Methods for Multiple Attribute Decision Making with Interval-Valued Pythagorean Fuzzy Information

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Abstract: Interval-valued Pythagorean fuzzy numbers (IVPFNs) can easily describe the incomplete and indeterminate information by degrees of membership and non-membership, and the Hamy mean (HM) operator and dual HM (DHM) operators are a good tool for dealing with multiple attribute decision making (MADM) problems because it can capture the interrelationship among the multi-input arguments. Motivated by the studies regarding the HM operator and dual HM operator, we expand the HM operator and dual HM (DMM) operator to process the interval-valued Pythagorean fuzzy numbers (IVPFNs) and then to solve the MADM problems. Firstly, we propose some HM and DHM operators with IVPFNs. Moreover, we present some new methods to solve MADM problems with the IVPFNs. Finally, an applicable example is given.

Keywords: multiple attribute decision making (MADM); Hamy mean (HM) operator; dual Hamy mean (DHM) operator; interval-valued Pythagorean fuzzy numbers (IVPFNs)

1. Introduction

Atanassov [1] gave the intuitionistic fuzzy set (IFS) based on the fuzzy set [2] such that their sum is not greater than one. After their existence, researchers have applied these theories in different disciplines [3–17] and found that they are more profitable to handle the uncertainties during the analysis. The above theories have been successfully defined, but in some cases, it is not possible to handle the situation by IFS. For instance, if a decision maker (DM) may take the membership degrees of any element as 0.8 and 0.5, then, clearly their sum is not less than one. Hence, under such cases, IFS can have some deficiencies. In order to resolve this, a Pythagorean fuzzy set (PFS) [18,19], an extension of IFSs, has emerged as a good tool for describing the indeterminacy in uncertain multiple attribute decision making (MADM). For this set, the condition of the sum of the degrees is replaced with their sum of squares is less than one; hence, PFS is more general than the IFS. Further, it is clear that $0.8^2 + 0.5^2 \leq 1$; hence, PFS stands for such cases. After this discovery, Zhang and Xu [20] presented the PFS TOPSIS for MADM. Zhang [21] presented a PFS similarity measure for solving MADM. Peng and Yang [22] proposed some fundamental properties for PFNs. Reformat and Yager [23] used the PFSs in solving the recommender system. Zeng et al. [24] proposed a hybrid method for Pythagorean fuzzy multiple-criteria decision making. Garg [25,26] proposed some generalized PFS aggregation operators based on Einstein Operations. Zhang [27] extended the PFS to the interval-valued PFSs (IVPFSs). Garg [28] presented some aggregation operators with interval-valued Pythagorean fuzzy numbers (IVPFNs). Also, a new accuracy function has been presented to rank the IVPFNs. However, a novel accuracy function [28], correlation coefficient [29], improved accuracy function [30], and improved score function [31] have been defined under the PFS and IVPFS, and have been used to solve the
MADM. Recently, some confidence operators with PFNs have been developed by Garg [32]. Other scholars also studied the MADM under the PFS or IVPFS [24,27,30,31,33–41].

Both Bonferroni mean (BM) operators [42–47] and the Heronian mean (HM) [48–53] operators consider the interrelationships of aggregated arguments. The Hamy mean (HM) [54] operator can consider interrelationships among any number of arguments; both BM operator and Maclaurin symmetric mean [55] are the special cases of HM operator. Qin and Liu [56] proposed some HM operators with 2-tuple linguistic information. Because IVPFNs can easily describe the fuzzy information, and the HM operator and dual HM (DMM) operator can capture interrelationships among any number of arguments, it is necessary to expand the HM and DHM operator to deal with the IVPFNs. The purpose of this paper is to propose some HM and DHM operators with IVPFNs, then to study some properties of these operators, and apply them to solve MADM problems with IVPFNs.

To achieve this goal, the rest of this paper is set out as follows. Section 2 introduces some basic definitions of IVPFSs. In Section 3, we propose some HM and DHM operators with IVPFNs. In Section 4, we propose two MADM methods for IVPFNs with the interval-valued Pythagorean fuzzy weighted Hamy mean (IVPFWHM) operator and interval-valued Pythagorean fuzzy weighted dual Hamy mean (IVPFWDHM) operator. In Section 5, an illustrative example is given. In Section 6, we give some conclusions of this study.

2. Basic Concepts

In this section, we introduce some fundamental concepts of IVPFSs and HM, which will be used in the next section. These concepts are based on a fixed set $X$.

2.1. Pythagorean Fuzzy Set (PFS)

**Definition 1 ([18,19]).** A PFS $P$ is defined as

$$P = \{ (x, \mu_P(x), \nu_P(x)) | x \in X \}$$

where the functions $\mu_P : X \to [0,1]$ and $\nu_P : X \to [0,1]$ define the degrees of membership and non-membership of the element $x \in X$ to $P$, such that for each $x$, the condition $(\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$, holds.

**Definition 2 ([33]).** The $T = (\mu, \nu)$ be called as Pythagorean fuzzy number (PFN) and defined the score and accuracy functions as $S(T) = \mu^2 - \nu^2$ and $H(T) = \mu^2 + \nu^2$. In order to compare two or more PFNs $T_1$ and $T_2$, a comparison law is defined as:

1. If $S(T_1) < S(T_2)$, then $T_1 < T_2$;
2. If $S(T_1) = S(T_2)$, then,
   - (i) If $H(T_1) = H(T_2)$, then $T_1 = T_2$;
   - (ii) If $H(T_1) < H(T_2)$, then $T_1 < T_2$.

