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# On Metric Dimensions of Symmetric Graphs Obtained by Rooted Product

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**Abstract:** Let $G = (V, E)$ be a connected graph and $d(x, y)$ be the distance between the vertices $x$ and $y$ in $G$. A set of vertices $W$ resolves a graph $G$ if every vertex is uniquely determined by its vector of distances to the vertices in $W$. A metric dimension of $G$ is the minimum cardinality of a resolving set of $G$ and is denoted by $dim(G)$. In this paper, Cycle, Path, Harary graphs and their rooted product as well as their connectivity are studied and their metric dimension is calculated. It is proven that metric dimension of some graphs is unbounded while the other graphs are constant, having three or four dimensions in certain cases.

**Keywords:** metric dimension; basis; resolving set; cycle; path; Harary graphs; rooted product

**MSC:** 05C15; 05C62; 05C12; 05C07; 05C10; 05C90

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## 1. Introduction

In a connected graph $G(V, E)$, where $V$ is the set of vertices and $E$ is the set of edges, the distance $d(u, v)$ between two vertices $u, v \in V$ is the length of shortest path between them. Let $W = \{w_1, w_2, ..., w_k\}$ be an order set of vertices of $G$ and let $v$ be a vertex of $G$. The representation $r(v|W)$ of $v$ with respect to $W$ is the k-tuple $(d(v, w_1), d(v, w_2), d(v, w_3), ..., d(v, w_k)\}$, where $W$ is called a resolving set [1] or locating set [2] if every vertex of $G$ is uniquely identified by its distances from the vertices of $W$, or equivalently, if distinct vertices of $G$ have distinct representations with respect to $W$. A resolving set of minimum cardinality is called a basis for $G$ and cardinality is the metric dimension of $G$, denoted by $dim(G)$ [3]. The concept of resolving set and metric basis have previously appeared in the literature [4–6].

For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph $G$, the $i^{th}$ component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that $W$ is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \setminus W$.

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [2,7] and studied independently by Harary and Melter in [5]. Application of this invariant to the navigation of robots in networks are discussed in [8] and application to chemistry is discussed in [1], while application to the problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures, are given in [6].

Let $F$ be a family of connected graphs. If all graphs in $F$ have the same metric dimension, then $F$ is called a family with constant metric dimension [9]. A connected graph $G$ has $dim(G) = 1$ if and only if $G$ is a path [1], cycle $C_n$ have metric dimension 2 for every $n \geq 3$, also honeycomb networks [10] have metric dimension 3.

Metric dimension is a parameter that has appeared in various applications of graph theory, as diverse as pharmaceutical chemistry [1,11], robot navigation [8,12] and combinatorial optimization [13], to name a few. A chemical compound can be represented by more than one suggested structure but only one of them, which expresses the physical and chemical properties of compound, is acceptable. The chemists require mathematical representation for a set of chemical compounds in a way that gives distinct representations to distinct compounds. As described in [1,11], the structure of chemical compounds can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [1,6,14–17].

Other families of graphs with unbounded metric dimension are regular bipartite graphs [4], wheel graph and jahangir graph [18].

Our main aim of this paper is to compute the metric dimension of graphs obtained from the rooted product graphs. For this purpose, we need the following definitions.

**Definition 1** ([19])**.** *A rooted graph is a graph in which one vertex has been distinguished as the root. Both directed and undirected versions of rooted graphs have been studied, and there are also variant definitions that allow multiple roots.*

**Definition 2** ([20])**.** *Let $H$ be a labelled graph on $n$ vertices. Let $G$ be a sequence of $n$ rooted graphs $G_1, G_2, ...G_n$. The graph $H(G)$ obtained by identifying the root of $G_i$ with the $i^{th}$ vertex of $H$. The graph $H(G)$ is called the rooted product of $H$ by $G$.*

**Definition 3** ([19])**.** *The Harary graph $H_{m,n}$ is defined as follows and depicted in Figure 1:*

***Case 1.*** *$m$ is even. Let $m = 2r$, then $H_{2r,n}$ is constructed as follows: It has vertices $0, 1, \ldots, n-1$ and two vertices $i$ and $j$ are joined if $i - r \leq j \leq i + r$ (where addition is taken modulo $n$).*

***Case 2.*** *$m$ is odd and $n$ is even. Let $m = 2r + 1$, then the $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex $i$ to vertex $i + (n/2)$ for $1 \leq i \leq n/2$.*

***Case 3.*** *$m, n$ are odd. Let $m = 2r + 1$, then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex 0 to vertices $(n-1)/2$ and $(n+1)/2$ and vertex $i$ to vertex $i + (n+1)/2$ for $1 \leq i \leq (n-1)/2$.*

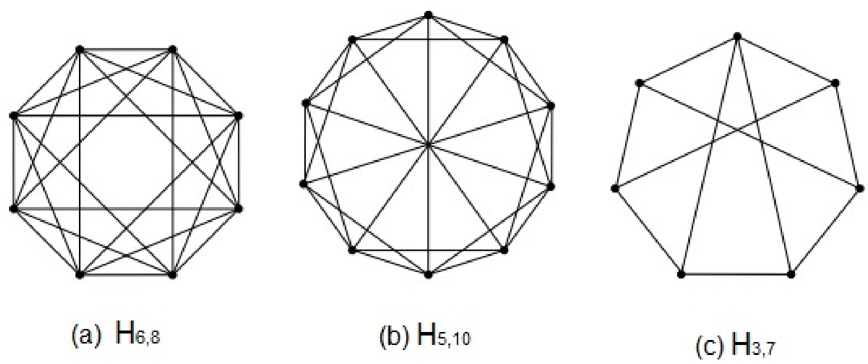


**Figure 1.** (**a**) The Harary graph $H_{6,8}$; (**b**) The Harary graph $H_{5,10}$; (**c**) The Harary graph $H_{3,7}$.

## 2. The Rooted Product of Harary Graphs with Cycle Graph

Suppose $C_3^i, 1 \leq i \leq n$ , $C_4^i, 1 \leq i \leq n$ and $C_5^i, 1 \leq i \leq n$ be n copies of $C_3$, $C_4$and $C_5$ having vertices $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 3\}$, $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 4\}$ and $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 5\}$ respectively and $\{v_1, v_2, v_3, ..., v_n\}$ be the set of vertices of $H_{m,n}$. By definition of rooted product $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 3\}$, $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 4\}$ and $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 5\}$ will be sets of vertices of $H_{m,n}(C_3)$,$H_{m,n}(C_4)$ and $H_{m,n}(C_5)$ respectively with indices taken modulo $n$. After rooted product, it is considered that all the cycles share $\{v_i^1, 1 \leq i \leq n\}$ with $H_{m,n}$.

More preciously, the graphs $H_{m,n}(C_3)$, $H_{m,n}(C_4)$ and $H_{m,n}(C_5)$ are the rooted product of Harary graphs $H_{m,n}$ by cycles $C_3$, $C_4$ and $C_5$ respectively.

Now we present our main results on metric dimension of $H_{m,n}(C_3)$, $H_{m,n}(C_4)$ and $H_{m,n}(C_5)$.

