The Double Roman Domination Numbers of Generalized Petersen Graphs \( P(n,2) \)

Huiqin Jiang 1, Pu Wu 2, Zehui Shao 2  *, Yongsheng Rao 2 and Jia-Bao Liu 3,4,5  *  

1 Key Laboratory of Pattern Recognition and Intelligent Information Processing, Institutions of Higher Education of Sichuan Province, Chengdu University, Chengdu 610106, China; hq.jiang@hotmail.com
2 Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China; puwu1997@126.com (P.W.); zshao@gzhu.edu.cn (Z.S.); rsheng@gzhu.edu.cn (Y.R.)
3 School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China
4 * Correspondence: liujiabao@ahju.edu.cn or liujiabaoad@163.com

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Abstract: A double Roman dominating function (DRDF) \( f \) on a given graph \( G \) is a mapping from \( V(G) \) to \( \{0,1,2,3\} \) in such a way that a vertex \( u \) for which \( f(u) = 0 \) has at least a neighbor labeled 3 or two neighbors both labeled 2 and a vertex \( u \) for which \( f(u) = 1 \) has at least a neighbor labeled 2 or 3. The weight of a DRDF \( f \) is the value \( w(f) = \sum_{u \in V(G)} f(u) \). The minimum weight of a DRDF on a graph \( G \) is called the double Roman domination number \( \gamma_{dR}(G) \) of \( G \). In this paper, we determine the exact value of the double Roman domination number of the generalized Petersen graphs \( P(n,2) \) by using a discharging approach.

Keywords: double Roman domination; discharging approach; generalized Petersen graphs

1. Introduction

In this paper, only graphs without multiple edges or loops are considered. For two vertices \( u \) and \( v \) of a graph \( G \), we say \( u \sim v \) in \( G \) if \( uv \in E(G) \). For positive integer \( k \) and \( u, v \in V(G) \), let \( d(u, v) \) be the distance between \( u \) and \( v \) and \( N_k(v) = \{ u \mid d(u, v) = k \} \). The neighborhood of \( v \) in \( G \) is defined to be \( N_1(v) \) (or simply \( N(v) \)). The closed neighborhood \( N[v] \) of \( v \) in \( G \) is defined to be \( N[v] = \{ v \} \cup N(v) \).

For a vertex subset \( S \subseteq V(G) \), we denote by \( G[S] \) the subgraph induced by \( S \). For a positive integer \( n \), we denote \( [n] = \{1,2,\ldots,n\} \). For a set \( S = \{x_1,x_2,\ldots,x_n\} \), if \( x_i = x_j \) for some \( i \) and \( j \), then \( S \) is considered as a multiset. Otherwise, \( S \) is an ordinary set.

For positive integer numbers \( n \) and \( k \) with \( n \) at least \( 2k + 1 \), the generalized Petersen graph \( P(n,k) \) is a graph with its vertex set \( \{u_i\mid i = 1,2,\ldots,n\} \cup \{v_i\mid i = 1,2,\ldots,n\} \) and its edge set the union of \( \{u_iu_{i+1},u_iv_i,v_iv_{i+k}\} \) for \( 1 \leq i \leq n \), where subscripts are reduced modulo \( n \) (see [1]).

A subset \( D \) of the vertex set of a graph \( G \) is a dominating set if every vertex in \( V(G) \setminus D \) has at least one neighbor in \( D \). The domination number, denoted by \( \gamma(G) \), is the minimum number of vertices over all dominating sets of \( G \).

There have been more than 200 papers studying various domination on graphs in the literature [2–6]. Among them, Roman domination and double Roman domination appear to be a new variety of interest [3,7–15].

A double Roman dominating function (DRDF) \( f \) on a given graph \( G \) is a mapping from \( V(G) \) to \( \{0,1,2,3\} \) in such a way that a vertex \( u \) for which \( f(u) = 0 \) has at least a neighbor labeled 3 or two neighbors both labeled 2 and a vertex \( u \) for which \( f(u) = 1 \) has at least a neighbor labeled 2 or 3. The weight of a DRDF \( f \) is the value \( w(f) = \sum_{u \in V(G)} f(u) \). The minimum weight of a DRDF on a graph \( G \) is called the double Roman domination number \( \gamma_{dR}(G) \) of \( G \). A DRDF \( f \) of \( G \) with \( w(f) = \gamma_{dR}(G) \)
Lemma 1. \(\gamma_{DR}(G)\)-function. Given a DRDF \(f\) of \(G\), we denote \(E^f_{\{x_1,x_2\}} = \{uv \in E(G) | \{f(u), f(v)\} = \{x_1, x_2\}\}\). A graph \(G\) is a double Roman Graph if \(\gamma_{DR}(G) = 3\gamma(G)\).

In [7], Beeler et al. obtained the following results:

**Proposition 1 ([7]).** In a double Roman dominating function of weight \(\gamma_{DR}(G)\), no vertex needs to be assigned the value one.

By Proposition 1, we now consider the DRDF of a graph \(G\) in which there exists no vertex assigned with one in the following.

Given a DRDF \(f\) of a graph \(G\), suppose \((V^f_0, V^f_2, V^f_3)\) is the ordered partition of the vertex set of \(G\) induced by \(f\) in such a way that \(V^f_i = \{v : f(v) = i\}\) for \(i = 0, 2, 3\). It can be seen that there is a 1-1 mapping between \(f\) and \((V^f_0, V^f_2, V^f_3)\), and we write \(f = (V^f_0, V^f_2, V^f_3)\), or simply \((V_0, V_2, V_3)\). Given a DRDF \(f\) of \(P(n,2)\) and letting \(w_i \in \{0,2,3\}\) for \(i = 1,2,3\) with \(w_1 \geq w_2 \geq w_3\), we write \(\forall w_i = \{w_1,w_2,w_3\} = \{f(x_1), f(x_2), f(x_3)\}\), where \(N(x) = \{x_1, x_2, x_3\}\).