**Example 1.** Let $t_1 = (0.7, 0.5), t_2 = (0.5, 0.1), t_3 = (0.4, 0.1)$ be three PFNs, according to Definitions 1 and 2, we get $S(t_1) = 0.7^2 - 0.5^2 = 0.2400$, $S(t_2) = 0.5^2 - 0.1^2 = 0.2400$, $S(t_3) = 0.4^2 - 0.1^2 = 0.1500$, $S(t_1) = S(t_2) > S(t_3)$, $H(t_1) = 0.7^2 + 0.5^2 = 0.7400, H(t_2) = 0.5^2 + 0.1^2 = 0.2600, S(t_1) > S(t_2) > S(t_3)$; therefore, $t_1 > t_2 > t_3$.

2.2. Interval-Valued Pythagorean Fuzzy Set (IVPFS)

Zhang [27] extended the PFS to the IVPFSs which is defined, as follows, over the fixed set $X$. 

**Definition 2.**
Definition 3 ([27]). An IVPFS $\tilde{b}$ is defined as:

$$\tilde{b} = \{ (x, (\mu_\tilde{b}(x), v_\tilde{b}(x))) | x \in X \}$$  \hspace{1cm} (2)

where $\mu_\tilde{b}(x) = [\mu^Z_\tilde{b}(x), \mu^Y_\tilde{b}(x)]$, $v_\tilde{b}(x) = [v^Z_\tilde{b}(x), v^Y_\tilde{b}(x)]$ are the interval numbers of $[0, 1]$ with the condition $0 \leq (\mu^Z_\tilde{b}(x))^2 + (\mu^Y_\tilde{b}(x))^2 \leq 1$, $\forall x \in X$. The pair $\tilde{a} = (\mu_\tilde{a}, v_\tilde{a})$ is called an IVPF number (IVPFN), where $\mu_\tilde{a}, v_\tilde{a} \subseteq [0, 1]$ and $(\mu^Z_\tilde{a})^2 + (\mu^Y_\tilde{a})^2 \leq 1$.

Definition 4 ([28]). For three IVPFNs $\tilde{b}_1 = ([\mu^Z_{b_1}, \mu^Y_{b_1}], [v^Z_{b_1}, v^Y_{b_1}])$, $\tilde{b}_2 = ([\mu^Z_{b_2}, \mu^Y_{b_2}], [v^Z_{b_2}, v^Y_{b_2}])$, and $\tilde{b} = ([\mu^Z_{b}, \mu^Y_{b}], [v^Z_{b}, v^Y_{b}])$, the basic operational laws are defined as follows:

$$\tilde{b}_1 \oplus \tilde{b}_2 = \left( \sqrt{\left(\frac{\mu^Z_{b_1}}{\mu^Y_{b_1}} \right)^2 + \left(\frac{\mu^Z_{b_2}}{\mu^Y_{b_2}} \right)^2 - \left(\frac{\mu^Z_{b_1}}{\mu^Y_{b_1}} \right)^2 \left(\frac{\mu^Z_{b_2}}{\mu^Y_{b_2}} \right)^2}, \left[\frac{v^Z_{b_1}}{v^Y_{b_1}}, \frac{v^Z_{b_2}}{v^Y_{b_2}}\right] \right);$$

$$\tilde{b}_1 \odot \tilde{b}_2 = \left( \frac{\mu^Z_{b_1} \mu^Z_{b_2} \mu^Y_{b_1} \mu^Y_{b_2}}{\mu^Y_{b_1} \mu^Y_{b_2}}, \frac{\mu^Y_{b_1} \mu^Y_{b_2} - \left(\frac{\mu^Z_{b_1}}{\mu^Y_{b_1}} \right)^2 \left(\frac{\mu^Z_{b_2}}{\mu^Y_{b_2}} \right)^2}{\mu^Y_{b_1} \mu^Y_{b_2}} \right);$$

$$\tilde{b}_1 = \left( \frac{\mu^Z_{b_1}}{\mu^Y_{b_1}}, \frac{\mu^Y_{b_1}}{\mu^Y_{b_1}} \right);$$

$$\tilde{b}^k = \left( \left[\frac{\mu^Z_{b}}{\mu^Y_{b}}\right]^k, \left[\frac{\mu^Y_{b}}{\mu^Y_{b}}\right]^k \right);$$

Example 2. Suppose that $t_1 = ([0.3, 0.6], [0.1, 0.4])$, $t_2 = ([0.3, 0.7], [0.2, 0.4])$, and $t_3 = ([0.5, 0.7], [0.3, 0.5])$ be three IVPFNs, and $k = 0.5$, then we have:

$$t_1 \oplus t_2 = (\sqrt{0.3^2 + 0.3^2 - 0.3^2 \times 0.3^2}, \sqrt{0.6^2 + 0.7^2 - 0.6^2 \times 0.7^2}, [0.1 \times 0.2, 0.4 \times 0.4])$$

$$= ([0.4146, 0.8207], [0.0200, 0.1600])$$

$$t_1 \otimes t_2 = (\sqrt{0.3^2 \times 0.6 \times 0.7}, [0.1^2 + 0.2^2 - 0.1^2 \times 0.2^2, [0.4 + 0.4^2 - 0.4^2 \times 0.4^2])$$