**Theorem 1.** *If $G_1 \cong H_{m,n}(C_3)$, then there exists a resolving set $W$ of $G_1$ such that*

$$\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_1), \;\; \{v_i^3; 1 \leq i \leq n\} \nsubseteq W(G_1),$$

$$|W(G_1)| \geq n.$$

**Proof.** As $d(v_i^2, v_i^1) = d(v_i^3, v_i^1), \forall, 1 \leq i \leq n$

and $d(v_i^2, v_k^j) = d(v_i^3, v_k^j), \forall, 1 \leq i, k \leq n, k \neq i$ and $j = 1, 2, 3$.

$\Rightarrow$ either $v_i^2 \in W(G_1)$ or $v_i^3 \in W(G_1)$.

To minimize the cardinality of $W(G_1)$, we can say without loss of any generality:

$\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_1)$ and $\{v_i^3; 1 \leq i \leq n\} \nsubseteq W(G_1)$.

$\Rightarrow |W(G_1)| \geqslant n$.
This conclude the proof. □

**Theorem 2.** *If $G_1 \cong H_{m,n}(C_3)$ and $W$ be a minimum resolving set of $G_1$ then $|W(G_1)| = n$.*

**Proof.** From Theorem 1, we have $|W(G_1)| \geqslant n$. Now to prove the reverse inequality, i.e., $|W(G_1)| \leq n$, we proceed as follows:

If we take $\{v_i^1, v_i^3; 1 \leq i \leq n\} \cap W = \phi$, then $W$ is also resolving set.

By Theorem 1, $v_i^2 \in W$ for all $1 \leq i \leq n$ and $d(v_i^2, v_i^1) \neq d(v_i^2, v_k^1), \forall\, 1 \leq i, k \leq n, i \neq k$.

$\Rightarrow r(v_i^1|W) \neq r(v_i^1|W), \forall\, 1 \leq i, k \leq n, i \neq k$.

As $v_i^2 \in W \Rightarrow v_{i+1}^2 \in W$ and $d(v_{i+1}^2, v_i^1) \neq d(v_{i+1}^2, v_i^3), \forall\, 1 \leq i \leq n$

$\Rightarrow r(v_i^1|W) \neq r(v_i^3|W), \forall, 1 \leq i \leq n,$

$r(v_i^1|W) \neq r(v_i^2|W)$ as $\{v_i^2; 1 \leq i \leq n\} \subseteq W$

also $r(v_i^3|W) \neq r(v_i^2|W)$ as $\{v_i^2; 1 \leq i \leq n\} \subseteq W$

$d(v_i^3, v_i^2) \neq d(v_k^3, v_i^2), \forall, 1 \leq i,k \leq n, i \neq k.$

$\Rightarrow r(v_i^3|W) \neq r(v_k^3|W), \forall, 1 \leq i,k \leq n, i \neq k.$

So we conclude that $W \setminus \{v_i^1; 1 \leq i \leq n\}$ is also the resolving set. This shows that $|W(G_1)| \leq n$. Hence, the required result is proved. □

**Theorem 3.** *If $G_2 \cong H_{m,n}(C_4)$, then there exists a resolving set $W$ of $G_2$ such that*

$$\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_2), \quad \{v_i^4; 1 \leq i \leq n\} \nsubseteq W(G_2), \quad |W(G_2)| \geq n$$

**Proof.** As $d(v_i^2, v_i^1) = d(v_i^4, v_i^1), \forall, 1 \leq i \leq n$

and $d(v_i^2, v_k^j) = d(v_i^4, v_k^j), \forall, 1 \leq i,k \leq n, k \neq i$ and $j = 1,2,3$.

$\Rightarrow$ either $v_i^2 \in W(G_2)$ or $v_i^4 \in W(G_2)$.

To minimize the cardinality of $W(G_2)$, we can say without loss of any generality:

$\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_2)$ and $\{v_i^4; 1 \leq i \leq n\} \nsubseteq W(G_2)$.

$\Rightarrow |W(G_2)| \geqslant n.$
This conclude the proof. □

**Theorem 4.** *If $G_2 \cong H_{m,n}(C_4)$ and $W$ be a minimum resolving set of $G_2$ then $|W(G_2)| = n$.*

**Proof.** From Theorem 3, $|W(G_2)| \geqslant n$. Now to prove the reverse inequality, i.e., $|W(G_2)| \leq n$, we proceed as follows:

If we take $\{v_i^1, v_i^3, v_i^4; 1 \leq i \leq n\} \cap W = \phi$ then $W$ is also resolving set.

By theorem 3, $v_i^2 \in W$ for all $1 \leq i \leq n$

and $d(v_i^2, v_i^1) \neq d(v_i^2, v_k^1), \forall, 1 \leq i,k \leq n, i \neq k.$

$\Rightarrow r(v_i^1|W) \neq r(v_k^1|W), \forall, 1 \leq i,k \leq n, i \neq k.$

and $r(v_i^1|W) \neq r(v_i^2|W)\ \forall\ 1 \leq i \leq n$, by definition of resolving set.

$d(v_i^2, v_i^1) \neq d(v_i^2, v_i^4), \forall, 1 \leq i \leq n.$

$\Rightarrow r(v_i^1|W) \neq r(v_i^4|W), \forall, 1 \leq i \leq n.$

As $v_i^2 \in W$ for all $1 \leq i \leq n \Rightarrow v_{i+1}^2 \in W$.

$d(v_{i+1}^2, v_i^3) \neq d(v_{i+1}^2, v_i^1), \forall\ 1 \leq i \leq n.$

$\Rightarrow r(v_i^1|W) \neq r(v_i^3|W), \forall, 1 \leq i \leq n.$

$r(v_i^2|W) \neq r(v_k^2|W), \forall, 1 \leq i,k \leq n, i \neq k$, by definition of resolving set.

$r(v_i^2|W) \neq r(v_k^3|W), \forall, 1 \leq i,k \leq n, i \neq k$, by definition of resolving set.

$r(v_i^2|W) \neq r(v_k^4|W), \forall, 1 \leq i,k \leq n, i \neq k$, by definition of resolving set.

As $d(v_i^2, v_i^3) \neq d(v_i^2, v_k^3), \forall\, 1 \leq i,k \leq n,\, i \neq k$.

$\Rightarrow r(v_i^3|W) \neq r(v_k^3|W), \forall, 1 \leq i,k \leq n,\, i \neq k$.

$d(v_i^2, v_i^3) \neq d(v_i^2, v_i^4), \forall, 1 \leq i \leq n$.

$\Rightarrow r(v_i^4|W) \neq r(v_i^3|W), \forall, 1 \leq i \leq n$.

$d(v_i^2, v_i^4) \neq d(v_i^2, v_k^4), \forall, 1 \leq i,k \leq n,\, i \neq k$.

$\Rightarrow r(v_i^4|W) \neq r(v_k^4|W), \forall, 1 \leq i,k \leq n,\, i \neq k$.

$\Rightarrow$ representation of all the vertices is unique if $\{v_i^1, v_i^3; 1 \leq i \leq n\} \nsubseteq W(G_2)$

In addition, from theorem 3, $\{v_i^4; 1 \leq i \leq n\} \nsubseteq W(G_2)$ and $\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_2)$.