Now, we will use \(f(.) = q^t\) to represent the value scope \(f(.) \geq q\) for an integer \(q\). We say a path \(t_1t_2\cdots t_k\) is a path of type \(c_1 - c_2 - \cdots - c_q\) if \(f(t_i) = c_i\) for \(i \in [k]\). Let \(H\) be a subgraph induced by five vertices \(s_1, s_2, s_3, s_4, s_5\) with \(s_1 \sim s_2, s_2 \sim s_3, s_3 \sim s_4, s_3 \sim s_5\) satisfying \(f(s_3) = 0\) and \(f(s_1) = a, f(s_2) = b, f(s_4) = c, f(s_5) = d\) for some \(a,b,c,d \in \{0,2,3\}\), then we say \(H\) is a subgraph of type \(a - b - c\).

Let \(W\) be a subgraph induced by four vertices \(s_1, s_2, s_3, s_4\) with \(s_1 \sim s_2, s_2 \sim s_3, s_3 \sim s_4\), satisfying \(f(s_1) = a, f(s_2) = 0, f(s_3) = b\), and \(f(s_4) = c\) for some \(a,b,c \in \{0,2,3\}\), then we say \(W\) is a subgraph of type \(a - b - c\).

In the graph \(P(n,2)\), we will denote the set of vertices of \(\{u_i,v_i\}\) with \(L^{(i)}\). For a given DRDF \(f\) of \(P(n,2)\), let \(w_f(L^{(i)})\) denote the weight of \(L^{(i)}\), that is \(w_f(L^{(i)}) = \sum_{u \in V(L^{(i)})} f(u)\). Let \(B_i = \{L^{(i-2)}, L^{(i-1)}, L^{(i)}, L^{(i+1)}, L^{(i+2)}\}\), where the subscripts are taken modulo \(n\). We define \(w_f(B_i) = \sum_{j=-2}^{2} w_f(L^{(i+j)})\), and:

\[
f(B_i) = f \left( \begin{array}{cccc} u_{i-2} & u_{i-1} & u_i & u_{i+1} \\ v_{i-2} & v_{i-1} & v_i & v_{i+1} \end{array} \right).
\]

Motivation: Beeler et al. [7] put forward an open problem about characterizing the double Roman graphs. As an interesting family of graphs, the domination and its variations of generalized Petersen graphs have attracted considerable attention [1,16]. Therefore, it is interesting to characterize the double Roman graphs in generalized Petersen graphs. In this paper, we focus on finding the double Roman graphs in \(P(n,2)\).

2. Double Roman Domination Number of \(P(n,2)\)

2.1. Upper Bound for the Double Roman Domination Number of \(P(n,2)\)

**Lemma 1.** If \(n \geq 5\), then:

\[
\gamma_{DR}(P(n,2)) \leq \begin{cases} \left\lfloor \frac{5n}{3} \right\rfloor, & n \equiv 0 \pmod{5}, \\
\left\lfloor \frac{5n}{3} \right\rfloor + 1, & n \equiv 1,2,3,4 \pmod{5}. \end{cases}
\]

**Proof.** We consider the following five cases.

Case 1: \(n \equiv 0 \pmod{5}\).

Let:

\[
P_3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.
\]
Then, by repeating the pattern of \( P_5 \), we obtain a DRDF of weight \( 8k \) of \( P(5k, 2) \), and the upper bound is obtained.

Case 2: \( n \equiv 1 \pmod{5} \).

If \( n = 6 \), let:

\[
P_6 = \begin{bmatrix}
0 & 2 & 0 & 2 & 0 & 0 \\
2 & 0 & 0 & 2 & 3
\end{bmatrix}.
\]

Then, the pattern \( P_6 \) induces a DRDF of weight 11 of \( P(6, 2) \), and the desired upper bound is obtained.

If \( n \geq 11 \), let:

\[
P_{11} = \begin{bmatrix}
2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 3 & 2
\end{bmatrix}.
\]

Then, by repeating the leftmost five columns of the pattern of \( P_{11} \), we obtain a DRDF of weight \( 8k + 3 \) of \( P(5k + 1, 2) \), and the desired upper bound is obtained.

Case 3: \( n \equiv 2 \pmod{5} \).

If \( n = 7 \), let:

\[
P_7 = \begin{bmatrix}
2 & 0 & 2 & 0 & 0 & 3 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 2
\end{bmatrix}.
\]

Then, the pattern \( P_7 \) induces a DRDF of weight 13 of \( P(7, 2) \), and the desired upper bound is obtained.

If \( n \geq 12 \), let:

\[
P_{12} = \begin{bmatrix}
2 & 0 & 2 & 0 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 2
\end{bmatrix}.
\]

Then, by repeating the leftmost five columns of the pattern of \( P_{12} \), we obtain a DRDF of weight \( 8k + 6 \) of \( P(5k + 2, 2) \), and the desired upper bound is obtained.

Case 4: \( n \equiv 3 \pmod{5} \).