$$= ([0.0900, 0.4200], [0.2227, 0.5426])$$

$$kt_3 = (\sqrt{1 - (1 - 0.5)0.5, [0.3^{0.5}, 0.5^{0.5}]})$$

$$= ([0.5412, 0.6725], [0.5477, 0.7071])$$

$$(t_3)^k = ([0.5^{0.5}, 0.5^{0.5}], [\sqrt{1 - (1 - 0.3)0.5}, \sqrt{1 - (1 - 0.5)0.5}]$$

$$= ([0.7071, 0.8367], [0.4042, 0.5412])$$

$$t_1^c = ([0.3, 0.5], [0.5, 0.7])$$

Based on the Definition 4, Garg [28] derived the following properties easily.
Theorem 1. Let \( \tilde{b}_1 = \left( \left[ \mu_{\tilde{b}_1}, \tilde{v}_{\tilde{b}_1} \right], \left[ \tilde{v}_{\tilde{b}_1}, \tilde{\mu}_{\tilde{b}_1} \right] \right) \) and \( \tilde{b}_2 = \left( \left[ \mu_{\tilde{b}_2}, \tilde{v}_{\tilde{b}_2} \right], \left[ \tilde{v}_{\tilde{b}_2}, \tilde{\mu}_{\tilde{b}_2} \right] \right) \) be two IVPFNs, and \( k, k_1, k_2 > 0, \) be three real numbers, then:

\[
\begin{align*}
\tilde{b}_1 \oplus \tilde{b}_2 &= \tilde{b}_2 \oplus \tilde{b}_1; \\
\tilde{b}_1 \otimes \tilde{b}_2 &= \tilde{b}_2 \otimes \tilde{b}_1; \\
k(\tilde{b}_1 + \tilde{b}_2) &= k\tilde{b}_1 + k\tilde{b}_2; \\
(\tilde{b}_1 \otimes \tilde{b}_2)^k &= (\tilde{b}_1)^k \otimes (\tilde{b}_2)^k; \\
k_1 \tilde{b}_1 + k_2 \tilde{b}_2 &= (k_1 + k_2)\tilde{b}_1; \\
\left( \tilde{b}_1 \right)^{k_1} \left( \tilde{b}_2 \right)^{k_2} &= (\tilde{b}_1)^{k_1} \cdot (\tilde{b}_2)^{k_2}.
\end{align*}
\]

Definition 5. For an IVPFN \( \tilde{b} = \left( \left[ \mu_{\tilde{b}}, \tilde{v}_{\tilde{b}} \right], \left[ \tilde{v}_{\tilde{b}}, \tilde{\mu}_{\tilde{b}} \right] \right), \) the score and accuracy functions of it are defined as

\[
S(\tilde{b}) = \frac{1}{4} \left[ \left( 1 + \left( \mu_{\tilde{b}} \right)^2 - \left( \tilde{v}_{\tilde{b}} \right)^2 \right) + \left( 1 + \left( \mu_{\tilde{b}} \right)^2 - \left( \tilde{\mu}_{\tilde{b}} \right)^2 \right) \right]
\]

and

\[
H(\tilde{b}) = \frac{\left( \mu_{\tilde{b}} \right)^2 + \left( \tilde{v}_{\tilde{b}} \right)^2 + \left( \tilde{\mu}_{\tilde{b}} \right)^2}{2}.
\]

respectively. Further, in order to compare two different IVPFNs \( \tilde{b}_1 \) and \( \tilde{b}_2, \) an order relation is defined as:

1. if \( S(\tilde{b}_1) < S(\tilde{b}_2), \) then \( \tilde{b}_1 < \tilde{b}_2. \)
2. if \( S(\tilde{b}_1) = S(\tilde{b}_2), \) then,

   i. if \( H(\tilde{b}_1) = H(\tilde{b}_2), \) then \( \tilde{b}_1 = \tilde{b}_2. \)
   
   ii. if \( H(\tilde{b}_1) < H(\tilde{b}_2), \) then \( \tilde{b}_1 < \tilde{b}_2. \)

Example 3. Let \( t_1 = (0.4, 0.7), t_2 = (0.5, 0.5), \) and \( t_3 = (0.3, 0.4), \) be three IVPFNs, according to Definition 5, we get:

\[
\begin{align*}
S(t_1) &= (1 + 0.4^2 - 0^2) + (1 + 0.7^2 - 0.5^2) = 0.6000, \\
S(t_2) &= (1 + 0.5^2 - 0.1^2) + (1 + 0.5^2 - 0.5^2) = 0.6000, \\
S(t_3) &= (1 + 0.3^2 - 0.2^2) + (1 + 0.4^2 - 0.5^2) = 0.4900, \\
\end{align*}
\]

\[
\begin{align*}
H(t_1) &= \frac{0.4^2 + 0.7^2 + 0.5^2}{2} = 0.4500, \\
H(t_2) &= \frac{0.5^2 + 0.5^2 + 0.1^2 + 0.3^2}{2} = 0.3000 \\
S(t_1) > S(t_2) > S(t_3), \text{ therefore } t_1 > t_2 > t_3.
\end{align*}
\]

2.3. HM Operator

Definition 6 ([54]). The HM operator is defined as the following:

\[
\text{HM}^{(x)}(b_1, b_2, \ldots, b_k) = \frac{\sum_{1 \leq i_1 < \ldots < i_x \leq k} \binom{x}{i} b_{i_1} \cdots b_{i_x}}{\binom{k}{x}}
\]
where \( x \) is a parameter and \( x \) are \( x \) integer values taken from the set \( \{1, 2, \cdots, k\} \) of \( k \) integer values, and \( C_k^x \) denotes the binomial coefficient and \( C_k^x = \frac{k!}{x!(k-x)!} \).