Hence $W = \{v_i^2; 1 \leq i \leq n\}$ is the minimum resolving set of $G_2$. This shows that $|W(G_2)| = n$. □

**Theorem 5.** *If $G_3 \cong H_{m,n}(C_5)$, then there exists a resolving set $W$ of $G_3$ such that*

$$\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_3) \quad and \quad |W(G_3)| \geq n.$$

**Proof.** As $d(v_i^2, v_k^j) = d(v_i^5, v_k^j), \forall, 1 \leq i,k \leq n,\, i \neq k$ and $1 \leq j \leq 5$.

and $d(v_i^2, v_i^1) = d(v_i^5, v_i^1), \forall, 1 \leq i \leq n$.

$d(v_i^2, v_i^3) \neq d(v_i^5, v_i^3), \forall, 1 \leq i \leq n,$

$d(v_i^2, v_i^4) \neq d(v_i^5, v_i^4)\ \forall\, 1 \leq i \leq n$.

$\Rightarrow$ For all $1 \leq i \leq n$, $r(v_i^2|W) = r(v_i^5|W), \forall$, resolving sets $W$ in which

$\{v_i^2, v_i^3, v_i^4, v_i^5\} \cap W = \phi$.

and For all $1 \leq i \leq n$, $r(v_i^3|W) = r(v_i^4|W), \forall$, resolving sets $W$ in which

$\{v_i^2, v_i^3, v_i^4, v_i^5\} \cap W = \phi$.

To make the representation unique, we can say $\{v_i^2, v_i^3, v_i^4, v_i^5\} \cap W \neq \phi$. Without loss of any generality we can assume that $\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_3) \Rightarrow |W(G_3)| \geq n$. This concludes the proof. □

**Theorem 6.** *If $G_3 \cong H_{m,n}(C_5)$ and $W$ be a minimum resolving set of $G_3$ then $|W(G_3)| = n$.*

**Proof.** From Theorem 5, $|W(G_3)| \geqslant n$. Now to prove the reverse inequality, i.e., $|W(G_3)| \leq n$, we proceed as follows:

If we take $\{v_i^1, v_i^3, v_i^4, v_i^5; 1 \leq i \leq n\} \cap W = \phi$, then $W$ is also resolving set.

By Theorem 5, $v_i^2 \in W$ for all $1 \leq i \leq n$

and $d(v_i^2, v_i^1) \neq d(v_i^2, v_k^1), \forall, 1 \leq i, k \leq n, i \neq k$.

$\Rightarrow r(v_i^1|W) \neq r(v_k^1|W), \forall, 1 \leq i, k \leq n, i \neq k$.

and $r(v_k^1|W) \neq r(v_i^2|W)$ $\forall, 1 \leq i, k \leq n$ by definition of resolving set.

$d(v_i^1, v_{i+1}^2) \neq d(v_i^3, v_{i+1}^2), \forall, 1 \leq i \leq n$.

and $d(v_i^1, v_i^2) \neq d(v_k^3, v_i^2), \forall, 1 \leq i, k \leq n$ and $i \neq k$.

$r(v_i^1|W) \neq r(v_k^3|W)$ $\forall$ $1 \leq i, k \leq n$.

As $d(v_i^1, v_i^2) \neq d(v_k^4, v_i^2), \forall, 1 \leq i, k \leq n$.

$\Rightarrow r(v_i^1|W) \neq r(v_k^4|W), \forall, 1 \leq i, k \leq n$.

As $d(v_i^1, v_i^2) \neq d(v_k^5, v_i^2), \forall, 1 \leq i, k \leq n$.

$\Rightarrow r(v_i^1|W) \neq r(v_k^5|W), \forall$ $1 \leq i, k \leq n$.

$r(v_i^2|W) \neq r(v_k^j|W)$ $\forall$ $1 \leq i, k \leq n$ , $1 \leq j \leq 5$, by definition of resolving set.

As $d(v_i^3, v_i^2) \neq d(v_k^3, v_i^2), \forall, 1 \leq i, k \leq n$ $i \neq k$.

$\Rightarrow r(v_i^3|W) \neq r(v_k^3|W), \forall, 1 \leq i, k \leq n$ $i \neq k$.

$d(v_i^3, v_i^2) \neq d(v_k^4, v_i^2), \forall, 1 \leq i, k \leq n$.

$\Rightarrow r(v_i^3|W) \neq r(v_k^4|W), \forall, 1 \leq i, k \leq n$.

$d(v_i^3, v_i^2) \neq d(v_k^5, v_i^2), \forall, 1 \leq i, k \leq n$.

$\Rightarrow r(v_i^3|W) \neq r(v_k^5|W), \forall, 1 \leq i, k \leq n$.

As $d(v_i^4, v_i^2) \neq d(v_k^4, v_i^2), \forall, 1 \leq i, k \leq n$ $i \neq k$.

$\Rightarrow r(v_i^4|W) \neq r(v_k^4|W), \forall, 1 \leq i, k \leq n$ $i \neq k$.

$d(v_i^4, v_{i+1}^2) \neq d(v_i^5, v_{i+1}^2), \forall, 1 \leq i \leq n$.

and $d(v_i^4, v_i^2) \neq d(v_k^5, v_i^2)$, $\forall, 1 \leq i, k \leq n$ and $i \neq k$.

$r(v_i^4|W) \neq r(v_k^5|W)$ $\forall, 1 \leq i, k \leq n$.

$d(v_i^5, v_i^2) \neq d(v_k^5, v_i^2), \forall, 1 \leq i, k \leq n\ i \neq k$.

$\Rightarrow r(v_i^5|W) \neq r(v_k^5|W), \forall, 1 \leq i, k \leq n\ i \neq k$.

Hence $W \setminus \{v_i^1, v_i^3, v_i^4, v_i^5; 1 \leq i \leq n\}$ is the resolving set.
This shows that $|W(G_3)| = n$. $\square$

## 3. The Rooted Product of Harary Graphs with Path Graph

The graph $H_{4,n}(P_m)$ is the rooted product of Harary graph $H_{4,n}$ by path $P_m$, see Figure 2. To construct the graph $H_{4,n}(C_3)^c$ we first construct rooted product of Harary graph $H_{4,n}$ by cycle $C_3$ as shown in Figure 3a and then connect the remaining two vertices of each rooted $C_3$ with both neighboring $C_3$ as shown in Figure 3b.

The graphs $H_{4,n}(P_m)$ and $H_{4,n}(C_3)^c$ are an important class of graphs, which can be used in the design of local area networks [18].

Now we present our main results on metric dimension of $H_{4,n}(P_m)$ and $H_{4,n}(C_3)^c$.To calculate metric dimension of $H_{4,n}(C_3)^c$, $H_{4,n}(P_m)$ and $(P_2 \times P_k)(C_4)^c$ we need the following result of khuller et al. [8].

