(\lambda, x, y) + \epsilon.$$

Since ϵ can be any positive real number, we must have $t \leq \Gamma^{\downarrow}(\lambda, x, y)$. Suppose that $t < \Gamma^{\downarrow}(\lambda, x, y)$. Using the first case of Part (ii) of Proposition 17, this also contradicts $M(x, y, t) = 1 - \lambda$. Therefore, we must have $t = \Gamma^{\downarrow}(\lambda, x, y)$.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Using the argument of Part (i) by considering $M(x, y, t) = 1 - \lambda$, we can similarly obtain $t \leq \Gamma^{\downarrow}(\lambda, y, x)$. Suppose that $t < \Gamma^{\downarrow}(\lambda, y, x)$. Using the second case of Part (ii) of Proposition 17, this contradicts $M(x, y, t) = 1 - \lambda$. Therefore, we must have $t = \Gamma^{\downarrow}(\lambda, y, x)$.

Finally, Part (iii) follows from Proposition 17 immediately by considering the contrapositive statements. This completes the proof. □

Proposition 19. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition. Given any fixed $x, y \in X$, we define*

$$\Omega^{\downarrow}(0+, x, y) \equiv \bigcap_{0 < \lambda \leq 1} \Omega^{\downarrow}(\lambda, x, y).$$

If $\{t > 0 : M(x, y, t) = 1\} \neq \emptyset$, then

$$\Omega^{\downarrow}(0+, x, y) = \{t > 0 : M(x, y, t) = 1\}.$$

Proof. If $M(x, y, t) = 1$, then $M(x, y, t) \geq 1 - \lambda$ for all $\lambda \in (0, 1]$, which says that

$$\{t > 0 : M(x, y, t) = 1\} \subseteq \bigcap_{0 < \lambda \leq 1} \Omega^{\downarrow}(\lambda, x, y) = \Omega^{\downarrow}(0+, x, y).$$

On the other hand, for $t \in \Omega^{\downarrow}(0+, x, y)$, i.e., $M(x, y, t) \geq 1 - \lambda$ for all $\lambda \in (0, 1]$, we have $M(x, y, t) \geq 1$, i.e., $M(x, y, t) = 1$. This completes the proof. □

In order to establish the triangle inequalities for the infimum type of dual fuzzy semi-metric, we need to provide a useful lemma.

Lemma 3. *Suppose that the t-norm $*$ is left-continuous at one with respect to the first or second component. For any $a \in (0,1)$ and any $p \in \mathbb{N}$, there exists $r \in (0,1)$ such that $\overbrace{r * r * \cdots * r}^{p \text{ times}} > a$.*

Proof. If $r_n \uparrow 1$, then, using Part (ii) of Proposition 1, we have $r_n * r_n \uparrow 1$. Therefore, for any $0 < a < 1$, there exists $r \in (0,1)$ such that $a < r * r$. In general, using the increasing property of the t-norm, we have

$$
\begin{aligned}
a &< r_1 * r_1 \\
&\leq r_1 * (r_2 * r_2) = r_1 * r_2 * r_2 \\
&\leq r_1 * r_2 * (r_3 * r_3) = r_1 * r_2 * r_3 * r_3 \\
&\vdots \\
&\leq r_1 * r_2 * r_3 * \cdots * r_{p-1} * r_p * r_p \\
&\leq r_1 * r_2 * r_3 * \cdots * r_{p-1} * r_p * 1 \\
&= r_1 * r_2 * r_3 * \cdots * r_{p-1} * r_p.
\end{aligned}
$$

Let

$$r = \max\left\{r_1, \cdots, r_p\right\}.$$

According to the the increasing property of the t-norm, we obtain

$$a < r * r * \cdots * r.$$

This completes the proof. □

Theorem 1. *(Triangle inequalities for the dual fuzzy semi-metric) Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the t-norm $*$ is left-continuous at one in the first or second component.*

(i) *Suppose that M satisfies the $\bowtie$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$ such that*

$$\Gamma^{\downarrow}(\mu, x_1, x_p) \leq \Gamma^{\downarrow}(\lambda, x_1, x_2) + \Gamma^{\downarrow}(\lambda, x_2, x_3) + \Gamma^{\downarrow}(\lambda, x_3, x_4) + \cdots + \Gamma^{\downarrow}(\lambda, x_{p-1}, x_p). \tag{17}$$

(ii) *Suppose that M satisfies the $\triangleright$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$ such that*

$$
\begin{aligned}
&\max\left\{\Gamma^{\downarrow}(\mu, x_1, x_p), \Gamma^{\downarrow}(\mu, x_p, x_1)\right\} \\
&\quad \leq \Gamma^{\downarrow}(\lambda, x_1, x_2) + \Gamma^{\downarrow}(\lambda, x_3, x_2) + \Gamma^{\downarrow}(\lambda, x_4, x_3) + \cdots + \Gamma^{\downarrow}(\lambda, x_p, x_{p-1}).
\end{aligned}
$$

(iii) *Suppose that M satisfies the $\triangleleft$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$ such that*

$$
\begin{aligned}
&\max\left\{\Gamma^{\downarrow}(\mu, x_1, x_p), \Gamma^{\downarrow}(\mu, x_p, x_1)\right\} \\
&\quad \leq \Gamma^{\downarrow}(\lambda, x_2, x_1) + \Gamma^{\downarrow}(\lambda, x_2, x_3) + \Gamma^{\downarrow}(\lambda, x_3, x_4) + \cdots + \Gamma^{\downarrow}(\lambda, x_{p-1}, x_p).
\end{aligned}
$$

(iv) *Suppose that M satisfies the $\diamond$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$ such that the following inequalities are satisfied.*

- *If p is even, then*

$$\begin{aligned}\Gamma^{\downarrow}(\lambda, x_p, x_1) \leq \Gamma^{\downarrow}(\lambda, x_1, x_2) + \Gamma^{\downarrow}(\lambda, x_2, x_3) + \Gamma^{\downarrow}(\lambda, x_4, x_3) + \Gamma^{\downarrow}(\lambda, x_4, x_5) + \Gamma^{\downarrow}(\lambda, x_6, x_5) \\ + \Gamma^{\downarrow}(\lambda, x_6, x_7) + \cdots + \Gamma^{\downarrow}(\lambda, x_p, x_{p-1})\end{aligned} \tag{18}$$

and

$$\begin{aligned}\Gamma^{\downarrow}(\lambda, x_1, x_p) \leq \Gamma^{\downarrow}(\lambda, x_2, x_1) + \Gamma^{\downarrow}(\lambda, x_3, x_2) + \Gamma^{\downarrow}(\lambda, x_3, x_4) + \Gamma^{\downarrow}(\lambda, x_5, x_4) + \Gamma^{\downarrow}(\lambda, x_5, x_6) \\ + \Gamma^{\downarrow}(\lambda, x_7, x_6) + \cdots + \Gamma^{\downarrow}(\lambda, x_{p-1}, x_p).\end{aligned}$$

- *If p is odd, then*

$$\begin{aligned}\Gamma^{\downarrow}(\lambda, x_1, x_p) \leq \Gamma^{\downarrow}(\lambda, x_1, x_2) + \Gamma^{\downarrow}(\lambda, x_2, x_3) + \Gamma^{\downarrow}(\lambda, x_4, x_3) + \Gamma^{\downarrow}(\lambda, x_4, x_5) + \Gamma^{\downarrow}(\lambda, x_6, x_5) \\ + \Gamma^{\downarrow}(\lambda, x_6, x_7) + \cdots + \Gamma^{\downarrow}(\lambda, x_{p-1}, x_p)\end{aligned}$$

and

$$\begin{aligned}\Gamma^{\downarrow}(\lambda, x_p, x_1) \leq \Gamma^{\downarrow}(\lambda, x_2, x_1) + \Gamma^{\downarrow}(\lambda, x_3, x_2) + \Gamma^{\downarrow}(\lambda, x_3, x_4) + \Gamma^{\downarrow}(\lambda, x_5, x_4) + \Gamma^{\downarrow}(\lambda, x_5, x_6) \\ + \Gamma^{\downarrow}(\lambda, x_7, x_6) + \cdots + \Gamma^{\downarrow}(\lambda, x_p, x_{p-1}).\end{aligned}$$