The properties of the operator are shown as follows:

(i) when \( b_i = b(i = 1, 2, \cdots, k) \), \( \text{HM}^{(x)}(b_1, b_2, \cdots, b_k) = b \);

(ii) when \( b_i \leq \pi_i(i = 1, 2, \cdots, k) \), \( \text{HM}^{(x)}(b_1, b_2, \cdots, b_k) \leq \text{HM}^{(x)}(\pi_1, \pi_2, \cdots, \pi_k) \);

(iii) when \( \min\{b_i\} \leq \text{HM}^{(x)}(b_1, b_2, \cdots, b_k) \leq \max\{b_i\} \).

Two particular cases of the HM operator are given as follows:

(1) when \( x = 1 \), \( \text{HM}^{(k)}(b_1, b_2, \cdots, b_k) = \frac{1}{k} \sum b_i \), it becomes the arithmetic mean operator.

(2) when \( x = k \), \( \text{HM}^{(k)}(b_1, b_2, \cdots, b_k) = \left( \prod_{i=1}^{k} b_i \right)^{\frac{1}{x}} \), it becomes the arithmetic mean operator. Which is the arithmetic averaging operator.

3. Certain HM and DHM Operators with IVPFNs

In this section, we shall develop certain Hamy mean operators with IVPFNs.

3.1. IVPFHM Operator

The HM operator has usually been utilized in the situation of an interaction relationship. Next, we extend the HM operator to IVPFSS. From Definitions 4 and 6, we can obtain:

Definition 7. Let \( \tilde{b}_i = \left( [\mu_i^x, \nu_i^x], [\varpi_i^x, \psi_i^x] \right) \) \((i = 1, 2, \cdots, k)\) be a set of IVPFNs, then the interval-valued Pythagorean Fuzzy Hamy mean (IVPFHM) operator is defined as:

\[
\text{IVPFHM}^{(x)} \left( \tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k \right) = \frac{1}{\sum_{i=1}^{k} b_i} \left( \prod_{j=1}^{k} b_j \right)^{\frac{1}{x}} 
\]

where \( x \) is a parameter and \( x = 1, 2, \cdots, k \), \( i_1, i_2, \cdots, i_k \) are \( x \) integer values taken from the set \( \{1, 2, \cdots, k\} \) of \( k \) integer values, and \( C_k^x \) denotes the binomial coefficient and \( C_k^x = \frac{k!}{x!(k-x)!} \).

Based on the operations of the IVPFN described, we can obtain Theorem 2.

Theorem 2. Let \( \tilde{b}_i = \left( [\mu_i^x, \nu_i^x], [\varpi_i^x, \psi_i^x] \right) \) \((i = 1, 2, \cdots, k)\) be a set of IVPFNs, then their aggregated result by utilizing IVPFHM operator is also an IVPFN, and

\[
\text{IVPFHM}^{(x)} \left( \tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k \right) = \frac{1}{\sum_{i=1}^{k} b_i} \left( \prod_{j=1}^{k} b_j \right)^{\frac{1}{x}} 
\]

Proof.

\[
\prod_{j=1}^{k} \tilde{b}_j = \left( \prod_{i=1}^{x} (\mu_i^x), \prod_{i=1}^{x} (\nu_i^x) \right), \left[ \sqrt{1 - \prod_{j=1}^{x} \left(1 - (v_j^x)^2\right)}, \sqrt{1 - \prod_{j=1}^{x} \left(1 - (v_j^x)^2\right)} \right]
\]
\[
\left( \sum_{j=1}^{n} b_j \right) \left( \prod_{j=1}^{n} \mu_j \right) \left( \prod_{j=1}^{n} v_j \right) = \left( \left( \prod_{j=1}^{n} \mu_j^\frac{1}{2} \right) \left( \prod_{j=1}^{n} v_j^\frac{1}{2} \right) \right) \left( \left( \prod_{j=1}^{n} \left( 1 - (v_j^2) \right) \right) \left( \prod_{j=1}^{n} \left( 1 - (\nu_j^2) \right) \right) \right)
\]

Therefore,
\[
\left( \sum_{j=1}^{n} b_j \right) \left( \prod_{j=1}^{n} \mu_j \right) \left( \prod_{j=1}^{n} v_j \right) = \left( \left( \prod_{j=1}^{n} \mu_j^\frac{1}{2} \right) \left( \prod_{j=1}^{n} v_j^\frac{1}{2} \right) \right) \left( \left( \prod_{j=1}^{n} \left( 1 - (v_j^2) \right) \right) \left( \prod_{j=1}^{n} \left( 1 - (\nu_j^2) \right) \right) \right)
\]

And then, we can know:
\[
0 \leq \left( \prod_{j=1}^{n} \left( 1 - \left( \prod_{j=1}^{n} \mu_j^\frac{1}{2} \left( 1 - (v_j^2) \right) \right) \right) \right) \frac{1}{\sqrt{n}} \leq 1
\]

We can obtain \((\mu_p(x))^2 + (\nu_p(x))^2 \leq 1\) from the definition of IVPFS, so
\[
0 \leq 1 - \left( \prod_{j=1}^{n} \left( 1 - (\nu_j^2) \right) \right) \frac{1}{\sqrt{n}} \leq 1
\]

We complete the proof. \(\Box\)

In the following, we give some properties of the IVPFMM operator.