If \( n \geq 8 \), let:

\[
P_8 = \begin{bmatrix}
2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 2 & 2
\end{bmatrix}.
\]

Then, by repeating the leftmost five columns of the pattern of \( P_8 \), we obtain a DRDF of weight \( 8k + 6 \) of \( P(5k + 3, 2) \), and the desired upper bound is obtained.

Case 5: \( n \equiv 4 \pmod{5} \).

If \( n \geq 9 \), let:

\[
P_9 = \begin{bmatrix}
2 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 3 & 2
\end{bmatrix}.
\]

Then, by repeating the leftmost five columns of the pattern of \( P_9 \), we obtain a DRDF of weight \( 8k + 8 \) of \( P(5k + 4, 2) \), and the desired upper bound is obtained.

\[
2.2. \text{Lower Bound for Double Roman Domination Number of } P(n, 2)
\]

Lemma 2. Let \( f \) be a \( \gamma_{dr} \)-function of \( P(n, 2) \) with \( n \geq 5 \). Then, \( w_f(B_i) \geq 4 \).

Proof. Since \( u_i, v_i, u_{i+1} \) and \( u_{i-1} \) need to be double Roman dominated by vertices in \( B_i \), we have \( w_f(B_i) \geq 3 \). Now, we will show that \( w_f(B_i) \neq 3 \). Otherwise, it is clear that \( f(u_i) = 3 \), and \( f(x) = 0 \) for any \( x \in B_i \setminus \{u_i\} \). Since \( v_{i \pm 1}, u_{i \pm 2} \) and \( v_{i \pm 2} \) need to be double Roman dominated, we have
\[ f(u_{i\pm3}) = f(v_{i\pm3}) = f(v_{i\pm4}) = 3. \] Now, we can obtain a DRDF \( f' \) from \( f \) by letting \( f'(u_{i-2}) = 2, f'(u_{i-3}) = 0 \) and \( f'(v) = f(v) \) for \( v \in V(P(n,2)) \setminus \{u_{i-2}, u_{i-3}\} \). Then, we have \( w(f') < w(f) \), a contradiction (see Figure 1). Therefore, \( w_f(B_i) \geq 4. \)

**Figure 1.** Construct a function \( f' \) from \( f \) used in Lemma 2.

**Lemma 3.** Let \( f \) be a \( \gamma_{dR} \)-function of \( P(n,2) \) with \( n \geq 5 \). Then, for any \( i \in [n] \), it is impossible that \( f(v_{i-1}) = f(v_i) = f(v_{i+1}) = 3 \) and \( f(x) = 0 \) for any \( x \in B_i \setminus \{v_{i-1}, v_i, v_{i+1}\} \).

**Proof.** Suppose to the contrary that \( f(v_{i-1}) = f(v_i) = f(v_{i+1}) = 3 \) and \( f(x) = 0 \) for \( x \in B_i \setminus \{v_{i-1}, v_i, v_{i+1}\} \). Then, we have \( f(u_{i\pm3}) = 3 \). Now, we can obtain a DRDF \( f' \) from \( f \) by letting \( f'(u_{i-1}) = 2, f'(v_{i-1}) = 0 \) and \( f'(v) = f(v) \) for \( v \in V(P(n,2)) \setminus \{v_{i-1}, u_{i-3}\} \). Then, we have \( w(f') < w(f) \), a contradiction (see Figure 2). \( \square \)

**Figure 2.** Construct a function \( f' \) from \( f \) in Lemma 3.

**Lemma 4.** Let \( f \) be a \( \gamma_{dR} \)-function of \( P(n,2) \) with \( n \geq 5 \). Then, for each \( x \in V_3^{000} \), there exists a neighbor \( y \) of \( x \) such that \( y \in V_0^{320} \cup V_0^{330} \cup V_0^{332} \cup V_0^{332} \cup V_0^{333}, \) or equivalently, it is impossible that for any \( x \in V_3^{000} \), \( f(z) = 0 \) for any \( z \in N_2(x) \).
Proof. Suppose to the contrary that there is a vertex \( x \in V_3^{000} \) such that \( y \in V_0^{300} \) for every neighbor \( y \) of \( x \). Now, it is sufficient to consider the following two cases.

Case 1: \( x = u_i \) for some \( i \).

In this case, we have \( f(u_i) = 3 \) and \( f(x) = 0 \) for \( x \in B_i \setminus \{u_i\} \). Then, we have \( w_f(B_i) = 3 < 4 \), contradicting Lemma 2.

Case 2: \( x = v_i \) for some \( i \).

In this case, since \( u_{i+1} \) and \( u_{i+2} \) need to be double Roman dominated, we have \( f(v_{i+1}) = 3 \) and \( f(u_{i+3}) = 3 \). By Lemma 3, such a case is impossible. \( \square \)

Discharging procedure: Let \( f \) be a DRDF of \( P(n, 2) \). We set the initial charge of every vertex \( x \) be \( s(x) = f(x) \). We use the discharging procedure, leading to a final charge \( s' \), defined by applying the following rules:

- **R1:** Each \( s(x) \) for which \( s(x) = 3 \) transmits 0.8 charge to each neighbor \( y \) with \( y \in V_0^{300} \) transmits 0.6 charge to each neighbor \( y \) with \( y \in V_0^{320} \cup V_0^{330} \cup V_0^{322} \cup V_0^{332} \cup V_0^{333} \).
- **R2:** Each \( s(x) \) for which \( s(x) = 2 \) transmits 0.4 charge to each neighbor \( y \) with \( y \in V_0 \).