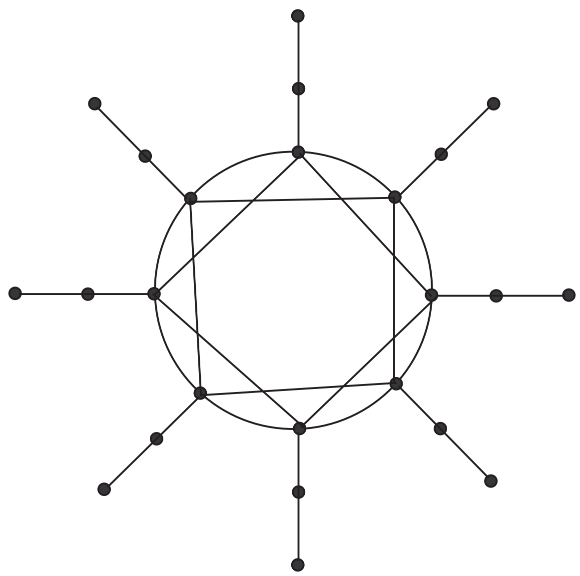

**Figure 2.** The graph $H_{4,8}(P_3)$.

**Theorem 7.** *Let $G$ be a graph with minimum metric dimension 2 and let $\{u, v\} \subset V$ be the metric basis in $G$. Then the following statements are true:*

*(a)* *There is a unique shortest path between $u$ and $v$.*
*(b)* *The degree of each $u$ and $v$ is at most 3.*

**Theorem 8.** *For $G \cong H_{4,n}(C_3)^c$, where $H_{4,n}$ be a 4-regular Harary graph with $n \geq 5$ and $C_3$ is the cycle of length 3; then we have $dim(G) = 3$ when $n \equiv 0, 1, 3(mod4)$ and $dim(G) \leq 4$ otherwise.*

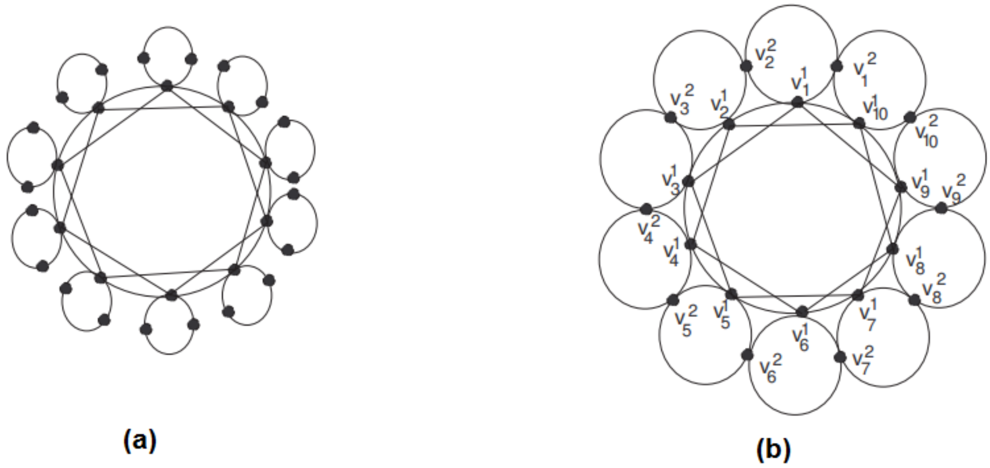


**Figure 3.** (**a**) The graph of $H_{4,10}(C_3)$; (**b**) The graph of $H_{4,10}(C_3)^c$.

**Proof. Case-I** when $n \equiv 0$ (mod4) i.e., $n = 4k, k \geq 2$ and $k \in N$.

be the resolving set of $G$ then $r(v_2^1|W) = (1,2,1)$,

$r(v_3^1|W) = (1,1,1)$,$r(v_4^1|W) = (2,1,2)$,$r(v_6^1|W) = (2,1,3)$,$r(v_7^1|W) = (1,1,3)$,

$r(v_8^1|W) = (1,2,2)$,$r(v_1^2|W) = (1,3,2)$,$r(v_2^2|W) = (1,3,1)$,$r(v_4^2|W) = (2,2,1)$,

$r(v_5^2|W) = (3,1,2)$,$r(v_6^2|W) = (3,1,3)$,$r(v_7^2|W) = (2,2,4)$,$r(v_8^2|W) = (2,2,3)$,

For $n = 12$ let $W = \{v_1^1, v_4^1, v_7^1\}$ be the resolving set of $G$ then

$r(v_2^1|W) = (1,3,1)$,$r(v_3^1|W) = (1,2,1)$,$r(v_5^1|W) = (2,1,1)$,$r(v_6^1|W) = (3,1,1)$,

$r(v_8^1|W) = (3,1,2)$,$r(v_9^1|W) = (2,1,3)$,$r(v_{10}^1|W) = (2,2,3)$,$r(v_{11}^1|W) = (1,2,3)$,

$r(v_{12}^1|W) = (1,3,2)$,$r(v_1^2|W) = (1,4,3)$,$r(v_2^2|W) = (1,4,2)$,$r(v_3^2|W) = (2,3,2)$,

$r(v_4^2|W) = (2,3,1)$,$r(v_5^2|W) = (3,2,1)$,$r(v_6^2|W) = (3,2,2)$,$r(v_7^2|W) = (4,1,2)$,

$r(v_8^2|W) = (4,1,3)$,$r(v_9^2|W) = (3,2,3)$,$r(v_{10}^2|W) = (3,2,4)$,$r(v_{11}^2|W) = (2,3,4)$,

$r(v_{12}^2|W) = (2,3,3)$.

For $n \geq 16$ let $W = \{v_1^2, v_4^2, v_{2k+2}^2\}$ be the resolving set of $G$ then

$$r(v_{2i}^1|W) = \begin{cases} (i+1, i-1, k+2-i) & 1 \leq i \leq k \\ (i-1, i-1, 1) & i = k+1 \\ (2k+1-i, 2k+3-i, i-k) & k+2 \leq i \leq 2k \end{cases}$$

$$r(v_{2i+1}^1|W) = \begin{cases} (i+1, i, k+1-i) & 2 \leq i \leq k \\ (2k+1-i, 2k+2-i, i-k+1) & k+1 \leq i \leq 2k-1 \end{cases}$$

$$r(v_{2i}^2|W) = \begin{cases} (i+1, i, k+3-i) & 4 \le i \le k-1 \\ (2k+2-i, 2k+4-i, i+1-k) & k+3 \le i \le 2k-1 \end{cases}$$

$$r(v_{2i+1}^2|W) = \begin{cases} (i+2, i, k+2-i) & 3 \le i \le k-1 \\ (2k+2-i, 2k+3-i, i+1-k) & k+2 \le i \le 2k-2 \end{cases}$$

$r(v_1^1|W) = (1,2,k+1)$,$r(v_2^1|W) = (2,2,k+1)$,$r(v_3^1|W) = (2,1,k)$,

$r(v_{2k}^2|W) = (k+1,k,2)$,$r(v_{2k+1}^2|W) = (k+2,k,1)$,$r(v_{2k+3}^2|W) = (k+1,k+1,1)$,

$r(v_{2k+4}^2|W) = (k,k+2,2)$,$r(v_{4k}^2|W) = (1,4,k+1)$,$r(v_{4k-1}^2|W) = (2,4,k)$,

$r(v_2^2|W) = (1,2,k+2)$,$r(v_3^2|W) = (2,1,k+1)$,$r(v_5^2|W) = (4,1,k)$.