**Property 1. (Idempotency.)** Let \(\tilde{b} = (\mu^Y, \nu^Y)\), then
\[
\text{IVPFHM}(^s)\left(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\right) = \tilde{b}
\]
Proof.

\[
\text{IVPFHM}^{(3)}(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_k) = \left( \left\{ \left( 1 - \frac{1}{\prod_{1 \leq h < \cdots < l \leq k} \left( 1 - \left( \prod_{j=1}^{i} \mu_j^l \right)^2 \right) \right)^{\frac{1}{y_i}} \right\} \right) \left( \left\{ \left( 1 - \frac{1}{\prod_{1 \leq h < \cdots < l \leq k} \left( 1 - \left( \prod_{j=1}^{i} \nu_j^l \right)^2 \right) \right)^{\frac{1}{y_i}} \right\} \right) \right)
\]  

\[
= \left( \left\{ \left( 1 - \frac{1}{\prod_{1 \leq h < \cdots < l \leq k} \left( 1 - \left( \prod_{j=1}^{i} \mu_j^l \right)^2 \right) \right)^{\frac{1}{y_i}} \right\} \right) \left( \left\{ \left( 1 - \frac{1}{\prod_{1 \leq h < \cdots < l \leq k} \left( 1 - \left( \prod_{j=1}^{i} \nu_j^l \right)^2 \right) \right)^{\frac{1}{y_i}} \right\} \right)
\]  

\[
= \left( \left\{ \sqrt{1 - \left( \prod_{j=1}^{i} \mu_j^l \right)^2} \right\} \right) \left( \left\{ \sqrt{1 - \left( \prod_{j=1}^{i} \nu_j^l \right)^2} \right\} \right) \right)
\]  

\[
\text{Property 2. (Monotonicity.) Let } \hat{b}_i = \left( \left[ \mu_i^2, \mu_i^\gamma \right], \left[ v_i^2, v_i^\gamma \right] \right) \text{ and } \tilde{c}_i = \left( \left[ \mu_i^{\tilde{c}}, \mu_i^{\tilde{c}^\gamma} \right], \left[ v_i^{\tilde{c}}, v_i^{\tilde{c}^\gamma} \right] \right) \text{ for } i = 1, 2, \ldots, k \text{ be two sets of IVPFNs, If } \left( \mu_i^2 \right)^2 + \left( \mu_i^\gamma \right)^2 \leq \left( \mu_i^{\tilde{c}} \right)^2 + \left( \mu_i^{\tilde{c}^\gamma} \right)^2 \text{ and } \left( v_i^2 \right)^2 + \left( v_i^\gamma \right)^2 \geq \left( v_i^{\tilde{c}} \right)^2 + \left( v_i^{\tilde{c}^\gamma} \right)^2, \text{ then,}
\]

\[
\text{IVPFHM}^{(3)}(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_k) \leq \text{IVPFHM}^{(3)}(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_k)
\]  

Proof.

\[
\left( \prod_{j=1}^{i} \mu_j^l \right)^{\frac{1}{y_i}} \leq \left( \prod_{j=1}^{i} \mu_j^{\tilde{c}_i} \right)^{\frac{1}{y_i}}
\]  

\[
\prod_{1 \leq h < \cdots < l \leq k} \left( 1 - \left( \prod_{j=1}^{i} \mu_j^l \right)^2 \right)^{\frac{1}{y_i}} \leq \prod_{1 \leq h < \cdots < l \leq k} \left( 1 - \left( \prod_{j=1}^{i} \mu_j^{\tilde{c}_i} \right)^2 \right)^{\frac{1}{y_i}}
\]  

Therefore,

\[
\sqrt{1 - \left( \prod_{j=1}^{i} \mu_j^l \right)^2} \leq \sqrt{1 - \left( \prod_{j=1}^{i} \mu_j^{\tilde{c}_i} \right)^2}
\]  

Similarly, we also can obtain

\[
\left( \prod_{1 \leq h < \cdots < l \leq k} \left( 1 - \left( \prod_{j=1}^{i} \nu_j^l \right)^2 \right) \right)^{\frac{1}{y_i}} \geq \left( \prod_{1 \leq h < \cdots < l \leq k} \left( 1 - \left( \prod_{j=1}^{i} \nu_j^{\tilde{c}_i} \right)^2 \right) \right)^{\frac{1}{y_i}}
\]  

And

\[
\sqrt{1 - \left( \prod_{j=1}^{i} \nu_j^l \right)^2} \leq \sqrt{1 - \left( \prod_{j=1}^{i} \nu_j^{\tilde{c}_i} \right)^2}
\]
If \( \text{four IVPFNs} \). (suppose \( x = \mathsf{Mathematics} \))

\[ \text{Property 3.} \]

\[ (\mu_{ij}^z)^2 < (\mu_{ij}^x)^2 \text{ and } (v_{ij}^z)^2 > (v_{ij}^x)^2 \text{ then} \]

\[ \text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_k) < \text{IVPFHM}^{(x)}(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_k); \]

\[ \text{if } (\mu_{ij}^z)^2 + (\mu_{ij}^y)^2 = (\mu_{ij}^x)^2 \text{ and } (v_{ij}^z)^2 + (v_{ij}^y)^2 = (v_{ij}^x)^2 \text{ then} \]