**Proposition 2.** If \( n \geq 5 \), then \( \gamma_{\text{DR}}(P(n, 2)) \geq \lceil \frac{8n}{5} \rceil \).

Proof. Assume \( f \) is a \( \gamma_{\text{DR}} \)-function of \( P(n, 2) \). We use the above discharging procedure. Now, it is sufficient to consider the following three cases.

Case 1: By Lemma 4, there exists a vertex \( z \) with \( f(z) \geq 2 \) for some \( z \in N_2(x) \), for any \( x \in V_0^{000} \). Therefore, by rule R1, for each \( v \in V_0^{000} \), the final charge \( s'(v) \) is at least \( 3 - 0.6 - 0.8 - 0.8 = 0.8 \). For each \( v \in V_3 \setminus V_0^{000} \), then the final charge \( s'(v) \) is at least \( 3 - 0.8 - 0.8 = 1.4 \).

Case 2: By rule R2, for each \( v \in V_2 \), the final charge \( s'(v) \) is at least \( 2 - 0.4 - 0.4 - 0.4 = 0.8 \).

Case 3: For each \( v \in V_0^{300} \), the final charge \( s'(v) \) is 0.8 by rule R1. For each \( v \in V_0 \setminus V_0^{300} \), the final charge \( s'(v) \) is at least 0.8 by rules R1 and R2.

From the above, we have:

\[
s'(v) \geq 0.8 \text{ for any } v \in P(n, 2).
\]

Hence, \( w(f) = \sum_{v \in V(P(n, 2))} s(v) = \sum_{v \in V(P(n, 2))} s'(v) \geq 0.8 \times 2n = \frac{8n}{5} \). Since \( w(f) \) is an integer, we have \( w(f) \geq \lceil \frac{8n}{5} \rceil \). \( \square \)

By using the above discharging rules, we have the following lemma immediately, and the proof is omitted.

**Lemma 5.** Let \( f \) be a \( \gamma_{\text{DR}} \)-function of \( P(n, 2) \) with \( n \geq 5 \). If we use the above discharging procedure for \( f \) on \( P(n, 2) \), then:

- (a) if there exists a path \( P \) of type 222 - 222 or type 222 - 222 - 0 - 3 - 0 - 222 - 0 - 3 - 0 - 222 - 0 - 3 - 0 - 232 - 0 - 3 - 0 - 233 - 0 - 3 - 0 - 233 - 0 - 3 - 0 - 233, then \( \sum_{v \in V(P)} s'(v) - 0.8 \) \( \geq \).
- (b) if there exist a path \( P_1 \) of type 222 - 222 and a path \( P_2 \) of type 222 - 0 - 3 - 0 - 232 - 0 - 3 - 0 - 233 - 0 - 3 - 0 - 233 - 0 - 3 - 0 - 233, then \( \sum_{v \in V(P_1)} s'(v) - 0.8 \) \( \geq \).
- (c) if there exists a subgraph \( H \) of type 222 - 222 - 0 - 232, then \( \sum_{v \in V(H)} s'(v) - 0.8 \) \( \geq \).
- (d) if there exist a path \( P \) of type 222 - 222 - 0 - 3 - 0 - 232 - 0 - 3 - 0 - 233 - 0 - 3 - 0 - 233 - 0 - 3 - 0 - 233, then \( \sum_{v \in V(P)} s'(v) - 0.8 \) \( \geq \).
- (e) if there exist three paths \( P_1, P_2, P_3 \) of type 222 - 222 - 0 - 3 - 0 - 232 - 0 - 3 - 0 - 233 - 0 - 3 - 0 - 233 - 0 - 3 - 0 - 233, then \( \sum_{v \in V(P_1) \cup V(P_2) \cup V(P_3)} s'(v) - 0.8 \) \( \geq \).
Lemma 6. Let $f$ be a $\gamma_{dR}$ function of $P(n, 2)$ with weight $\left\lceil \frac{8n}{5} \right\rceil$, then there exists no edge $uv \in E(P(n, 2))$ for which $uv \in E_{\{2,2\}}^f \cup E_{\{2,3\}}^f \cup E_{\{3,3\}}^f$.

Proof. First, we have:

$$
\gamma_{dR}(P(n, 2)) = w(f) = \left\lceil \frac{8n}{5} \right\rceil \leq \frac{8n + 4}{5} = \frac{8n}{5} + 0.8,
$$

and so:

$$
w(f) - \frac{8n}{5} \leq 0.8.
$$

We use the above discharging procedure for $f$ on $P(n, 2)$, and similar to the proof of Proposition 2, we have:

$$
w(f) = \sum_{v \in V(P(n, 2))} s'(v),
$$

and so:

$$
\sum_{v \in V(P(n, 2))} (s'(v) - \frac{4}{5}) \leq 0.8 \quad (2)
$$

By Lemma 5a and Equation (2), we have that there exists no edge $uv \in E_{\{2,2\}}^f \cup E_{\{2,3\}}^f \cup E_{\{3,3\}}^f$.

Now, suppose to the contrary that there exists an edge $uv \in E_{\{2,2\}}^f$, and it is sufficient to consider the following three cases.

Case 1: $f(u_i) = f(u_{i+1}) = 2$.

We have $f(u_{i-1}) = f(u_{i+2}) = f(v_{i+1}) = f(v_i) = 0$. Otherwise, there exists a path $P$ of type $2 - 2 - 2$ or type $2^+ - 3$. By Lemma 5a, we have $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

Since $u_{i+2}$ needs to be double Roman dominated, we have $\{f(u_{i+3}), f(v_{i+2})\} = \{0, 2\}$. Otherwise, $f(x) = 3$ for some $x \in \{u_{i+3}, v_{i+2}\}$ or $f(u_{i+3}) = f(v_{i+2}) = 2$.