Proof. To prove Part (i), if $\mu = 1$, then $\Gamma^{\downarrow}(1, x_1, x_p) = 0$ by (12), and the result is obvious. Therefore, we assume $\mu \in (0, 1)$. According to Lemma 3, there exists $\lambda \in (0, 1)$ such that

$$(1 - \lambda) * \cdots * (1 - \lambda) > 1 - \mu. \tag{19}$$

Given any $\epsilon > 0$, we have

$$\begin{aligned}&M\left(x_1, x_p, \Gamma^{\downarrow}(\lambda, x_1, x_2) + \Gamma^{\downarrow}(\lambda, x_2, x_3) + \cdots + \Gamma^{\downarrow}(\lambda, x_{p-1}, x_p) + (p-1)\epsilon\right) \\ &\quad \geq M\left(x_1, x_2, \Gamma^{\downarrow}(\lambda, x_1, x_2) + \epsilon\right) * \cdots * M\left(x_{p-1}, x_p, \Gamma^{\downarrow}(\lambda, x_{p-1}, x_p) + \epsilon\right) \\ &\qquad \text{(by the first observation of Remark 1)} \\ &\quad \geq (1 - \lambda) * \cdots * (1 - \lambda) \text{ (by (15) and the increasing property of the t-norm)} \\ &\quad > 1 - \mu \text{ (by (19))}.\end{aligned}$$

By the definition of Γ_λ, we obtain

$$\Gamma^{\downarrow}(\lambda, x_1, x_2) + \Gamma^{\downarrow}(\lambda, x_2, x_3) + \cdots + \Gamma^{\downarrow}(\lambda, x_{p-1}, x_p) + (p-1)\epsilon \geq \Gamma^{\downarrow}(\mu, x_1, x_p).$$

By taking $\epsilon \to 0+$, we obtain the inequality (17). Part (ii) can be obtained by applying the second observation of Remark 1 to the above argument. Part (iii) can be obtained by applying the third observation of Remark 1 to the above argument.

To prove Part (iv), we first assume that p is even. Then

$$\begin{aligned}
&M\Big(x_p, x_1, \Gamma^{\downarrow}(\lambda, x_2, x_1) + \Gamma^{\downarrow}(\lambda, x_3, x_2) + \Gamma^{\downarrow}(\lambda, x_3, x_4) + \Gamma^{\downarrow}(\lambda, x_5, x_4) + \Gamma^{\downarrow}(\lambda, x_5, x_6) \\
&\qquad + \Gamma^{\downarrow}(\lambda, x_7, x_6) + \cdots + \Gamma^{\downarrow}(\lambda, x_{p-1}, x_p) + (p-1)\epsilon\Big) \\
&\geq M\Big(x_1, x_2, \Gamma^{\downarrow}(\lambda, x_2, x_1) + \epsilon\Big) * M\Big(x_2, x_3, \Gamma^{\downarrow}(\lambda, x_3, x_2) + \epsilon\Big) \\
&\quad * M\Big(x_4, x_3, \Gamma^{\downarrow}(\lambda, x_3, x_4) + \epsilon\Big) * M\Big(x_4, x_5, \Gamma^{\downarrow}(\lambda, x_5, x_4) + \epsilon\Big) \\
&\quad * M\Big(x_6, x_5, \Gamma^{\downarrow}(\lambda, x_5, x_6) + \epsilon\Big) * M\Big(x_6, x_7, \Gamma^{\downarrow}(\lambda, x_7, x_6) + \epsilon\Big) \\
&\quad * \cdots * M\Big(x_{p-1}, x_p, \Gamma^{\downarrow}(\lambda, x_p, x_{p-1}) + \epsilon\Big) \\
&\qquad \text{(by the fourth observation of Remark 1)} \\
&\geq (1-\lambda) * \cdots * (1-\lambda) \text{ (by (16) and the increasing property of the t-norm)} \\
&> 1 - \mu \text{ (by (19))},
\end{aligned}$$

which implies the inequality (18) by the above similar argument. The other inequalities can be similarly obtained. This completes the proof. □

The main convergence theorem is presented below.

Theorem 2. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the t-norm $*$ is left-continuous at one with respect to the first or second component. Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Then, we have the following properties.*

(i) $x_n \xrightarrow{M^{\triangleright}} x$ *as* $n \to \infty$ *if and only if* $\Gamma^{\downarrow}(\lambda, x_n, x) \to 0$ *as* $n \to \infty$ *for all* $\lambda \in (0,1)$.

(ii) $x_n \xrightarrow{M^{\triangleleft}} x$ *as* $n \to \infty$ *if and only if* $\Gamma^{\downarrow}(\lambda, x, x_n) \to 0$ *as* $n \to \infty$ *for all* $\lambda \in (0,1)$.

Proof. It suffices to prove Part (i). Suppose that $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$, which says that, given any $t > 0$ and $\delta > 0$, there exists $n_{t,\delta} \in \mathbb{N}$ such that $|M(x_n, x, t) - 1| < \delta$ for $n \geq n_{t,\delta}$, which also says that, for any fixed $\lambda \in (0,1)$, given any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $|M(x_n, x, \epsilon/2) - 1| < \lambda$ for $n \geq n_\epsilon$. Therefore, we obtain $M(x_n, x, \epsilon/2) > 1 - \lambda$ for $n \geq n_\epsilon$, i.e.,

$$\Gamma^{\downarrow}(\lambda, x_n, x) \leq \frac{\epsilon}{2} < \epsilon$$

for $n \geq n_\epsilon$ by the definition of Γ_λ. This shows that $\Gamma^{\downarrow}(\lambda, x_n, x) \to 0$ as $n \to \infty$. For the converse, suppose that $\Gamma^{\downarrow}(\lambda, x_n, x) \to 0$ as $n \to \infty$ for all $\lambda \in (0,1)$. Given any $\delta > 0$ and $\lambda \in (0,1)$, there exists $n_{\delta,\lambda} \in \mathbb{N}$ such that $|\Gamma^{\downarrow}(\lambda, x_n, x)| < \delta$ for all $n \geq n_{\delta,\lambda}$, which also says that, for any fixed $t > 0$, given any $\epsilon \in (0,1)$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$\Gamma^{\downarrow}\left(\frac{\epsilon}{2}, x_n, x\right) = \left|\Gamma^{\downarrow}\left(\frac{\epsilon}{2}, x_n, x\right)\right| < t$$

for $n \geq n_\epsilon$. Therefore, we obtain

$$M(x_n, x, t) \geq 1 - \frac{\epsilon}{2} > 1 - \epsilon$$

for $n \geq n_\epsilon$ by Part (i) of Proposition 17. This shows that $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$, and the proof is complete. □

Example 6. *Continued from Examples 2 and 5, we have*

$$x_n \xrightarrow{M^{\triangleright}} x \text{ if and only if } \lim_{n\to\infty} d(x_n, x) = 0$$
$$x_n \xrightarrow{M^{\triangleleft}} x \text{ if and only if } \lim_{n\to\infty} d(x, x_n) = 0$$
$$\Gamma^{\downarrow}(\lambda, x, y) = \left(\frac{1}{\lambda} - 1\right) \cdot d(x, y).$$

It is clear to see that $x_n \xrightarrow{M^{\triangleright}} x$ *if and only if* $\Gamma^{\downarrow}(\lambda, x_n, x) \to 0$, *and* $x_n \xrightarrow{M^{\triangleleft}} x$ *if and only if* $\Gamma^{\downarrow}(\lambda, x, x_n) \to 0$ *for all* $\lambda \in (0, 1)$.