\[ \text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_k) < \text{IVPFHM}^{(x)}(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_k); \]

\[ \text{if } (\mu_{ij}^z)^2 + (\mu_{ij}^y)^2 = (\mu_{ij}^x)^2 \text{ and } (v_{ij}^z)^2 + (v_{ij}^y)^2 = (v_{ij}^x)^2 \text{ then} \]

\[ \text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_k) = \text{IVPFHM}^{(x)}(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_k). \]

\[ \square \]

\[ \text{Property 3. (Boundedness.) Let } \tilde{b}_i = ([\mu_{ij}^z, \mu_{ij}^y], [v_{ij}^z, v_{ij}^y]), \tilde{b}^+ = ([\mu_{\text{maxi}}, \mu_{\text{maxi}}^y], [v_{\text{maxi}}, v_{\text{maxi}}^y]) \text{ and} \]

\[ \begin{align*}
\tilde{b}^- &= ([\mu_{\text{mini}}, \mu_{\text{mini}}^y], [v_{\text{mini}}, v_{\text{mini}}^y]) (i = 1, 2, \ldots, k) \text{ be a set of IVPFNs. According to the process of} \\
&\text{property of monotonicity and idempotency, it is easy to obtain that,} \\
\end{align*} \]

\[ \tilde{b}_i^- < \text{IVPFHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_k) < \tilde{b}_i^+ \]

\[ \text{(22)} \]

\[ \text{Example 4.} \]

\[ \text{Let } \tilde{b}_1 = ([0.1, 0.4], [0.2, 0.5]), \tilde{b}_2 = ([0.4, 0.6], [0.3, 0.4]), \tilde{b}_3 = ([0.5, 0.7], [0.2, 0.3]), \]

\[ \tilde{b}_4 = ([0.3, 0.5], [0.2, 0.4]) \text{ be four IVPFNs. Then we use the proposed IVPFHJ operator to aggregate four IVPFNs. (suppose } x = 2). \]

\[ \text{IVPFHM}^{(y)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) \]

\[ = \left( \left( \prod_{1 \leq i < j \leq 4} \left( 1 - \left( \prod_{1 \leq j \leq 4} \left( \frac{1}{2} \left( \frac{1 - \left( \prod_{j > 1} \mu_{ij}^z \right)^2}{2} \right) \right) \right) \right)^{\frac{1}{2}} \right) \right) \]

\[ \left[ \left( \prod_{1 \leq i < j \leq 4} \left( 1 - \left( \prod_{j > 1} \mu_{ij}^y \right)^2 \right) \right) \right]^{\frac{1}{2}} \]
Finally, we get $IVPFHM^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) = ([0.3171, 0.5497], [0.1317, 0.2401])$.

3.2. IVPFWHM Operator

It can be seen that the IVPFHM operator doesn’t consider the importance of the aggregated arguments. To overcome the limitation of IVPFHM, we develop the interval-valued Pythagorean fuzzy weighted HM (IVPFWHM) operator as follows.

**Definition 8.** Let $\tilde{b}_i = ([\mu^i, \nu^i], [\sigma^i, \tau^i])$ $(i = 1, 2, \cdots, k)$ be a set of IVPFNs with weight vector being $\omega = (\omega_1, \omega_2, \cdots, \omega_k)^T$, $\omega_i \in [0, 1]$ and $\sum_{k=1}^{k} \omega_i = 1$, then the IVPFWHM operator is given as:

$$
IVPFWBM^{(x)}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k) = \left\{ \begin{array}{ll}
\text{Joining} \\
\bigoplus_{i=1}^{k} \frac{1}{\omega_i} \left( \sum_{j=1}^{m} \omega_j \left( \frac{\tilde{b}_j}{\sqrt{\omega_j}} \right) \right)^{\frac{1}{2}} & (1 \leq x < k) \\
\text{Joining} & (x = k)
\end{array} \right. 
$$

(23)

Based on the operations of the IVPFN described, we can obtain Theorem 3.

**Theorem 3.** Let $\tilde{b}_i = ([\mu^i, \nu^i], [\sigma^i, \tau^i])$ $(i = 1, 2, \cdots, k)$ be a collection of IVPFNs, then their aggregated result of IVPFWHM operator is also an IVPFN, and

$$
IVPFWBM^{(x)}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k) = \frac{\left(\prod_{i=1}^{k} \omega_i \left( \sum_{j=1}^{m} \omega_j \left( \frac{\tilde{b}_j}{\sqrt{\omega_j}} \right) \right) \right)^{\frac{1}{2}}}{\sum_{i=1}^{k} \omega_i \left( \sum_{j=1}^{m} \omega_j \left( \frac{\tilde{b}_j}{\sqrt{\omega_j}} \right) \right)^{\frac{1}{2}}}
$$

(24)
or

\[
\text{IVPFW}(x; \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_k) = \frac{1}{\hat{b}_1^{1-\omega_1}} \left[ \prod_{i=1}^{k} (\hat{\mu}^2) \prod_{i=1}^{k} (\mu^2) \right],
\]

\[
\frac{1}{\sqrt{1 - \prod_{i=1}^{k} (1 - (\nu^2)^2) \prod_{i=1}^{k} (1 - (\nu^2)^2)}},
\]

\[
(x = k)
\]

Proof.