If $f(x) = 3$ for some $x \in \{u_{i+3}, v_{i+2}\}$, there exists a path $P$ of type $2 - 2 - 0 - 3$. By Lemma 5a, we have $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

If $f(u_{i+3}) = f(v_{i+2}) = 2$, there exists a subgraph $H$ of type $2 - 2 - 0 - 2$. By Lemma 5c, we have $\sum_{v \in V(H)} (s'(v) - 0.8) \geq 1.2$, contradicting Equation (2).

Now, it is sufficient to consider the following two cases.

Case 1.1: $f(v_{i+2}) = 2$, $f(u_{i+3}) = 0$.

To double Roman dominate $v_{i+1}$, we have $f(v_{i+3}) \geq 2$ or $f(v_{i-1}) \geq 2$. First, we have $f(v_{i+3}) \neq 3$ and $f(v_{i-1}) \neq 3$. Otherwise, $u_{i+1}v_{i+1}v_{i+3}$ or $u_{i+1}v_{i+1}v_{i-1}$ is a path $P$ of type $2 - 2 - 0 - 3$. By Lemma 5a, we have $\sum_{v \in V(P)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

Now, we have that it is impossible $f(v_{i+3}) = f(v_{i-1}) = 2$. Otherwise, the set $\{u_i, u_{i+4}, v_{i+1}, v_{i+3}, v_{i-1}\}$ induces a subgraph $H$ of type $2 - 2 - 0 - \frac{2}{2}$. By Lemma 5c, we have $\sum_{v \in V(H)} (s'(v) - 0.8) \geq 1.2$, contradicting Equation (2).

Therefore, we have $\{f(v_{i+3}), f(v_{i-1})\} = \{0, 2\}$. Now, it is sufficient to consider the following two cases.

Case 1.1.1: $f(v_{i+3}) = 2$, $f(v_{i-1}) = 0$.

Since $v_{i-1}$ and $u_{i-1}$ need to be double Roman dominated, we have $f(v_{i-3}) = 3$, $f(u_{i-2}) = 2^+$. Then, there exists a path $P_1$ of type $2 - 2$ and a path $P_2$ of type $2^+ - 0 - 3$. By Lemma 5b, we have $\sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1$, contradicting Equation (2).

Case 1.1.2: $f(v_{i+3}) = 0$, $f(v_{i-1}) = 2$.
Since \( u_{i+3} \) and \( v_{i+3} \) need to be double Roman dominated, we have \( f(u_{i+4}) = f(v_{i+5}) = 3 \). Then, there exist a path \( P_1 \) of type 2 – 2 and a path \( P_2 \) of type 3 – 0 – 3. By Lemma 5b, \( \sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Case 1.2: \( f(v_{i+2}) = 0, f(u_{i+3}) = 2 \).

Since \( v_{i+2} \) needs to be double Roman dominated, we have \( f(v_{i+4}) = 3 \). Then, there exist a path \( P_1 \) of type 2 – 2 and a path \( P_2 \) of type 2 – 0 – 3. By Lemma 5b, \( \sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Case 2: \( f(v_i) = f(u_i) = 2 \).

We have \( f(u_{i+1}) = f(v_{i+2}) = 0 \). Otherwise, there exists a path \( P \) of type 2 – 2 – 2 or type 2\(^+\) – 3. By Lemma 5a, we have \( \sum_{v \in V(P)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Since \( u_{i+1} \) needs to be double Roman dominated, we have \( \{ f(u_{i+2}), f(v_{i+1}) \} = \{ 0, 2 \} \). Otherwise, by Lemma 5a or Lemma 5c, we obtain a contradiction with Equation (2).

Now, we consider the following two subcases.

Case 2.1: \( f(v_{i+1}) = 2, f(u_{i+2}) = 0 \).

Since \( v_{i+2} \) needs to be double Roman dominated, we have \( f(v_{i+4}) = 3 \). Then, there exist a path \( P_1 \) of type 2 – 2 and a path \( P_2 \) of type 2 – 0 – 3. By Lemma 5b, \( \sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Case 2.2: \( f(v_{i+1}) = 0, f(u_{i+2}) = 2 \).

Since \( v_{i+1} \) needs to be double Roman dominated, we have \( f(v_{i+3}) = 3 \) for some \( x \in \{ v_{i+3}, v_{i-1} \} \) or \( f(v_{i+3}) = f(v_{i-1}) = 2 \). If \( f(x) = 3 \) for some \( x \in \{ v_{i+3}, v_{i-1} \} \), there exist a path \( P_1 \) of type 2 – 2 and a path \( P_2 \) of type 2 – 0 – 3. By Lemma 5b, \( \sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

If \( f(v_{i+3}) = f(v_{i-1}) = 2 \), then by Lemma 5b,c, we have \( u_{i-2} = 0 \). Since \( u_{i-2} \) needs to be double Roman dominated, we have \( f(u_{i-3}) = 3 \). Then, there exist a path \( P_1 \) of type 2 – 2 and a path \( P_2 \) of type 2 – 0 – 3. By Lemma 5b, \( \sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Case 3: \( f(v_{i+1}) = f(v_{i-1}) = 2 \).

We have \( f(u_{i+1}) = f(v_{i+3}) = 0 \). Otherwise, there exists a path \( P \) of type 2 – 2 – 2 or type 2\(^+\) – 3. By Lemma 5a, we have \( \sum_{v \in V(P)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Since \( u_i \) needs to be double Roman dominated, we have \( f(u_i) = 2 \) or \( f(v_i) = 3 \).