Since distinct vertices have distinct representation, $dim(G) \le 3$ in this case. However, by Theorem 1 no two vertices can resolve $G$ into distinct representation so $dim(G) > 2$. Hence $dim(G) = 3$.

**Case-II** when $n \equiv 1$ (mod4) i.e., $n = 4k+1, k \in N$.

For $n = 5$ let $W = \{v_1^2, v_4^2, v_4^1\}$ be the resolving set of $G$ then $r(v_1^1|W) = (1,3,1)$,

$r(v_2^1|W) = (2,2,1)$,$r(v_3^1|W) = (3,1,1)$,$r(v_5^1|W) = (1,2,1)$,$r(v_2^2|W) = (1,2,2)$,

$r(v_3^2|W) = (2,1,2)$,$r(v_5^2|W) = (1,1,1)$.

For $n \ge 9$ let $W = \{v_1^2, v_4^2, v_{2k+1}^1\}$ be the resolving set of $G$ then

$$r(v_{2i}^1|W) = \begin{cases} (i+1, i-1, k+1-i) & 2 \le i \le k \\ (2k+2-i, 2k+3-i, i-k) & k+2 \le i \le 2k \end{cases}$$

$$r(v_{2i+1}^1|W) = \begin{cases} (i+1, i, k-i) & 1 \le i \le k-1 \\ (2k+1-i, 2k+3-i, i-k) & k+2 \le i \le 2k \end{cases}$$

$$r(v_{2i}^2|W) = \begin{cases} (i+1, i, k+2-i) & 4 \le i \le k \\ (2k+3-i, 2k+4-i, i-k) & k+2 \le i \le 2k-1 \end{cases}$$

$$r(v_{2i+1}^2|W) = \begin{cases} (i+2, i, k+1-i) & 3 \le i \le k \\ (2k+2-i, 2k+4-i, i-k+1) & k+2 \le i \le 2k-1 \end{cases}$$

$r(v_{2k+2}^1|W) = (k+1,k,1)$,$r(v_{2k+3}^1|W) = (k,k+1,1)$,$r(v_2^1|W) = (2,2,k)$,

$r(v_1^1|W) = (1,2,k)$,$r(v_{2k+2}^2|W) = (k+2,k+1,1)$,$r(v_{2k+3}^2|W) = (k+1,k+1,2)$,

$r(v_{4k}^2|W) = (2,3,k)$,$r(v_{4k+1}^2|W) = (1,3,k)$,$r(v_2^2|W) = (1,2,k+1)$,

$r(v_3^2|W) = (2,1,k)$,$r(v_5^2|W) = (4,1,k-1)$,$r(v_6^2|W) = (4,2,k-1)$.

Since distinct vertices have distinct representation, $dim(G) \le 3$ in this case. However, by Theorem 1 no two vertices can resolve $G$ into distinct representation so $dim(G) > 2$. Hence $dim(G) = 3$.

**Case-III** when $n \equiv 3$ (mod4) i.e., $n = 4k+3, k \in N$.

For $n = 7$ let $W = \{v_1^2, v_6^2, v_7^2\}$ be the resolving set of $G$ then $r(v_1^1|W) = (1,2,2)$,

$r(v_2^1|W) = (2,3,2)$,$r(v_3^1|W) = (2,2,3)$,$r(v_4^1|W) = (3,2,2)$,$r(v_5^1|W) = (2,1,2)$,

$r(v_6^1|W) = (2,1,1)$,$r(v_7^1|W) = (1,2,1)$,$r(v_2^2|W) = (1,3,2)$,$r(v_3^2|W) = (2,3,3)$,

$r(v_4^2|W) = (3,2,3)$,$r(v_5^2|W) = (3,1,2)$.

For $n = 11$ let $W = \{v_1^2, v_4^2, v_7^1\}$ be the resolving set of $G$ then

$r(v_1^1|W) = (1,2,3)$,$r(v_2^1|W) = (2,2,3)$,$r(v_3^1|W) = (2,1,2)$,$r(v_4^1|W) = (3,1,2)$,

$r(v_5^1|W) = (3,2,1)$,$r(v_6^1|W) = (4,2,1)$,$r(v_8^1|W) = (3,3,1)$,$r(v_9^1|W) = (2,4,1)$,

$r(v_{10}^1|W) = (2,3,2)$,$r(v_{11}^1|W) = (1,3,2)$,$r(v_2^2|W) = (1,2,4)$,$r(v_3^2|W) = (2,1,3)$,

$r(v_5^2|W) = (4,1,2)$,$r(v_6^2|W) = (4,2,2)$,$r(v_7^2|W) = (4,3,1)$,$r(v_8^2|W) = (4,4,1)$,

$r(v_9^2|W) = (3,4,2)$,$r(v_{10}^2|W) = (2,4,2)$,$r(v_{11}^2|W) = (1,4,3)$.

For $n \geq 15$ let $W = \{v_1^2, v_6^2, v_{2k+5}^2\}$ be the resolving set of $G$ then

$$r(v_{2i}^1|W) = \begin{cases} (i+1, i-2, k+3-i) & 3 \leq i \leq k+1 \\ (2k+3-i, 2k+5-i, i-k-1) & k+4 \leq i \leq 2k+1 \end{cases}$$

$$r(v_{2i+1}^1|W) = \begin{cases} (i+1, i-1, k+3-i) & 2 \leq i \leq k \\ (2k+2-i, 2k+5-i, i-k-1) & k+3 \leq i \leq 2k+1 \end{cases}$$

$$r(v_{2i}^2|W) = \begin{cases} (i+1, i-1, k+4-i) & 5 \leq i \leq k+1 \\ (2k+4-i, 2k+6-i, i-k-1) & k+4 \leq i \leq 2k \end{cases}$$

$$r(v_{2i+1}^2|W) = \begin{cases} (i+2, i-1, k+4-i) & 4 \leq i \leq k \\ (2k+3-i, 2k+6-i, i-k) & k+4 \leq i \leq 2k \end{cases}$$

$r(v_{2k+3}^1|W) = (k+1, k, 2)$,$r(v_{2k+4}^1|W) = (k+1, k, 1)$,$r(v_{2k+5}^1|W) = (k, k+1, 1)$,

$r(v_{2k+6}^1|W) = (k, k+1, 2)$,$r(v_1^1|W) = (1,3,k+1)$,$r(v_2^1|W) = (2,3,k+1)$,

$r(v_3^1|W) = (2,2,k+2)$,$r(v_4^1|W) = (3,2,k+1)$,$r(v_{2k+3}^2|W) = (k+2, k, 2)$,

$r(v_{2k+4}^2|W) = (k+2, k+1, 1)$,$r(v_{2k+6}^2|W) = (k+1, k+2, 1)$,

$r(v_{2k+7}^2|W) = (k, k+2, 2)$,$r(v_{4k+2}^2|W) = (2,5,k)$,$r(v_{4k+3}^2|W) = (1,5,k+1)$,

$r(v_2^2|W) = (1,4,k+2)$,$r(v_3^2|W) = (2,3,k+2)$,$r(v_4^2|W) = (3,2,k+2)$,

$r(v_5^2|W) = (4,1,k+2)$,$r(v_7^2|W) = (5,1,k+1)$,$r(v_8^2|W) = (5,2,k)$.