Next, we consider the Cauchy sequence that will be useful for studying the fixed point theorems in fuzzy semi-metric space.

Definition 9. *Let* (X, M) *be a fuzzy semi-metric space, and let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* X.

- *We say that* $\{x_n\}_{n=1}^{\infty}$ *is a* $>$*-Cauchy sequence in the metric sense if and only if, given any pair* (r, t) *with* $t > 0$ *and* $0 < r < 1$, *there exists* $n_{r,t} \in \mathbb{N}$ *such that* $M(x_m, x_n, t) > 1 - r$ *for all pairs* (m, n) *of integers* m *and* n *with* $m > n \geq n_{r,t}$.
- *We say that* $\{x_n\}_{n=1}^{\infty}$ *is a* $<$*-Cauchy sequence in the metric sense if and only if, given any pair* (r, t) *with* $t > 0$ *and* $0 < r < 1$, *there exists* $n_{r,t} \in \mathbb{N}$ *such that* $M(x_n, x_m, t) > 1 - r$ *for all pairs* (m, n) *of integers* m *and* n *with* $m > n \geq n_{r,t}$.
- *We say that* $\{x_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in the metric sense if and only if, given any pair* (r, t) *with* $t > 0$ *and* $0 < r < 1$, *there exists* $n_{r,t} \in \mathbb{N}$ *such that* $M(x_m, x_n, t) > 1 - r$ *and* $M(x_n, x_m, t) > 1 - r$ *for all pairs* (m, n) *of integers* m *and* n *with* $m, n \geq n_{r,t}$ *and* $m \neq n$.

Definition 10. *Let* (X, M) *be a fuzzy semi-metric space such that* M *satisfies the canonical condition, and let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* X.

- *Given any fixed* $\lambda \in (0, 1)$, *we say that* $\{x_n\}_{n=1}^{\infty}$ *is a* $(\lambda, >)$*-Cauchy sequence with respect to* $\Gamma^{\downarrow}$ *if and only if, given any* $\epsilon > 0$, *there exists* $n_{\epsilon} \in \mathbb{N}$ *such that* $m > n \geq n_{\epsilon}$ *implies* $\Gamma^{\downarrow}(\lambda, x_m, x_n) < \epsilon$.
- *Given any fixed* $\lambda \in (0, 1)$, *we say that* $\{x_n\}_{n=1}^{\infty}$ *is a* $(\lambda, <)$*-Cauchy sequence with respect to* $\Gamma^{\downarrow}$ *if and only if, given any* $\epsilon > 0$, *there exists* $n_{\epsilon} \in \mathbb{N}$ *such that* $n > m \geq n_{\epsilon}$ *implies* $\Gamma^{\downarrow}(\lambda, x_m, x_n) < \epsilon$.
- *Given any fixed* $\lambda \in (0, 1)$, *we say that* $\{x_n\}_{n=1}^{\infty}$ *is a* λ*-Cauchy sequence with respect to* $\Gamma^{\downarrow}$ *if and only if, given any* $\epsilon > 0$, *there exists* $n_{\epsilon} \in \mathbb{N}$ *such that* $m, n \geq n_{\epsilon}$ *implies* $\Gamma^{\downarrow}(\lambda, x_m, x_n) < \epsilon$ *and* $\Gamma^{\downarrow}(\lambda, x_n, x_m) < \epsilon$.

Theorem 3. *Let* (X, M) *be a fuzzy semi-metric space along with a t-norm* $*$ *such that* M *satisfies the canonical condition and the t-norm* $*$ *is left-continuous at one with respect to the first or second component. Suppose that* M *satisfies the* $\circ$*-triangle inequality for* $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* X. *Then, we have the following properties.*

(i) $\{x_n\}_{n=1}^{\infty}$ *is a* $>$*-Cauchy sequence in the metric sense if and only if it is a* $(\lambda, >)$*-Cauchy sequence with respect to* $\Gamma^{\downarrow}$ *for all* $\lambda \in (0, 1)$.

(ii) $\{x_n\}_{n=1}^{\infty}$ *is a* $<$*-Cauchy sequence in the metric sense if and only if it is a* $(\lambda, <)$*-Cauchy sequence with respect to* $\Gamma^{\downarrow}$ *for all* $\lambda \in (0, 1)$.

Proof. It suffices to prove Part (i). Suppose that $\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence in the metric sense, which says that, given any $t > 0$ and $\delta > 0$, there exists $n_{t,\delta} \in \mathbb{N}$ such that $m > n \geq n_{t,\delta}$ implies $M(x_m, x_n, t) > 1 - \delta$. In other words, for any fixed $\lambda \in (0, 1)$, given any $\epsilon > 0$, there exists $n_{\epsilon} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon}$ implies $M(x_m, x_n, \epsilon/2) > 1 - \lambda$. By the definition of Γ_{λ}, we obtain

$$\Gamma^{\downarrow}(\lambda, x_m, x_n) \leq \frac{\epsilon}{2} < \epsilon$$

for $m > n \geq n_\epsilon$, which shows that $\{x_n\}_{n=1}^{\infty}$ is a $(\lambda, >)$-Cauchy sequence with respect to $\Gamma^{\downarrow}$. For the converse, given any $\delta > 0$ and $\lambda \in (0, 1]$, there exists $n_{\delta,\lambda} \in \mathbb{N}$ such that $m > n \geq n_{\delta,\lambda}$ implies $\Gamma^{\downarrow}(\lambda, x_m, x_n) < \delta$, which also says that, for any fixed $t > 0$, given any $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$ such that $m > n \geq n_\epsilon$ implies $\Gamma^{\downarrow}(\epsilon/2, x_m, x_n) < t$. Therefore, we obtain

$$M(x_m, x_n, t) \geq 1 - \frac{\epsilon}{2} > 1 - \epsilon$$

for $m > n \geq n_\epsilon$ by Part (i) of Proposition 17. This shows that $\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence in the metric sense, and the proof is complete. □

Example 7. *Continued from Examples 1 and 5, we have*

$$M(x_m, x_n, t) > 1 - r \text{ if and only if } \frac{t}{t + d(x_m, x_n)} > 1 - r \text{ if and only if } \left(\frac{1}{\lambda} - 1\right) \cdot d(x_m, x_n) < t$$

and

$$\Gamma^{\downarrow}(\lambda, x_m, x_n) < \epsilon \text{ if and only if } \left(\frac{1}{\lambda} - 1\right) \cdot d(x_m, x_n) < \epsilon.$$

It is clear to see that $\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence in the metric sense if and only if it is a $(\lambda, >)$-Cauchy sequence with respect to $\Gamma^{\downarrow}$ for all $\lambda \in (0, 1)$.

5. Dual Metric Convergence Based on the Supremum

Recall that the infimum type of dual fuzzy semi-metric is based on the infimum in which the canonical conditions are assumed to be satisfied. The purpose for considering the canonical condition is to guarantee the infimum type of dual fuzzy semi-metric space is well-defined.

In this section, we shall consider the supremum type of dual fuzzy semi-metric based on the supremum. In this case, we should assume that the rational conditions is satisfied.