Case 3.1: \( f(u_i) = 2, f(v_i) = 0 \).

By Lemma 5b,c and Equation (2), we have \( f(u_{i+2}) = 0 \). Since \( v_i \) needs to be double Roman dominated, we have \( \{ f(v_{i-2}), f(v_{i+2}) \} = \{ 0, 2 \} \). Considering isomorphism, we without loss of generality assume \( f(v_{i+2}) = 2 \) and \( f(v_{i-2}) = 0 \). Since \( u_{i-2} \) needs to be double Roman dominated, \( f(u_{i-3}) = 3 \). Then, there exist a path \( P_1 \) of type 2 – 2 and a path \( P_2 \) of type 2 – 0 – 3. By Lemma 5b, \( \sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Case 3.2: \( f(u_i) = 0, f(v_i) = 3 \).

By Lemma 5a and Equation (2), we have \( f(v_{i+2}) = 0 \). Since \( u_{i+1} \) needs to be double Roman dominated, we have \( f(u_{i+2}) = 2 \). Then, there exist a path \( P_1 \) of type 2 – 2 and a path \( P_2 \) of type 2 – 0 – 3. By Lemma 5b, \( \sum_{v \in V(P_1) \cup V(P_2)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Therefore, the proof is complete. \( \square \)

**Lemma 7.** Let \( f \) be a \( \gamma_{DR} \)-function of \( P(n, 2) \) with weight \( \left[ \frac{n}{3} \right], n \in V_3^{000} \) and \( S = \{ x | x \in N_2(v), f(x) \geq 2 \} \), then \( 1 \leq |S| \leq 2 \).
Proof. We use the above discharging procedure for \( f \) on \( P(n, 2) \). By Lemma 4, we have \( |S| \geq 1 \). Now, suppose to the contrary that \( |S| \geq 3 \). By rules R1 and R2 and Equation (1), we have:

\[
\sum_{v \in V(P(n, 2))} (s'(v) - \frac{4}{5}) \geq \sum_{x \in N(v) \cup N_2(v)} (s'(x) - \frac{4}{5}) \geq 1,
\]

contradicting Equation (2).

\[\square\]

Lemma 8. If \( n \geq 5 \) and \( f \) is a \( \gamma_{DIR} \)-function of \( P(n, 2) \) with \( f(u_i) = 3 \) for some \( i \in [n] \), then \( w(f) \geq \left\lceil \frac{8n}{5} \right\rceil + 1 \).

Proof. Suppose to the contrary that there exists a \( \gamma_{DIR} \)-function \( f \) with \( w(f) = \left\lfloor \frac{8n}{5} \right\rfloor \) such that \( f(u_i) = 3 \) for some \( i \in [n] \). By Lemma 6, we have \( f(v_i) = f(u_{i+1}) = 0 \). Let \( S = \{ x \in N_2(v), f(x) \geq 2 \} \). By Lemma 7, we have \( |S| \in [1, 2] \). Therefore, we just need to consider the following two cases.

Case 1: \( |S| = 1 \).

We may w.l.o.g assume that \( \{ f(u_{i-2}), f(v_{i-1}), f(v_{i-2}) \} = \{ 0, 0, 2 \} \) or \( \{ 0, 0, 3 \} \) and \( f(v_{i+1}) = f(v_{i+2}) = f(v_{i+4}) = 3 \). Since \( u_{i+2}, v_{i+2} \) need to be double Roman dominated, we have \( f(u_{i+3}) = f(v_{i+4}) = 3 \). Then, suppose to the contrary that there exists a subgraph \( H \) of type \( 3 - 0 - 3 \) and a path \( P_i \) of type \( 2^+ - 0 - 3 - 0 - 2^+ \) or type \( 3 - 0 - 3 - 0 - 2^+ \). By Lemma 5a, we have \( \sum_{v \in V(P_i) \cup V(H)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Case 2: \( |S| = 2 \).

It is sufficient to consider the following cases.

Case 2.1: \( S \subseteq \{ v_{i-1}, v_{i-2}, u_{i-2} \} \) and \( f(v_{i+1}) = f(v_{i+2}) = f(u_{i+2}) = 0 \).

Since \( u_{i+2}, v_{i+2} \) need to be double Roman dominated, we have \( f(u_{i+3}) = f(v_{i+4}) = 3 \). Then, there exist a path \( P \) of type \( 3 - 0 - 3 - 0 - 3 - 0 - 2^+ \) and a subgraph \( H \) of type \( 2^+ - 0 - 3 - 0 - 2^+ \). By Lemma 5a, we have \( \sum_{v \in V(P) \cup V(H)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Case 2.2: \( S = \{ s_1, s_2 \}, s_1 \in \{ v_{i-1}, v_{i-2}, u_{i-2} \} \) and \( s_2 \in \{ v_{i+1}, v_{i+2}, u_{i+2} \} \).