\[
\left( \frac{\hat{x}}{\hat{b}_i} \right)^{\frac{1}{\omega_i}} = \left[ \left( \prod_{j=1}^{k} \hat{\mu}^2 \right)^{\frac{1}{\omega_i}}, \left( \prod_{j=1}^{k} \mu^2 \right)^{\frac{1}{\omega_i}} \right], \quad \left[ \sqrt{1 - \prod_{i=1}^{k} (1 - (\nu^2)^2)}, \sqrt{1 - \prod_{i=1}^{k} (1 - (\nu^2)^2)} \right]
\]

(26)

Thereafter,

\[
\left( 1 - \frac{\hat{x}}{\prod_{j=1}^{k} \omega_j} \right) \left( \frac{\hat{x}}{\hat{b}_i} \right)^{\frac{1}{\omega_i}} \left[ \left( \prod_{j=1}^{k} \hat{\mu}^2 \right)^{\frac{1}{\omega_i}}, \left( \prod_{j=1}^{k} \mu^2 \right)^{\frac{1}{\omega_i}} \right], \quad \left[ \sqrt{1 - \prod_{i=1}^{k} (1 - (\nu^2)^2)}, \sqrt{1 - \prod_{i=1}^{k} (1 - (\nu^2)^2)} \right]
\]

(27)

Moreover,

\[
\left( \left( \prod_{i=1}^{k} \hat{\mu}^2 \right)^{\frac{1}{\omega_i}}, \left( \prod_{i=1}^{k} \mu^2 \right)^{\frac{1}{\omega_i}} \right), \quad \left[ \sqrt{1 - \prod_{i=1}^{k} (1 - (\nu^2)^2)}, \sqrt{1 - \prod_{i=1}^{k} (1 - (\nu^2)^2)} \right]
\]

(28)

Therefore,

\[
\left( \prod_{i=1}^{k} \hat{\mu}^2 \right)^{\frac{1}{\omega_i}}, \left( \prod_{i=1}^{k} \mu^2 \right)^{\frac{1}{\omega_i}} \left[ \sqrt{1 - \prod_{i=1}^{k} (1 - (\nu^2)^2)}, \sqrt{1 - \prod_{i=1}^{k} (1 - (\nu^2)^2)} \right]
\]

(29)

For the second case, when \((x = k)\), we get

\[
\frac{1}{\hat{b}_i^{1-\omega_i}} = \left[ (\hat{\mu}^2)^{1-\omega_i}, (\mu^2)^{1-\omega_i} \right], \quad \left[ \sqrt{1 - (1 - (\nu^2)^2)^{1-\omega_i}}, \sqrt{1 - (1 - (\nu^2)^2)^{1-\omega_i}} \right]
\]

(30)
Then,

\[
\prod_{i=1}^{k} (\mu_i^{1-\omega_i}) = \left( \prod_{i=1}^{k} (\mu_i^{1-\omega_i}) \right) \prod_{i=1}^{k} (\mu_i^{1-\omega_i}) = \left( \prod_{i=1}^{k} (1-(\nu_i^{2})^2)^{\frac{1}{m_i-1}} \prod_{i=1}^{k} (1-(\nu_i^{2})^2)^{\frac{1}{m_i-1}} \right)
\]

(31)

and we can easily obtain,

\[
0 \leq 1 - \left( \prod_{1 \leq i < \cdots < k \leq k} (1-(\nu_i^{2})^2)^{\frac{1}{m_i-1}} \right)^{2} \leq 1
\]

(32)

\[
0 \leq \left( \prod_{1 \leq i < \cdots < k \leq k} (1-(\nu_i^{2})^2)^{\frac{1}{m_i-1}} \right)^{2} \leq 1
\]

(33)

and,

\[
0 \leq \left( \prod_{i=1}^{k} (\mu_i^{1-\omega_i}) \right) \leq 1
\]

(34)

\[
0 \leq \left( 1 - \prod_{i=1}^{k} (1-(\nu_i^{2})^2)^{\frac{1}{m_i-1}} \right)^{2} \leq 1
\]

(35)

Therefore,

\[
\left( \prod_{1 \leq i < \cdots < k \leq k} (1-(\nu_i^{2})^2)^{\frac{1}{m_i-1}} \right)^{2} \leq 1
\]

(36)

For the second case, \( x = k \), we can easily prove that it is kept. Therefore, the aggregation result produced by Definition 8 is still an IVPFN. Next, we shall deduce some desirable properties of the IVPFWHM operator.

In the following, we give some properties of the IVPFWHM operator.

**Property 4. (Idempotency.)** Let \( \tilde{b}_i = \left( [\mu_i^{1-\omega_i}, \mu_i^{1-\omega_i}], [\nu_i^{2}, \nu_i^{2}] \right) = ([\mu^{1-\omega_i}, \mu^{1-\omega_i}], [\nu^{2}, \nu^{2}]) = \tilde{b} \), and weight vector meets \( \omega = (\omega_1, \omega_2, \cdots, \omega_k)^{T} \), \( \omega_i \in [0, 1] \) and \( \sum_{i=1}^{k} \omega_i = 1 \), then,
IVPFWMH(x)(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_k) = \tilde{b} \quad (37)