Since distinct vertices have distinct representation, $dim(G) \leq 3$ in this case. However, by theorem 1 no two vertices can resolve $G$ into distinct representation so $dim(G) > 2$.

Hence $dim(G) = 3$.

**Case-IV** when $n \equiv 2(mod 4)$ i.e., $n = 4k+2, k \in N$.

For $n = 6$ let $W_1 = \{v_1^1, v_1^2, v_5^1\}$ be the subset of $V(G)$ and $r(v_2^1|W_1) = (2,1,2)$,

$r(v_3^1|W_1) = (2,1,1)$,$r(v_4^1|W_1) = (2,2,1)$,$r(v_6^1|W_1) = (1,1,1)$,$r(v_2^2|W_1) = (1,1,2)$,

$r(v_3^2|W_1) = (2,2,2)$,$r(v_4^2|W_1) = (3,2,2)$,$r(v_5^2|W_1) = (2,2,1)$,$r(v_6^2|W_1) = (1,2,1)$,

since $r(v_5^2|W_1) = r(v_4^1|W_1) \Rightarrow W = W_1 \cup \{v_4^1\}$ is the resolving set of $G$.

$\Rightarrow dim(G) \leq 4$

For $n = 10$ let $W_1 = \{v_1^2, v_4^2, v_8^2\}$ be the subset of $V(G)$ and $r(v_1^1|W_1) = (1,2,3)$,

$r(v_2^1|W_1) = (2,2,3)$,$r(v_3^1|W_1) = (2,1,3)$,$r(v_4^1|W_1) = (3,1,3)$,$r(v_5^1|W_1) = (3,2,2)$,

$r(v_6^1|W_1) = (3,2,2)$,$r(v_7^1|W_1) = (3,3,1)$,$r(v_8^1|W_1) = (2,3,1)$,$r(v_9^1|W_1) = (2,3,2)$,

$r(v_{10}^1|W_1) = (1,3,2)$,$r(v_2^2|W_1) = (1,2,4)$,$r(v_3^2|W_1) = (2,1,4)$,$r(v_5^2|W_1) = (4,1,3)$,

$r(v_6^2|W_1) = (4,2,2)$,$r(v_7^2|W_1) = (4,3,1)$,$r(v_9^2|W_1) = (2,4,1)$,

$r(v_{10}^2|W_1) = (1,4,2)$.

since $r(v_5^1|W_1) = r(v_6^1|W_1) \Rightarrow W = W_1 \cup \{v_5^1\}$ is the resolving set of $G$.

$\Rightarrow dim(G) \leq 4$

For $n \geq 14$ let $W = \{v_1^2, v_4^2, v_{2k+4}^2\}$ be the subset of $V(G)$ then

$$r(v_{2i}^1|W) = \begin{cases} (i+1, i-1, k+3-i) & 2 \leq i \leq k \\ (2k+2-i, 2k+4-i, i-k-1) & k+3 \leq i \leq 2k+1 \end{cases}$$

$$r(v_{2i+1}^1|W) = \begin{cases} (i+1, i, k+2-i) & 1 \leq i \leq k \\ (2k+2-i, 2k+3-i, i-k) & k+2 \leq i \leq 2k \end{cases}$$

$$r(v_{2i}^2|W) = \begin{cases} (i+1, i, k+4-i) & 4 \leq i \leq k \\ (2k+3-i, 2k+5-i, i-k) & k+4 \leq i \leq 2k \end{cases}$$

$$r(v_{2i+1}^2|W) = \begin{cases} (i+2, i, k+3-i) & 3 \leq i \leq k \\ (2k+3-i, 2k+4-i, i-k) & k+3 \leq i \leq 2k-1 \end{cases}$$

$r(v_{2k+2}^1|W) = (k+1, k, 2)$,$r(v_{2k+3}^1|W) = (k+1, k+1, 1)$,

$r(v_{2k+4}^1|W_1) = (k, k+1, 1)$,$r(v_1^1|W_1) = (1, 2, k+1)$,$r(v_2^1|W_1) = (2, 2, k+1)$,

$r(v_{2k+2}^2|W_1) = (k+2, k+1, 2)$,$r(v_{2k+3}^2|W_1) = (k+2, k+1, 1)$,

$r(v_{2k+5}^2|W_1) = (k+1, k+2, 1)$,$r(v_{2k+6}^2|W_1) = (k, k+2, 2)$,$r(v_{4k+1}^2|W_1) = (2, 4, k)$,

$r(v_{4k+2}^2|W_1) = (1, 4, k+1)$,$r(v_2^2|W_1) = (1, 2, k+2)$,$r(v_3^2|W_1) = (2, 1, k+2)$,

$r(v_5^2|W_1) = (4, 1, k+1)$,$r(v_6^2|W_1) = (4, 2, k+1)$. since $r(v_{2k+1}^1|W_1) = r(v_{2k+2}^1|W_1) \Rightarrow W = W_1 \cup \{v_{2k+1}^1\}$ is the resolving set of $G$.

$\Rightarrow dim(G) \leq 4$. This complete the proof. □

**Theorem 9.** *For $G \cong H_{4,n}(P_m)$ where $H_{4,n}$ be a 4-regular Harary graph with $n \geq 5$ and $P_m$ is the path of length $m-1$; then we have $dim(G) = 3$ when $n \equiv 0, 2, 3(mod4)$ and $dim(G) \leq 4$ otherwise.*

**Proof. Case-I** when $n \equiv 0(mod4)$ i.e., $n = 4k, k \geq 2$ and $k \in N$.
Let $W = \{v_1^1, v_2^1, v_{2k+1}^1\}$ be the resolving set of $G$ then

$$r(v_{2i}^j|W) = \begin{cases} (i+j-1, i+j-2, k-i+j) & 2 \leq i \leq k, 1 \leq j \leq m \\ (2k-i+j, 2k-i+j, i-k+j-1) & k+1 \leq i \leq 2k, 1 \leq j \leq m \end{cases}$$

$$r(v_{2i+1}^j|W) = \begin{cases} (i+j-1, i+j-1, k-i+j-1) & 1 \leq i \leq k-1, 1 \leq j \leq m \\ (2k-i+j-1, 2k-i+j, i-k+j-1) & k+1 \leq i \leq 2k-1, 1 \leq j \leq m \end{cases}$$

For $2 \leq j \leq m$,$r(v_1^j|W) = (j-1, j, k+j-1)$,

$r(v_2^j|W) = (j, j-1, k+j-1)$,$r(v_{2k+1}^j|W) = (k+j-1, k+j-1, j-1)$.

Since distinct vertices have distinct representation, $dim(G) \leq 3$ in this case. Now we prove that $dim(G) \neq 2$ when $n \equiv 0$(mod4). Since every vertex that lies on cycle has degree 5, by Theorem 1 we shall take the vertices on pendents uncommon to the cycle when $|W| = 2$. Without loss of generality we can say

$W = \{v_1^2, v_1^3\}$ and $W = \{v_1^2, v_i^2\}, 2 \leq i \leq 2k+1$ represent all possible cases in which $|W| = 2$ and in each case the following contradictions arise.
Take $W = \{v_1^2, v_1^3\}$ then $r(v_2^1/W) = r(v_{4k}^1|W) = (2, 3)$ a contradiction.