First, we have \( f(v_{i+1}) = 0 \). Otherwise, we may without loss of generality assume that \( f(v_{i+1}) \geq 2 \). Since \( u_{i+2}, v_{i+2} \) need to be double Roman dominated, we have \( f(u_{i+3}) = f(v_{i+4}) = 3 \). Then, there exist a path \( P \) of type \( 3 - 0 - 3 - 0 - 3 - 0 - 2^+ \) and a subgraph \( H \) of type \( 2^+ - 0 - 3 - 0 - 2^+ \). By Lemma 5d, we have \( \sum_{v \in V(P) \cup V(H)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Then, since \( v_{i+1}, v_{i-1} \) need to be double Roman dominated, we have \( f(v_{i+3}) = f(v_{i-3}) = 3 \). By Lemma 6, we have \( f(u_{i+3}) = f(u_{i-3}) = 0 \). Since \( u_{i+2}, v_{i+2} \) need to be double Roman dominated, we have \( \{ f(u_{i-2}), f(v_{i-2}) \} = \{ 0, 3 \} \) and \( \{ f(u_{i+2}), f(v_{i+2}) \} = \{ 0, 3 \} \).

It is impossible that \( f(v_{i+2}) + f(u_{i+2}) = 3 \) and \( f(v_{i+2}) + f(u_{i+2}) = 3 \). Otherwise, there exists a path \( P \) of type \( 3 - 0 - 3 - 0 - 3 - 0 - 2^+ \). By Lemma 5a, we have \( \sum_{v \in V(P)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

Then, we may without loss of generality assume that \( f(u_{i+2}) = 2 \) and \( f(v_{i-2}) = 3 \). Then, there exists a path \( P \) of type \( 3 - 0 - 2 - 0 - 3 - 0 - 3 - 0 - 3 - 0 - 2^+ \). By Lemma 5a, we have \( \sum_{v \in V(P)} (s'(v) - 0.8) \geq 1 \), contradicting Equation (2).

\[\square\]

Lemma 9. If \( n \geq 5 \) and \( f \) is a \( \gamma_{DIR} \)-function of \( P(n, 2) \) with \( f(v_i) = 3 \) for some \( i \in [n] \), then \( w(f) \geq \left\lceil \frac{8n}{5} \right\rceil + 1 \).

Proof. Suppose to the contrary that there exists a \( \gamma_{DIR} \)-function \( f \) with \( w(f) = \left\lfloor \frac{8n}{5} \right\rfloor \) such that \( f(v_i) = 3 \) for some \( i \in [n] \). By Lemma 6, we have \( f(u_i) = f(v_{i+2}) = 0 \). Let \( S = \{ x \in N_2(v), f(x) \geq 2 \} \). By Lemma 7, we have \( 1 \leq |S| \leq 2 \), and we just need to consider the following two cases.
Case 1: $|S| = 1$.

We may without loss of generality assume that $\{f(u_{i-1}), f(u_{i-2}), f(v_{i-4})\} = \{0, 0, 2\}$ or $\{0, 0, 3\}$ and $f(u_{i+1}) = f(u_{i+2}) = f(v_{i+4}) = 0$. Since $u_{i+1}$ and $u_{i+2}$ need to be double Roman dominated, we have $f(v_{i+1}) = f(u_{i+3}) = 3$, contradicting Lemma 8.

Case 2: $|S| = 2$.

Now, it is sufficient to consider the following two cases.

Case 2.2: $S \subseteq \{u_{i-1}, u_{i-2}, v_{i-4}\}$ and $f(u_{i+1}) = f(u_{i+2}) = f(v_{i+4}) = 0$.

Since $u_{i+1}, u_{i+2}$ need to be double Roman dominated, we have $f(v_{i+1}) = f(u_{i+3}) = 3$, contradicting Lemma 8.

Case 2.2.1: $S = \{s_1, s_2\}$, where $s_1 \in \{u_{i-1}, u_{i-2}, v_{i-4}\}$ and $s_2 \in \{u_{i+1}, u_{i+2}, v_{i+4}\}$.

By Lemma 8, $f(u_k) \neq 3$ for each $k \in \{1, 2, \cdots, n\}$, and thus, $\{f(u_{i+1}), f(u_{i+2}), f(u_{i-2}), f(u_{i-1})\} = \{0, 2\}$.

Then, we have $f(v_{i+4}) = f(v_{i-4}) = 0$. Otherwise, $f(v_{i+4}) \neq 0$ or $f(v_{i-4}) \neq 0$. By symmetry, we may assume without loss of generality that $f(v_{i+4}) \neq 0$. Thus, we have $f(u_{i+1}) = f(u_{i+2}) = 0$. Since $u_{i+1}, u_{i+2}$ need to be double Roman dominated, we have $f(v_{i+1}) = f(u_{i+3}) = 3$, contradicting Lemma 8.

Now, it is sufficient to consider the following three cases.

Case 2.2.2.1: $f(u_{i+1}) = f(u_{i-1}) = 2$.

By Lemma 6, we have $f(u_{i+2}) = f(v_{i+1}) = 0$. Since $u_{i+2}$ needs to be double Roman dominated and by Lemma 8, we have $f(u_{i+3}) = 2$. Since $v_{i+1}$ needs to be double Roman dominated, we have $f(v_{i+1}) \geq 2$. Thus, there exists an edge $e \in E_f^{(22)}$, a contradiction with Lemma 6.

Case 2.2.2.2: $f(u_{i+2}) = f(u_{i-2}) = 2$.

By Lemma 6, we have $f(u_{i+3}) = f(u_{i-1}) = 0$. Since $u_{i+1}, u_{i-1}$ need to be double Roman dominated, we have $f(v_{i+1}) = 2$. Thus, there exists an edge $e \in E_f^{(22)}$, a contradiction with Lemma 6.

Case 2.2.2.3: $f(u_{i+1}) = f(u_{i-2}) = 2$.