Definition 11. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition. Given any fixed $x, y \in X$ with $x \neq y$ and any fixed $\lambda \in [0, 1)$, we define the set*

$$\Omega^{\uparrow}(\lambda, x, y) = \{t > 0 : M(x, y, t) \leq 1 - \lambda\}$$

and the function $\Gamma^{\uparrow}(\lambda, \cdot, \cdot) : X \times X \to [0, +\infty)$ by:

$$\Gamma^{\uparrow}(\lambda, x, y) = \sup \Omega^{\uparrow}(\lambda, x, y) = \sup \{t > 0 : M(x, y, t) \leq 1 - \lambda\}.$$

The mapping $\Gamma^{\uparrow}$ from $(0, 1] \times X \times X$ into $[0, \infty)$ is also called the supremum type of dual fuzzy semi-metric.

Example 8. *Continued from Example 1, we have*

$$\Omega^{\uparrow}(\lambda, x, y) = \left\{t > 0 : \frac{t}{t + d(x, y)} \leq 1 - \lambda\right\} = \left\{t > 0 : t \leq \left(\frac{1}{\lambda} - 1\right) \cdot d(x, y)\right\}$$

and

$$\Gamma^{\uparrow}(\lambda, x, y) = \sup \Omega^{\uparrow}(\lambda, x, y) = \sup \left\{t > 0 : t \leq \left(\frac{1}{\lambda} - 1\right) \cdot d(x, y)\right\} = \left(\frac{1}{\lambda} - 1\right) \cdot d(x, y).$$

We need to claim that $\Omega^{\uparrow}(\lambda, x, y) \neq \emptyset$. Suppose that $\Omega^{\uparrow}(\lambda, x, y) = \emptyset$. Then, we must have $M(x, y, t) > 1 - \lambda$ for all $t > 0$. Therefore, we obtain

$$\lim_{t \to 0+} M(x, y, t) \geq 1 - \lambda > 0,$$

which contradicts the fact that M satisfies the the rational condition. This says that Definition 11 is well defined, which also says that $\Gamma^{\uparrow}(\lambda, x, y) > 0$. On the other hand, we have

$$\Gamma^{\uparrow}(0, x, y) = \sup\{t > 0 : M(x, y, t) \leq 1\} = \sup\{t > 0\} = +\infty.$$

If $\mu < \lambda$, then

$$\Omega^{\uparrow}(\lambda, x, y) \subseteq \Omega^{\uparrow}(\mu, x, y) \text{ and } \Gamma^{\uparrow}(\mu, x, y) \geq \Gamma^{\uparrow}(\lambda, x, y).$$

Proposition 20. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition. Given any fixed $x, y \in X$ with $x \neq y$, suppose that $\Gamma^{\uparrow}(\lambda, x, y) = +\infty$. Then, we have the following results.*

(i) If M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$, then $M(x, y, t) \leq 1 - \lambda$ for all $t > 0$.
(ii) If M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$, then $M(y, x, t) \leq 1 - \lambda$ for all $t > 0$.

Proof. Since $\Gamma^{\uparrow}(\lambda, x, y) = +\infty$, it follows that $M(x, y, t) \leq 1 - \lambda$ for sufficiently large $t > 0$ in the sense of $t \to \infty$. To prove Part (i), suppose that there exists $t_0 > 0$ such that $M(x, y, t_0) > 1 - \lambda$. Since $M(x, y, \cdot)$ is nondecreasing by Parts (i) and (ii) of Proposition 1, if t_1 is sufficiently large with $t_1 > t_0$, then

$$M(x, y, t_1) \geq M(x, y, t_0) > 1 - \lambda,$$

which contradicts the fact of $M(x, y, t) \leq 1 - \lambda$ for sufficiently large $t > 0$.

To prove Part (ii), suppose that there exists $t_0 > 0$ such that $M(y, x, t_0) > 1 - \lambda$. Since $M(x, y, \cdot)$ is symmetrically nondecreasing by Part (iii) of Proposition 1, if t_1 is sufficiently large with $t_1 > t_0$, then

$$M(x, y, t_1) \geq M(y, x, t_0) > 1 - \lambda,$$

which contradicts the fact of $M(x, y, t) \leq 1 - \lambda$ for sufficiently large $t > 0$. This completes the proof. □

Proposition 21. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational and canonical conditions. For any fixed $x, y \in X$ with $x \neq y$, we have $\Gamma^{\uparrow}(\lambda, x, y) < +\infty$ for all $\lambda \in (0, 1)$.*

Proof. Assume that $\Gamma^{\uparrow}(\lambda, x, y) = +\infty$, which means that $M(x, y, t) \leq 1 - \lambda$ for sufficiently large t in the sense of $t \to \infty$. Since M satisfies the canonical condition, we have

$$1 = \lim_{t \to \infty} M(x, y, t) \leq 1 - \lambda,$$

which contradicts $0 < \lambda < 1$. This completes the proof. □

Proposition 22. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical and rational conditions. For any fixed $x, y \in X$ with $x \neq y$, we have the following properties.*

(i) Suppose that M satisfies the strict $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then

$$\Gamma^{\uparrow}(\lambda, x, y) \leq \Gamma^{\downarrow}(\lambda, x, y)$$

for each $\lambda \in (0, 1)$.

(ii) Suppose that M satisfies the strict $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then

$$\Gamma^{\uparrow}(\lambda, x, y) \leq \Gamma^{\downarrow}(\lambda, y, x)$$

for each $\lambda \in (0, 1)$.

Proof. By Proposition 21, we have $\Gamma^{\uparrow}(\lambda, x, y) < +\infty$ for all $\lambda \in (0,1)$. By the concept of the supremum, given any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$M(x, y, t_\epsilon) \leq 1 - \lambda \text{ and } \Gamma^{\uparrow}(\lambda, x, y) - \epsilon < t_\epsilon.$$

To prove Part (i), using Parts (i) and (ii) of Proposition 18, we have $t_\epsilon \leq \Gamma^{\downarrow}(\lambda, x, y)$, which says that $\Gamma^{\uparrow}(\lambda, x, y) - \epsilon < \Gamma^{\downarrow}(\lambda, x, y)$. Since ϵ can be any positive real number, we obtain the desired inequality. Part (ii) can be similarly obtained by considering $t_\epsilon \leq \Gamma^{\downarrow}(\lambda, y, x)$. This completes the proof. □

Proposition 23. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition. Given any fixed $x, y \in X$ with $x \neq y$ and any fixed $\lambda \in (0,1)$, we have the following properties.*

(i) *Suppose that $\Gamma^{\uparrow}(\lambda, x, y) < +\infty$. For any $\epsilon > 0$, we have*

$$M\left(x, y, \Gamma^{\uparrow}(\lambda, x, y) + \epsilon\right) > 1 - \lambda. \tag{20}$$

(ii) *If $\Gamma^{\uparrow}(\lambda, x, y) < +\infty$ and $\epsilon > 0$ is sufficiently small such that $\Gamma^{\uparrow}(\lambda, x, y) > \epsilon$, then we have the following properties.*

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then*

$$M\left(x, y, \Gamma^{\uparrow}(\lambda, x, y) - \epsilon\right) \leq 1 - \lambda. \tag{21}$$

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then*

$$M\left(x, y, \Gamma^{\uparrow}(\lambda, y, x) - \epsilon\right) \leq 1 - \lambda. \tag{22}$$

Proof. To prove Part (i), given any $\epsilon > 0$, suppose that $M(x, y, \Gamma^{\uparrow}(\lambda, x, y) + \epsilon) \leq 1 - \lambda$. By the definition of $\Gamma^{\uparrow}$, we have $\Gamma^{\uparrow}(\lambda, x, y) \geq \Gamma^{\uparrow}(\lambda, x, y) + \epsilon$, which is a contradiction. Therefore, we must have $M(x, y, \Gamma^{\uparrow}(\lambda, x, y) + \epsilon) > 1 - \lambda$.