Take $W = \{v_1^2, v_{2i}^2\}, 1 \leq i \leq k$ then $r(v_{2k+1}^1|W) = r(v_{2k+2}^1|W) = (k+1, k+2-i)$ a contradiction.

Take $W = \{v_1^2, v_{2i+1}^2\}, 1 \leq i \leq k-1$ then $r(v_{2i+2}^1|W) = r(v_{2i+3}^1|W) = (i+2, 2)$ a contradiction.

Take $W = \{v_1^2, v_{2k+1}^2\}$, then $r(v_{2k}^1|W) = r(v_{2k+2}^1|W) = (k+1, 2)$ a contradiction.

hence $dim(G) = 3$.

**Case-II** when $n \equiv 2(mod4)$ i.e., $n = 4k+2, k \in N$.
Let $W = \{v_1^1, v_3^1, v_{2k+3}^1\}$ be the resolving set of $G$ then

$$r(v_{2i}^j|W) = \begin{cases} (i+j-1, i+j-2, k-i+j+1) & 2 \leq i \leq k+1, 1 \leq j \leq m \\ (2k-i+j+1, 2k-i+j+2, i-k+j-2) & k+2 \leq i \leq 2k+1, 1 \leq j \leq m \end{cases}$$

$$r(v_{2i+1}^{j}|W) = \begin{cases} (i+j-1, i+j-2, k-i+j) & 2 \leq i \leq k, 1 \leq j \leq m \\ (2k-i+j, 2k-i+j+1, i-k+j-2) & k+2 \leq i \leq 2k, 1 \leq j \leq m \end{cases}$$

For $2 \leq j \leq m$, $r(v_1^j|W) = (j-1, j, k+j-1)$,$r(v_2^j|W) = (j, j, k+j)$,

$$r(v_3^j|W) = (j, j-1, k+j-1), r(v_{2k+3}^j|W) = (k+j-1, k+j-1, j-1).$$

Since distinct vertices have distinct representation, $dim(G) \leq 3$ in this case. Now we prove that $dim(G) \neq 2$ when $n \equiv 2(\text{mod}4)$. Since every vertex lies that on cycle has degree 5, by theorem 1 we shall take the vertices on pendents uncommon to the cycle when $|W| = 2$. Without loss of generality we can say

$W = \{v_1^2, v_1^3\}$ and $W = \{v_1^2, v_i^2\}$, $2 \leq i \leq 2k+1$ represent all possible cases in which $|W| = 2$ and in each case the following contradictions arise. Take $W = \{v_1^2, v_1^3\}$ then $r(v_2^1|W) = r(v_{4k+2}^1|W) = (2, 3)$ a contradiction.

Take $W = \{v_1^2, v_2^2\}$, then $r(v_3^1|W) = r(v_{4k+2}^1|W) = (2, 2)$ a contradiction.

Take $W = \{v_1^2, v_{2i}^2\}$, $2 \leq i \leq k+1$ then $r(v_{2k+3}^1|W) = r(v_{2k+4}^1|W) = (k+1, k+3-i)$ a contradiction.

Take $W = \{v_1^2, v_{2i+1}^2\}$, $2 \leq i \leq k-1$ then $r(v_{2i+2}^1|W) = r(v_{2i+3}^1|W) = (i+2, 2)$ a contradiction.

Take $W = \{v_1^2, v_{2k+1}^2\}$ then $r(v_{2k}^1|W) = r(v_{2k+3}^1|W) = (k+1, 2)$ a contradiction.

hence $dim(G) = 3$.

**Case-III** when $n \equiv 3(mod4)$ i.e., $n = 4k+3$, $k \in N$.

Let $W = \{v_1^1, v_2^1, v_{2k+2}^1\}$ be the resolving set of $G$ then

$$r(v_{2i}^{j}|W) = \begin{cases} (i+j-1, i+j-2, k-i+j) & 2 \leq i \leq k, 1 \leq j \leq m \\ (2k-i+j+1, 2k-i+j+2, i-k+j-2) & k+2 \leq i \leq 2k+1, 1 \leq j \leq m \end{cases}$$

$$r(v_{2i+1}^{j}|W) = \begin{cases} (i+j-1, i+j-1, k-i+j) & 1 \leq i \leq k, 1 \leq j \leq m \\ (2k-i+j+1, 2k-i+j+1, i-k+j-1) & k+1 \leq i \leq 2k+1, 1 \leq j \leq m \end{cases}$$

For $2 \leq j \leq m$, $r(v_1^j|W) = (j-1, j, k+j)$,$r(v_2^j|W) = (j, j-1, k+j-1)$, $r(v_{2k+2}^j|W) = (k+j, k+j-1, j-1)$ .

Since distinct vertices have distinct representation, $dim(G) \leq 3$ in this case. Now we prove that $dim(G) \neq 2$ when $n \equiv 2$ (mod4). Since every vertex that lies on cycle has degree 5, by Theorem 1 we shall take the vertices on pendents uncommon to the cycle when $|W| = 2$. Without loss of generality we can say:

$W = \{v_1^2, v_1^3\}$ and $W = \{v_1^2, v_i^2\}$, $2 \leq i \leq 2k+2$ represent all possible cases in which $|W| = 2$ and in each case the following contradictions arise. Take $W = \{v_1^2, v_1^3\}$ then $r(v_2^1|W) = r(v_{4k+3}^1|W) = (2, 3)$ a contradiction.

Take $W = \{v_1^2, v_2^2\}$, then $r(v_3^1|W) = r(v_{4k+3}^1|W) = (2, 2)$ a contradiction.

Take $W = \{v_1^2, v_{2i}^2\}, 2 \leq i \leq k+1$ then $r(v_{2i-1}^1|W) = r(v_{2i-2}^1|W) = (i,2)$ a contradiction.

Take $W = \{v_1^2, v_{2i+1}^2\}, 1 \leq i \leq k$ then $r(v_{2i+2}^1|W) = r(v_{2i+3}^1|W) = (i+2,2)$ a contradiction.

hence $dim(G) = 3$.

**Case-IV** when $n \equiv 1(mod4)$ i.e., $n = 4k+1, k \in N$.
Let $W = \{v_1^1, v_2^1, v_{2k+2}^1, v_{2k+3}^1\}$ be the resolving set of $G$ then

$$r(v_{2i}^j|W) = \begin{cases} (i+j-1, i+j-2, k-i+j, k-i+j+1) & 2 \leq i \leq k, 1 \leq j \leq m \\ (2k-i+j, 2k-i+j+1, k-i+j+2, k-ij+2) & k+2 \leq i \leq 2k, 1 \leq j \leq m \end{cases}$$

$$r(v_{2i+1}^j|W) = \begin{cases} (i+j-1, i+j-1, k-i+j, k-i+j) & 1 \leq i \leq k, 1 \leq j \leq m \\ (2k-i+j, 2k-i+j, k-i+j+3, k-i+j+2) & k+2 \leq i \leq 2k, 1 \leq j \leq m \end{cases}$$

For $2 \leq j \leq m$, $r(v_1^j|W) = (j-1, j, k+j-1, k+j-1)$,
$r(v_2^j|W) = (j, j-1, k+j-1, k+j-1)$,$r(v_{2k+2}^j|W) = (k+j-1, k+j-1, j-1, j)$,

$r(v_{2k+3}^j|W) = (k+j-1, k+j-1, j, j-1)$. Since distinct vertices have distinct representation so $dim(G) \leq 4$ in this case. This complete the proof. □

## 4. The Rooted Product of Ladder Graph with Cycle Graph

To construct the graph $G \cong (P_2 \times P_k)(C_4)^c$ we first construct rooted product of ladder graph $(P_2 \times P_k)$ by cycle $C_4$ as shown in Figure 4a and then connect each rooted $C_4$ with both neighboring $C_4$ as shown in Figure 4b.