By Lemma 6, we have $f(u_{i+3}) = f(v_{i+1}) = f(u_{i+2}) = 0$. Since $u_{i+2}$ needs to be double Roman dominated, we have $f(u_{i+3}) = 2$. By Lemma 6, we have $f(v_{i+3}) = f(u_{i+4}) = 0$. Since $v_{i+4}$ needs to be double Roman dominated and by Lemma 8, we have $f(u_{i+5}) = 2$. Since $v_{i+3}$ needs to be double Roman dominated, we have $f(v_{i+5}) \geq 2$. Thus, there exists an edge $e \in E_f^{(22)}$, a contradiction with Lemma 6. □

Lemma 10. Let $n \geq 5$ and $n \not\equiv 0 \pmod{5}$. If $f$ is a $\gamma_{DR}$-function of $P(n, 2)$, then $w(f) \geq \left\lceil \frac{8n}{5} \right\rceil + 1$.

Proof. Suppose to the contrary that $w(f) = \left\lceil \frac{8n}{5} \right\rceil$. By Lemmas 8 and 9, we have $|V_3| = 0$. Now, we have:

Claim 1. $|V_2 \cap N(v)| = 2$ for any $v \in V(P(n, 2))$ with $f(v) = 0$.

Proof. Suppose to the contrary that there exists a vertex $v \in V(P(n, 2))$ with $f(v) = 0$ and $|V_2 \cap N(v)| = 3$. We consider the following two cases.

Case 1: $v = u_i$ for some $i \in [n]$.  

Since \(|V_2 \cap N(v)| = 3\), we have \(f(u_{i-1}) = f(u_{i+1}) = f(v_i) = 2\). By Lemma 6, we have \(f(u_{i\pm 2}) = 0\), \(f(v_{i\pm 1}) = 0\) and \(f(v_{i\pm 2}) = 0\). Since \(v_{i+1}\) needs to be double Roman dominated, we have \(f(v_{i+3}) = 2\). Since \(u_{i+2}\) needs to be double Roman dominated, we have \(f(u_{i+3}) = 2\). Since \(v_{i+3}u_{i+3} \in E_{(2,2)}\), contradicting Lemma 6.

Case 2: \(v = v_i\) for some \(i \in [n]\).

Since \(|V_2 \cap N(v)| = 3\), we have \(f(v_{i-2}) = f(v_{i+2}) = f(u_i) = 2\). By Lemma 6, we have \(f(u_{i\pm 1}) = f(u_{i\pm 2}) = f(v_{i\pm 4}) = 0\). Since \(u_{i-1}\) needs to be double Roman dominated, we have \(f(v_{i-1}) = 2\). Since \(u_{i-1}\) needs to be double Roman dominated, we have \(f(v_{i-1}) = 2\). Since \(v_{i+1}v_{i-1} \in E_{(2,2)}\), contradicting Lemma 6. \(\square\)

We assume without loss of generality that \(f(u_i) = 2\). By Lemma 6, we have \(f(u_{i-1}) = 0\), \(f(v_i) = 0\) and \(f(u_{i+1}) = 0\). Since \(v_i\) needs to be double Roman dominated, we assume without loss of generality that \(f(v_{i-2}) = 2\). By Claim 1, we have \(f(v_{i+2}) = 0\). Since \(f(v_{i-2}) = 2\), together with Lemma 6, we have \(f(u_{i-2}) = 0\). Since \(u_{i-1}\) needs to be double Roman dominated, we have \(f(v_{i-1}) = 2\). Then, by Lemma 6, we have \(f(v_{i+1}) = 0\). Since \(v_{i+2}\) needs to be double Roman dominated, we have \(f(u_{i+2}) = 2\). That is to say, we have:

\[
f(B_i) = f \left( \begin{array}{cccc}
u_{i-2} & u_{i-1} & u_i & u_{i+1} & u_{i+2} \\
v_{i-2} & v_{i-1} & v_i & v_{i+1} & v_{i+2} \end{array} \right) = \begin{pmatrix}0 & 0 & 2 & 0 & 2 \\2 & 2 & 0 & 0 & 0 \end{pmatrix}.
\]

By repeatedly applying Claim 1 and Lemma 6, \(f(x)\) can be determined for each \(x \in B_{i+5}\), and we have \(f(B_i) = f(B_{i+5})\). It is straightforward to see that \(w(f) = \lceil \frac{8n}{5} \rceil\) only if \(n \equiv 0\) (mod 5), a contradiction. \(\square\)

3. Conclusions

By Lemma 1, Proposition 2 and Lemma 10, we have

**Theorem 1.** If \(n \geq 5\), then:

\[
\gamma_{dR}(P(n, 2)) = \begin{cases} 
\lfloor \frac{8n}{5} \rfloor, & n \equiv 0 \pmod{5} \\
\lfloor \frac{8n}{5} \rfloor + 1, & n \equiv 1, 2, 3, 4 \pmod{5}.
\end{cases}
\]

**Remark 1.** Beeler et al. [7] proposed the concept of the double Roman domination. They showed that \(2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)\). Moreover, they suggested to find double Roman graphs.

In [17], it was proven that:

**Theorem 2.** If \(n \geq 5\), then \(\gamma(P(n, 2)) = \lceil \frac{3n}{2} \rceil\).

Therefore, we have that \(P(n, 2)\) is not double Roman for all \(n \geq 5\).

In fact, there exist many double Roman graphs among Petersen graph \(P(n, k)\). For example, in [12], it was shown that \(P(n, 1)\) is a double Roman graph for any \(n \not\equiv 2 \pmod{4}\). Therefore, it is interesting to find other Petersen graphs that are double Roman.

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