To prove Part (ii), we consider the following cases.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. By the concept of the supremum for $\Gamma^{\uparrow}(\lambda, x, y)$, given any $\epsilon > 0$ with $\Gamma^{\uparrow}(\lambda, x, y) > \epsilon$, there exists $t_\epsilon > 0$ such that $M(x, y, t_\epsilon) \leq 1 - \lambda$ and $t_\epsilon > \Gamma^{\uparrow}(\lambda, x, y) - \epsilon$. Since the mapping $M(x, y, \cdot)$ is nondecreasing by Parts (i) and (ii) of Proposition 1, we have

$$M\left(x, y, \Gamma^{\uparrow}(\lambda, x, y) - \epsilon\right) \leq M(x, y, t_\epsilon) \leq 1 - \lambda.$$

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. By the concept of supremum for $\Gamma^{\uparrow}(\lambda, y, x)$, given any $\epsilon > 0$ with $\Gamma^{\uparrow}(\lambda, y, x) > \epsilon$, there exists $t_\epsilon > 0$ such that $M(y, x, t_\epsilon) \leq 1 - \lambda$ and $t_\epsilon > \Gamma^{\uparrow}(\lambda, y, x) - \epsilon$. Since the mapping $M(x, y, \cdot)$ is symmetrically nondecreasing by Parts (ii) and (iii) of Proposition 1, we have

$$M\left(x, y, \Gamma^{\uparrow}(\lambda, y, x) - \epsilon\right) \leq M(y, x, t_\epsilon) \leq 1 - \lambda.$$

This completes the proof. □

Proposition 24. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition. Given any fixed $x, y \in X$ with $x \neq y$ and any fixed $\lambda \in (0,1)$, we have the following properties.*

(i) *Assume that $\Gamma^{\uparrow}(\lambda, x, y) < t$. Then, the following statements hold true.*

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then, $M(x, y, t) > 1 - \lambda$.*

- *Suppose that M satisfies the* $\circ$*-triangle inequality for* $\circ \in \{\triangleright, \triangleleft, \diamond\}$*. Then,* $M(y,x,t) > 1-\lambda$.

(ii) *The following statements hold true.*

- *Suppose that M satisfies the* $\circ$*-triangle inequality for* $\circ \in \{\bowtie, \triangleright, \triangleleft\}$*. If* $t < \Gamma^{\uparrow}(\lambda,x,y)$*, then* $M(x,y,t) \leq 1-\lambda$.
- *Suppose that M satisfies the* $\circ$*-triangle inequality for* $\circ \in \{\triangleright, \triangleleft, \diamond\}$*. If* $\Gamma^{\uparrow}(\lambda,y,x) = +\infty$ *or* $t < \Gamma^{\uparrow}(\lambda,x,y) < +\infty$*, then* $M(x,y,t) \leq 1-\lambda$.

Proof. To prove Part (i), if $t > \Gamma^{\uparrow}(\lambda,x,y)$, then there exists $\epsilon > 0$ such that $t \geq \Gamma^{\uparrow}(\lambda,x,y)+\epsilon$.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Since the mapping $M(x,y,\cdot)$ is nondecreasing by Parts (i) and (ii) of Proposition 1, using (20), we obtain

$$M(x,y,t) \geq M\left(x,y,\Gamma^{\uparrow}(\lambda,x,y)+\epsilon\right) > 1-\lambda.$$

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Since the mapping $M(x,y,\cdot)$ is symmetrically nondecreasing by Part (iii) of Proposition 1, using (20), we obtain

$$M(y,x,t) \geq M\left(x,y,\Gamma^{\uparrow}(\lambda,x,y)+\epsilon\right) > 1-\lambda.$$

To prove Part (ii), we consider the following cases.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Assume that $\Gamma^{\uparrow}(\lambda,x,y) = +\infty$. By Part (i) of Proposition 20, we obtain the desired inequality. Assume that $\Gamma^{\uparrow}(\lambda,x,y) < +\infty$. Since $t < \Gamma^{\uparrow}(\lambda,x,y)$, there exists $\epsilon > 0$ such that $0 \leq t \leq \Gamma^{\uparrow}(\lambda,x,y)-\epsilon$. Using (21), we obtain

$$M(x,y,t) \leq M\left(x,y,\Gamma^{\uparrow}(\lambda,x,y)-\epsilon\right) \leq 1-\lambda.$$

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. If $\Gamma^{\uparrow}(\lambda,y,x) = +\infty$, then using Part (ii) of Proposition 20, we obtain the desired inequality. If $t < \Gamma^{\uparrow}(\lambda,x,y) < +\infty$, then there exists $\epsilon > 0$ such that $0 \leq t \leq \Gamma^{\uparrow}(\lambda,x,y)-\epsilon$. Using (22), we obtain

$$M(x,y,t) \leq M\left(y,x,\Gamma^{\uparrow}(\lambda,x,y)-\epsilon\right) \leq 1-\lambda.$$

This completes the proof. □

Proposition 25. *Let* (X,M) *be a fuzzy semi-metric space along with a t-norm* $*$ *such that M satisfies the rational condition. Given any fixed* $x,y \in X$ *with* $x \neq y$ *and any fixed* $\lambda \in (0,1)$*, we have the following properties.*

(i) *Assume that* $M(x,y,t) \leq 1-\lambda$*. Then, the following statements hold true.*

- *Suppose that M satisfies the* $\circ$*-triangle inequality for* $\circ \in \{\bowtie, \triangleright, \triangleleft\}$*. Then,* $t \leq \Gamma^{\uparrow}(\lambda,x,y)$.
- *Suppose that M satisfies the* $\circ$*-triangle inequality for* $\circ \in \{\triangleright, \triangleleft, \diamond\}$*. Then,* $t \leq \Gamma^{\uparrow}(\lambda,y,x)$.

(ii) *The following statements hold true.*

- *Suppose that M satisfies the* $\circ$*-triangle inequality for* $\circ \in \{\bowtie, \triangleright, \triangleleft\}$*. If* $M(x,y,t) > 1-\lambda$*, then* $\Gamma^{\uparrow}(\lambda,x,y) < +\infty$ *and* $t \geq \Gamma^{\uparrow}(\lambda,x,y)$.
- *Suppose that M satisfies the* $\circ$*-triangle inequality for* $\circ \in \{\triangleright, \triangleleft, \diamond\}$*. We have the following results.*
  - *If* $M(x,y,t) > 1-\lambda$*, then* $\Gamma^{\uparrow}(\lambda,y,x) < +\infty$.
  - *If* $M(x,y,t) > 1-\lambda$ *and* $\Gamma^{\uparrow}(\lambda,x,y) < +\infty$*, then* $t \geq \Gamma^{\uparrow}(\lambda,x,y)$.

Proof. To prove Part (i), we consider the following cases.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Assume $\Gamma^{\uparrow}(\lambda, x, y) = +\infty$. It is clear $t \leq \Gamma^{\uparrow}(\lambda, x, y)$. Assume that $\Gamma^{\uparrow}(\lambda, x, y) < +\infty$. Using the contraposition of the first property of Part (i) of Proposition 24, it follows that if $M(x, y, t) \leq 1 - \lambda$, then $t \leq \Gamma^{\uparrow}(\lambda, x, y)$.
- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Assume $\Gamma^{\uparrow}(\lambda, y, x) = +\infty$. It is clear $t \leq \Gamma^{\uparrow}(\lambda, y, x)$. Assume that $\Gamma^{\uparrow}(\lambda, y, x) < +\infty$. Using the contraposition of second property of Part (i) of Proposition 24, it follows that if $M(x, y, t) \leq 1 - \lambda$, then $t \leq \Gamma^{\uparrow}(\lambda, y, x)$.

To prove Part (ii), we consider the following cases.

- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Assume $\Gamma^{\uparrow}(\lambda, x, y) = +\infty$. By Part (i) of Proposition 20, we have $M(x, y, t) \leq 1 - \lambda$ for all $t > 0$. Therefore, if $M(x, y, t) > 1 - \lambda$, then we must have $\Gamma^{\uparrow}(\lambda, x, y) < +\infty$. Using the contraposition of the first property of Part (ii) of Proposition 24, we obtain the desired result.
- Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. The results follow from the contraposition of second property of Part (ii) of Proposition 24.

This completes the proof. $\square$

Proposition 26. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition. Given any fixed $x, y \in X$ with $x \neq y$ and any fixed $\lambda \in (0, 1)$, we have the following properties.*

(i) Suppose that M is right-continuous with respect to the distance on $(0, \infty)$ and that $\Gamma^{\uparrow}(\lambda, x, y) < +\infty$. Then

$$M\left(x, y, \Gamma^{\uparrow}(\lambda, x, y)\right) \geq 1 - \lambda. \tag{23}$$

(ii) Suppose that M is left-continuous with respect to the distance on $(0, \infty)$. Then, we have the following results.

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$, and that $\Gamma^{\uparrow}(\lambda, x, y) < +\infty$. Then*

$$M\left(x, y, \Gamma^{\uparrow}(\lambda, x, y)\right) \leq 1 - \lambda. \tag{24}$$

- *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$, and that $\Gamma^{\uparrow}(\lambda, y, x) < +\infty$. Then*

$$M\left(x, y, \Gamma^{\uparrow}(\lambda, y, x)\right) \leq 1 - \lambda. \tag{25}$$

(iii) Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$, and that $\Gamma^{\uparrow}(\lambda, x, y) < +\infty$. If M is continuous with respect to the distance on $(0, \infty)$, then

$$M\left(x, y, \Gamma^{\uparrow}(\lambda, x, y)\right) = 1 - \lambda.$$

Proof. To prove Part (i), using the right-continuity of M and applying $\epsilon \to 0+$ to the inequality (20), we obtain (23). To prove Part (ii), using the left-continuity of M and applying $\epsilon \to 0+$ to the inequalities (21) and (22), we can obtain (24) and (25), respectively. Finally, Part (iii) follows from Parts (i) and (ii) immediately. This completes the proof. $\square$

Theorem 4. *(Triangle inequalities for the dual fuzzy semi-metric) Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition, and that the t-norm is left-continuous at one in the first or second component.*

(i) Suppose that M satisfies the $\bowtie$-triangle inequality. Given any fixed and distinct $x_1, x_2, \cdots, x_p \in X$ and any fixed $\mu \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that

$$\Gamma^{\uparrow}(\mu, x_1, x_p) \leq \Gamma^{\uparrow}(\lambda, x_1, x_2) + \Gamma^{\uparrow}(\lambda, x_2, x_3) + \cdots + \Gamma^{\uparrow}(\lambda, x_{p-1}, x_p). \tag{26}$$

(ii) *Suppose that M satisfies the $\rhd$-triangle inequality. Given any fixed and distinct $x_1, x_2, \cdots, x_p \in X$ and any fixed $\mu \in (0, 1]$, there exists $\lambda \in (0, 1)$ such that*

$$\max\left\{\Gamma^{\uparrow}(\mu, x_1, x_p), \Gamma^{\uparrow}(\mu, x_p, x_1)\right\} \leq \Gamma^{\uparrow}(\lambda, x_1, x_2) + \Gamma^{\uparrow}(\lambda, x_3, x_2) + \Gamma^{\uparrow}(\lambda, x_4, x_3) + \cdots + \Gamma^{\uparrow}(\lambda, x_p, x_{p-1}).$$

(iii) *Suppose that M satisfies the $\lhd$-triangle inequality. Given any fixed and distinct $x_1, x_2, \cdots, x_p \in X$ and any fixed $\mu \in (0, 1]$, there exists $\lambda \in (0, 1)$ such that*

$$\max\left\{\Gamma^{\uparrow}(\mu, x_1, x_p), \Gamma^{\uparrow}(\mu, x_p, x_1)\right\} \leq \Gamma^{\uparrow}(\lambda, x_2, x_1) + \Gamma^{\uparrow}(\lambda, x_2, x_3) + \Gamma^{\uparrow}(\lambda, x_3, x_4) + \cdots + \Gamma^{\uparrow}(\lambda, x_{p-1}, x_p).$$

(iv) *Suppose that M satisfies the $\diamond$-triangle inequality. Given any fixed and distinct $x_1, x_2, \cdots, x_p \in X$ and any fixed $\mu \in (0, 1]$, there exists $\lambda \in (0, 1)$ such that the following inequalities are satisfied.*

- *If p is even and $\Gamma^{\uparrow}(\lambda, x_1, x_p) < +\infty$, then*

$$\Gamma^{\uparrow}(\lambda, x_1, x_p) \leq \Gamma^{\uparrow}(\lambda, x_2, x_1) + \Gamma^{\uparrow}(\lambda, x_3, x_2) + \Gamma^{\uparrow}(\lambda, x_3, x_4) + \Gamma^{\uparrow}(\lambda, x_5, x_4) + \Gamma^{\uparrow}(\lambda, x_5, x_6) + \Gamma^{\uparrow}(\lambda, x_7, x_6) + \cdots + \Gamma^{\uparrow}(\lambda, x_{p-1}, x_p).$$

- *If p is even and $\Gamma^{\uparrow}(\lambda, x_p, x_1) < +\infty$, then*

$$\Gamma^{\uparrow}(\lambda, x_p, x_1) \leq \Gamma^{\uparrow}(\lambda, x_1, x_2) + \Gamma^{\uparrow}(\lambda, x_2, x_3) + \Gamma^{\uparrow}(\lambda, x_4, x_3) + \Gamma^{\uparrow}(\lambda, x_4, x_5) + \Gamma^{\uparrow}(\lambda, x_6, x_5) + \Gamma^{\uparrow}(\lambda, x_6, x_7) + \cdots + \Gamma^{\uparrow}(\lambda, x_p, x_{p-1}). \tag{27}$$

- *If p is odd and $\Gamma^{\uparrow}(\lambda, x_1, x_p) < +\infty$, then*

$$\Gamma^{\uparrow}(\lambda, x_1, x_p) \leq \Gamma^{\uparrow}(\lambda, x_1, x_2) + \Gamma^{\uparrow}(\lambda, x_2, x_3) + \Gamma^{\uparrow}(\lambda, x_4, x_3) + \Gamma^{\uparrow}(\lambda, x_4, x_5) + \Gamma^{\uparrow}(\lambda, x_6, x_5) + \Gamma^{\uparrow}(\lambda, x_6, x_7) + \cdots + \Gamma^{\uparrow}(\lambda, x_{p-1}, x_p)$$

- *If p is even and $\Gamma^{\uparrow}(\lambda, x_p, x_1) < +\infty$, then*

$$\Gamma^{\uparrow}(\lambda, x_p, x_1) \leq \Gamma^{\uparrow}(\lambda, x_2, x_1) + \Gamma^{\uparrow}(\lambda, x_3, x_2) + \Gamma^{\uparrow}(\lambda, x_3, x_4) + \Gamma^{\uparrow}(\lambda, x_5, x_4) + \Gamma^{\uparrow}(\lambda, x_5, x_6) + \Gamma^{\uparrow}(\lambda, x_7, x_6) + \cdots + \Gamma^{\uparrow}(\lambda, x_p, x_{p-1}).$$

Proof. According to Lemma 3, there exists $\lambda \in (0, 1)$ such that

$$(1 - \lambda) * \cdots * (1 - \lambda) > 1 - \mu. \tag{28}$$

Suppose that $\Gamma^{\uparrow}(\lambda, x_i, x_{i+1}) < +\infty$ for all $i = 1, \cdots, p-1$. Given any $\epsilon > 0$, we have

$$\begin{aligned} & M\left(x_1, x_p, \Gamma^{\uparrow}(\lambda, x_1, x_2) + \Gamma^{\uparrow}(\lambda, x_2, x_3) + \cdots + \Gamma^{\uparrow}(\lambda, x_{p-1}, x_p) + (p-1)\epsilon\right) \\ & \geq M\left(x_1, x_2, \Gamma^{\uparrow}(\lambda, x_1, x_2) + \epsilon\right) * \cdots * M\left(x_{p-1}, x_p, \Gamma^{\uparrow}(\lambda, x_{p-1}, x_p) + \epsilon\right) \\ & \quad \text{(by the first observation of Remark 1)} \\ & \geq (1 - \lambda) * \cdots * (1 - \lambda) \text{ (by (20) and the increasing property of the t-norm)} \\ & > 1 - \mu \text{ (by (28))}. \end{aligned} \tag{29}$$

To prove Part (i), we consider the following cases.