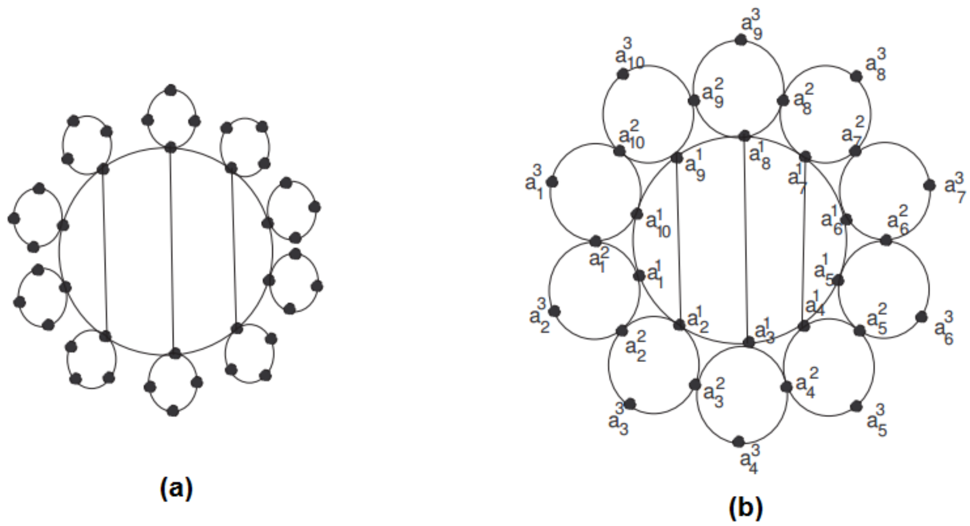


**Figure 4.** (**a**) The graph of $(P_2 \times P_5)(C_4)$; (**b**) The graph of $(P_2 \times P_5)(C_4)^c$.

**Theorem 10.** *For $G \cong (P_2 \times P_k)(C_4)^c$ where $C_4$ be a cycle of length 4 and $P_k$ is the path of length $k-1$; then we have $dim(G) = 3$.*

**Proof.** When $n = 2k, k \in N$ let $W = \{a_1^1, a_k^1, a_n^1\}$ be the resolving set of $G$ then

$$r(a_i^1|W) = \begin{cases} (i-1, i, k-i) & 2 \leq i \leq k-1 \\ (2k-i+1, 2k-i, i-k) & k+1 \leq i \leq 2k-1 \end{cases}$$

$$r(a_i^2|W) = \begin{cases} (i-1, i, k-i+1) & 2 \leq i \leq k \\ (2k-i+2, 2k-i+1, i-k) & k+2 \leq i \leq 2k \end{cases}$$

$$r(a_i^3|W) = \begin{cases} (i-1, i, k-i+2) & 3 \leq i \leq k \\ (2k-i+3, 2k-i+2, i-k) & k+2 \leq i \leq n \end{cases}$$

$r(a_1^2|W) = (1,1,k)$,$r(a_{k+1}^2|W) = (k,k,1)$,$r(a_1^3|W) = (2,2,k+1)$, $r(a_2^3|W) = (2,2,k)$,$r(a_{k+1}^3|W) = (k,k+1,2)$.

Since distinct vertices have distinct representation, $dim(G) \leq 3$. Now we prove that $dim(G) \neq 2$. Since all vertices have degree either 4 or 5 except $a_i^3$, by theorem 1 we can say $W = \{a_1^3, a_i^3\}$, $2 \leq i \leq n$ and $W = \{a_n^3, a_i^3\}$, $1 \leq i \leq n-1$ represent all possible cases in which $|W| = 2$ and in each case the following contradictions arise.

take $W = \{a_1^3, a_2^3\}$ then $r(a_n^1|W) = r(a_1^1|W) = (2,2)$, a contradiction.

take $W = \{a_1^3, a_3^3\}$ then $r(a_{n-1}^1|W) = r(a_n^1|W) = (2,3)$, a contradiction.

take $W = \{a_1^3, a_i^3\}$,$4 \leq i \leq k+1$ then $r(a_n^1|W) = r(a_1^1|W) = (2,2)$, a contradiction.

take $W = \{a_1^3, a_{k+2}^3\}$ then $r(a_{k+1}^1|W) = r(a_{k+3}^3|W) = (k,2)$, a contradiction.

take $W = \{a_1^3, a_i^3\}$,$k+3 \leq i \leq n$ then $r(a_k^1|W) = r(a_{k+1}^2|W) = (k+1, i-k)$ a contradiction.

take $W = \{a_n^3, a_i^3\}$,$1 \leq i \leq k-1$ then $r(a_{k+1}^1|W) = r(a_{k+2}^2|W) = (k-1, k+3-i)$, a contradiction.

take $W = \{a_n^3, a_k^3\}$ then $r(a_{k-1}^1|W) = r(a_{k-1}^3|W) = (k-1,2)$, a contradiction.

take $W = \{a_n^3, a_{k+1}^3\}$then $r(a_k^1|W) = r(a_k^3|W) = (k,2)$, a contradiction.

take $W = \{a_n^3, a_{k+2}^3\}$then $r(a_{k+1}^1|W) = r(a_{k+3}^3|W) = (k-1,2)$, a contradiction.

take $W = \{a_n^3, a_i^3\}$,$k+3 \leq i \leq n-1$ then $r(a_k^1|W) = r(a_{k+1}^2|W) = (k, i-k)$, a contradiction.

So $dim(G) \geq 3$. Combining both inequalities, we get $dim(G) = 3$. This conclude the proof. □

## 5. Conclusions

In the foregoing section, graphs $H_{4,n}(C_3)^c$, $(P_2 \times P_k)(C_4)^c$ and $H_{m,n}(C_i)$ for $i = 3,4,5$ are constructed. It is proven that metric dimension of $H_{4,n}(C_3)^c$ and $(P_2 \times P_k)(C_4)^c$ is either three or four in certain cases but the family of graphs $H_{m,n}(C_3)$ for $i = 3,4,5$ have unbounded metric dimension. This section is closed by raising the following open problems.

**Open Problem 1.** Determine the metric dimension of $H_{4,n}(C_3)$ and $(P_2 \times P_k)(C_4)$.
**Open Problem 2.** Determine the metric dimension of $H_{m,n}(C_n)$.

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