- Suppose that $\Gamma^{\uparrow}(\mu, x_1, x_p) = +\infty$. We are going to claim that there exists i_0 such that $\Gamma^{\uparrow}(\lambda, x_{i_0}, x_{i_0+1}) = +\infty$. Assume that $\Gamma^{\uparrow}(\lambda, x_i, x_{i+1}) < +\infty$ for all $i = 1, \cdots, p-1$. Using (29) and

Part (ii) of Proposition 25, it follows that $\Gamma^{\uparrow}(\mu, x_1, x_p) < +\infty$. This contradiction says that there exists i_0 such that $\Gamma^{\uparrow}(\lambda, x_{i_0}, x_{i_0+1}) = +\infty$. In this case, the inequality (26) holds true.

- Suppose that $\Gamma^{\uparrow}(\mu, x_1, x_p) < +\infty$. We also consider the following cases.
  - If there exists i_0 such that $\Gamma^{\uparrow}(\lambda, x_{i_0}, x_{i_0+1}) = +\infty$, then the inequality (26) also holds true.
  - We assume that $\Gamma^{\uparrow}(\lambda, x_i, x_{i+1}) < +\infty$ for all $i = 1, \cdots, p-1$. Using (29) and Part (ii) of Proposition 25 again, it follows that

$$\Gamma^{\uparrow}(\lambda, x_1, x_2) + \Gamma^{\uparrow}(\lambda, x_2, x_3) + \cdots + \Gamma^{\uparrow}(\lambda, x_{p-1}, x_p) + (p-1)\epsilon \geq \Gamma^{\uparrow}(\mu, x_1, x_p).$$

By taking $\epsilon \to 0+$, we obtain the desired inequality (26).

Part (ii) can be obtained by applying the second observation of Remark 1 to the above argument. Part (iii) can be obtained by applying the third observation of Remark 1 to the above argument.

To prove Part (iv), we first assume that p is even. Suppose that one of the terms in the right side of the inequality (27) is $+\infty$. Then, it is done. Therefore, we assume that each term in the right side of the inequality (27) is finite. Given any $\epsilon > 0$, we have

$$\begin{aligned}
& M\Big(x_p, x_1, \Gamma^{\uparrow}(\lambda, x_2, x_1) + \Gamma^{\uparrow}(\lambda, x_3, x_2) + \Gamma^{\uparrow}(\lambda, x_3, x_4) + \Gamma^{\uparrow}(\lambda, x_5, x_4) + \Gamma^{\uparrow}(\lambda, x_5, x_6) \\
&\qquad + \Gamma^{\uparrow}(\lambda, x_7, x_6) + \cdots + \Gamma^{\uparrow}(\lambda, x_{p-1}, x_p) + (p-1)\epsilon\Big) \\
&\geq M\Big(x_1, x_2, \Gamma^{\uparrow}(\lambda, x_2, x_1) + \epsilon\Big) * M\Big(x_2, x_3, \Gamma^{\uparrow}(\lambda, x_3, x_2) + \epsilon\Big) \\
&\quad * M\Big(x_4, x_3, \Gamma^{\uparrow}(\lambda, x_3, x_4) + \epsilon\Big) * M\Big(x_4, x_5, \Gamma^{\uparrow}(\lambda, x_5, x_4) + \epsilon\Big) \\
&\quad * M\Big(x_6, x_5, \Gamma^{\uparrow}(\lambda, x_5, x_6) + \epsilon\Big) * M\Big(x_6, x_7, \Gamma^{\uparrow}(\lambda, x_7, x_6) + \epsilon\Big) \\
&\quad * \cdots * M\Big(x_{p-1}, x_p, \Gamma^{\uparrow}(\lambda, x_p, x_{p-1}) + \epsilon\Big) \\
&\qquad \text{(by the fourth observation of Remark 1)} \\
&\geq (1-\lambda) * \cdots * (1-\lambda) \text{ (by (20) and the increasing property of the t-norm)} \\
&> 1-\mu \text{ (by (28))}.
\end{aligned}$$

Since $\Gamma^{\downarrow}(\lambda, x_p, x_1)$ is assumed to be finite, using Part (ii) of Proposition 25, we can similarly obtain the inequality (27). The other inequalities can be similarly obtained, and the proof is complete. □

Theorem 5. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition. Suppose that the t-norm $*$ is left-continuous at one with respect to the first or second component. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.*

(i) Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$.

- *$x_n \xrightarrow{M^{\triangleright}} x$ as $n \to \infty$ if and only if $\Gamma^{\uparrow}(\lambda, x_n, x) \to 0$ as $n \to \infty$ for all $\lambda \in (0,1)$.*
- *$x_n \xrightarrow{M^{\triangleleft}} x$ as $n \to \infty$ if and only if $\Gamma^{\uparrow}(\lambda, x, x_n) \to 0$ as $n \to \infty$ for all $\lambda \in (0,1)$.*

(ii) Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$.

- *If $x_n \xrightarrow{M^{\triangleright}} x$ as $n \to \infty$, then, given any fixed $\lambda \in (0,1)$, we have that $\Gamma^{\uparrow}(\lambda, x_n, x) < +\infty$ for all $n \in \mathbb{N}$ imply $\Gamma^{\uparrow}(\lambda, x_n, x) \to 0$ as $n \to \infty$.*
- *If $x_n \xrightarrow{M^{\triangleleft}} x$ as $n \to \infty$, then, given any fixed $\lambda \in (0,1)$, we have that $\Gamma^{\uparrow}(\lambda, x, x_n) < +\infty$ for all $n \in \mathbb{N}$ imply $\Gamma^{\uparrow}(\lambda, x, x_n) \to 0$ as $n \to \infty$.*
- *If $\Gamma^{\uparrow}(\lambda, x_n, x) \to 0$ as $n \to \infty$ for all $\lambda \in (0,1)$, then $x_n \xrightarrow{M^{\triangleleft}} x$ as $n \to \infty$.*
- *If $\Gamma^{\uparrow}(\lambda, x, x_n) \to 0$ as $n \to \infty$ for all $\lambda \in (0,1)$, then $x_n \xrightarrow{M^{\triangleright}} x$ as $n \to \infty$.*

Proof. To prove Part (i), it suffices to prove the first case. Suppose that $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$, which says that, given any $t > 0$ and $\delta > 0$, there exists $n_{t,\delta} \in \mathbb{N}$ such that $|M(x_n, x, t) - 1| < \delta$

for $n \geq n_{t,\delta}$. In other words, for any fixed $\lambda \in (0,1)$, given any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $|M(x_n, x, \epsilon/2) - 1| < \lambda$ for $n \geq n_\epsilon$, which says that $M(x_n, x, \epsilon/2) > 1 - \lambda$ for $n \geq n_\epsilon$. By Part (ii) of Proposition 25, we obtain

$$\Gamma^{\uparrow}(\lambda, x_n, x) \leq \frac{\epsilon}{2} < \epsilon$$

for $n \geq n_\epsilon$. This shows that $\Gamma^{\uparrow}(\lambda, x_n, x) \to 0$ as $n \to \infty$. For the converse, suppose that $\Gamma^{\uparrow}(\lambda, x_n, x) \to 0$ as $n \to \infty$ for all $\lambda \in (0,1)$. Given any $\delta > 0$ and $\lambda \in (0,1]$, there exists $n_{\delta,\lambda} \in \mathbb{N}$ such that $|\Gamma^{\uparrow}(\lambda, x_n, x)| < \delta$ for all $n \geq n_{\delta,\lambda}$. In other words, for any fixed $t > 0$, given any $\epsilon \in (0,1)$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$\Gamma^{\uparrow}(\epsilon, x_n, x) = |\Gamma^{\uparrow}(\epsilon, x_n, x)| < t$$

for $n \geq n_\epsilon$, which says that $M(x_n, x, t) > 1 - \epsilon$ for $n \geq n_\epsilon$ by Part (i) of Proposition 24. This shows that $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$. Part (ii) can be similarly obtained. This completes the proof. $\square$

Example 9. *Continued from Examples 2 and 8, we have*

$$\begin{aligned}
&x_n \xrightarrow{M^{\triangleright}} x \text{ if and only if } \lim_{n\to\infty} d(x_n, x) = 0 \\
&x_n \xrightarrow{M^{\triangleleft}} x \text{ if and only if } \lim_{n\to\infty} d(x, x_n) = 0 \\
&\Gamma^{\uparrow}(\lambda, x, y) = \left(\frac{1}{\lambda} - 1\right) \cdot d(x, y).
\end{aligned}$$

It is clear to see that $x_n \xrightarrow{M^{\triangleright}} x$ if and only if $\Gamma^{\uparrow}(\lambda, x_n, x) \to 0$ and $x_n \xrightarrow{M^{\triangleleft}} x$ if and only if $\Gamma^{\uparrow}(\lambda, x, x_n) \to 0$ for all $\lambda \in (0,1)$.

Definition 12. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition.*

- *Given any fixed $\lambda \in (0,1)$, we say that $\{x_n\}_{n=1}^{\infty}$ is a $(\lambda, >)$-Cauchy sequence with respect to $\Gamma^{\uparrow}$ if and only if, given any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $m > n \geq n_\epsilon$ implies $\Gamma^{\uparrow}(\lambda, x_m, x_n) < \epsilon$.*
- *Given any fixed $\lambda \in (0,1)$, we say that $\{x_n\}_{n=1}^{\infty}$ is a $(\lambda, <)$-Cauchy sequence with respect to $\Gamma^{\uparrow}$ if and only if, given any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $n > m \geq n_\epsilon$ implies $\Gamma^{\uparrow}(\lambda, x_m, x_n) < \epsilon$.*
- *Given any fixed $\lambda \in (0,1)$, we say that $\{x_n\}_{n=1}^{\infty}$ is a λ-Cauchy sequence with respect to $\Gamma^{\uparrow}$ if and only if, given any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $m, n \geq n_\epsilon$ implies $\Gamma^{\uparrow}(\lambda, x_m, x_n) < \epsilon$.*

Theorem 6. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition. Suppose that the t-norm $*$ is left-continuous at one with respect to the first or second component. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.*

(i) *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$.*

- *$\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence in the metric sense if and only if it is a $(\lambda, >)$-Cauchy sequence with respect to $\Gamma^{\uparrow}$ for all $\lambda \in (0,1)$.*
- *$\{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence in the metric sense if and only if it is a $(\lambda, <)$-Cauchy sequence with respect to $\Gamma^{\uparrow}$ for all $\lambda \in (0,1)$.*

(ii) *Suppose that M satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$.*

- *If $\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence in the metric sense, then, given any fixed $\lambda \in (0,1)$, $\Gamma^{\uparrow}(\lambda, x_m, x_n) < +\infty$ for all $m > n$ imply that it is a $(\lambda, >)$-Cauchy sequence with respect to $\Gamma^{\uparrow}$.*
- *If $\{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence in the metric sense then, given any fixed $\lambda \in (0,1)$, $\Gamma^{\uparrow}(\lambda, x_n, x_m) < +\infty$ for all $m > n$ imply that it is a $(\lambda, <)$-Cauchy sequence with respect to $\Gamma^{\uparrow}$.*
- *If $\{x_n\}_{n=1}^{\infty}$ is a $(\lambda, >)$-Cauchy sequence with respect to $\Gamma^{\uparrow}$ for all $\lambda \in (0,1)$, then it is a $<$-Cauchy sequence in metric sense.*

- *If* $\{x_n\}_{n=1}^{\infty}$ *is a* $(\lambda, <)$*-Cauchy sequence with respect to* $\Gamma^{\uparrow}$ *for all* $\lambda \in (0,1)$, *then it is a* $>$*-Cauchy sequence in the metric sense.*

Proof. To prove Part (i), it suffices to prove the first case. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence in the metric sense, which says that, given any $t > 0$ and $\delta > 0$, there exists $n_{t,\delta} \in \mathbb{N}$ such that $m > n \geq n_{t,\delta}$ implies $M(x_m, x_n, t) > 1 - \delta$. In other words, for any fixed $\lambda \in (0,1)$, given any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $m > n \geq n_\epsilon$ implies $M(x_m, x_n, \epsilon/2) > 1 - \lambda$. By Part (ii) of Proposition 25, we obtain

$$\Gamma^{\uparrow}(\lambda, x_m, x_n) \leq \frac{\epsilon}{2} < \epsilon$$

for $m > n \geq n_{t,\delta}$. This shows that $\{x_n\}_{n=1}^{\infty}$ is a $(\lambda, >)$-Cauchy sequence with respect to $\Gamma^{\uparrow}$. For the converse, given any $\delta > 0$ and $\lambda \in (0,1)$, there exists $n_{\delta,\lambda} \in \mathbb{N}$ such that $m > n \geq n_{\delta,\lambda}$ implies $\Gamma^{\uparrow}(\lambda, x_m, x_n) < \delta$. In other words, for any fixed $t > 0$, given any $\epsilon \in (0,1)$, there exists $n_\epsilon \in \mathbb{N}$ such that $m > n \geq n_\epsilon$ implies $\Gamma^{\uparrow}(\epsilon, x_m, x_n) < t$, which says that $M(x_m, x_n, t) > 1 - \epsilon$ for $n \geq n_\epsilon$ by Part (i) of Proposition 24. This shows that $\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence in the metric sense. Part (ii) can be similarly obtained. This completes the proof. □

Example 10. *Continued from Examples 1 and 8, we have*

$$M(x_m, x_n, t) > 1 - r \text{ if and only if } \frac{t}{t + d(x_m, x_n)} > 1 - r \text{ if and only if } \left(\frac{1}{\lambda} - 1\right) \cdot d(x_m, x_n) < t$$

and

$$\Gamma^{\uparrow}(\lambda, x_m, x_n) < \epsilon \text{ if and only if } \left(\frac{1}{\lambda} - 1\right) \cdot d(x_m, x_n) < \epsilon.$$

It is clear to see that $\{x_n\}_{n=1}^{\infty}$ *is a* $>$*-Cauchy sequence in the metric sense if and only if it is a* $(\lambda, >)$*-Cauchy sequence with respect to* $\Gamma^{\uparrow}$ *for all* $\lambda \in (0,1)$.

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