Property 5. (Monotonicity.) Let \( \tilde{b}_i = \left[ \left( \mu_{i}^{x}, \mu_{i}^{y}, v_{i}^{x}, v_{i}^{y} \right) \right] \) and \( \tilde{c}_i = \left[ \left( \mu_{i}^{x}, \mu_{i}^{y}, v_{i}^{x}, v_{i}^{y} \right) \right] \) \((i = 1, 2, \ldots, k)\) be two sets of IVPFNs, and weight vector meets \( \omega = (\omega_1, \omega_2, \ldots, \omega_k)^T, \omega_i \in [0, 1] \) and \( \sum_{i=1}^{k} \omega_i = 1. \) If \((\mu_{i}^{x})^2 + (\mu_{i}^{y})^2 \leq (\mu_{j}^{x})^2 + (\mu_{j}^{y})^2 \) and \((v_{i}^{x})^2 + (v_{i}^{y})^2 \geq (v_{j}^{x})^2 + (v_{j}^{y})^2 \) then,

\[
IVPFWMH_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_k) \leq IVPFWHM_{\omega}^{(x)}(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_k)
\quad (38)
\]

Property 6. (Boundedness.) Let \( \tilde{b}_i = \left[ \left( \mu_{i}^{x}, \mu_{i}^{y}, v_{i}^{x}, v_{i}^{y} \right) \right], \tilde{b}^+ = \left[ \left( \mu_{i}^{x}, \mu_{i}^{y}, v_{i}^{x}, v_{i}^{y} \right) \right] \) and \( \tilde{b} = \left[ \left( \mu_{i}^{x}, \mu_{i}^{y}, v_{i}^{x}, v_{i}^{y} \right) \right] \) \((i = 1, 2, \ldots, k)\) be a set of IVPFNs with weight vector being \( \omega = (\omega_1, \omega_2, \ldots, \omega_k)^T, \omega_i \in [0, 1], \sum_{i=1}^{k} \omega_i = 1, \) because of Property 5, then,

\[
\tilde{b}_i < IVPFWHM_{\omega}^{(x)}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_k) < \tilde{b}^+_i
\quad (39)
\]

Example 5. Let \( \tilde{b}_1 = ([0.1, 0.4], [0.2, 0.3]), \tilde{b}_2 = ([0.2, 0.4], [0.3, 0.5]), \tilde{b}_3 = ([0.3, 0.4], [0.2, 0.3]), \tilde{b}_4 = ([0.4, 0.7], [0.1, 0.3]) \) be four IVPFNs, the weighting vector of attributes be \( \omega = [0.2, 0.1, 0.3, 0.4]. \) Then we use the proposed IVPFWHM operator to aggregate four IVPFNs, (suppose \( x = 2 \).

\[
IVPFWMH_{\omega}^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) = \left[ \left( \mu_{i}^{x}, \mu_{i}^{y}, v_{i}^{x}, v_{i}^{y} \right) \right]
\]

= \( [0.2299, 0.4585], [0.5479, 0.6893] \)

Finally, we get \( IVPFWHM_{\omega}^{(2)}(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) = ([0.2299, 0.4585], [0.5479, 0.6893]) \).
3.3. IVPFDHM Operator

Wu et al. [32] proposed the dual Hamy mean (DHM) operator.

Definition 9. The DHM operator is defined as follows:

\[
\text{DHM}^{(x)}(b_1, b_2, \ldots , b_k) = \left( \prod_{1 \leq i_1 < \cdots < i_x \leq n} \left( \frac{\sum_{j=1}^{x} b_{i_j}}{x} \right) \right)^{\frac{1}{x}}
\]

(40)

where \( x \) is a parameter and \( x = 1, 2, \ldots , k \), \( i_1, i_2, \ldots , i_x \) are \( x \) integer values taken from the set \( \{1,2,\ldots,k\} \) of \( k \) integer values, and \( C_n^x \) denotes the binomial coefficient and \( C_n^x = \frac{k!}{x!(k-x)!} \).

In the following, we proposed the interval-valued Pythagorean fuzzy DHM (IVPFDHM) operator for IVPFNs.

Definition 10. Let \( \tilde{b}_i = ([\mu_i^x, \mu_i^y], [\nu_i^x, \nu_i^y]) \) \((i = 1, 2, \ldots, k)\) be a set of IVPFNs, then the IVPFDHM operator is:

\[
\text{IVPFDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \ldots , \tilde{b}_k) = \left( \prod_{1 \leq i_1 < \cdots < i_x \leq k} \left( \frac{\sum_{j=1}^{x} \tilde{b}_{i_j}}{x} \right) \right)^{\frac{1}{x}}
\]

(41)

where \( x \) is a parameter and \( x = 1, 2, \ldots , k \), \( i_1, i_2, \ldots , i_x \) are \( x \) integer values taken from the set \( \{1,2,\ldots,k\} \) of \( k \) integer values, and \( C_n^x \) denotes the binomial coefficient and \( C_n^x = \frac{k!}{x!(k-x)!} \).

Based on the operations of the IVPFN described, we can obtain Theorem 4.

Theorem 4. Let \( \tilde{b}_i = ([\mu_i^x, \mu_i^y], [\nu_i^x, \nu_i^y]) \) \((i = 1, 2, \ldots, k)\) be a collection of IVPFNs, then the aggregated value of IVPFDHM operator is also an IVPFN, and

\[
\text{IVPFDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \ldots , \tilde{b}_k) = \left( \prod_{1 \leq i_1 < \cdots < i_x \leq k} \left( \frac{\sum_{j=1}^{x} \tilde{b}_{i_j}}{x} \right) \right)^{\frac{1}{x}}
\]

(42)

Proof.

\[
\frac{\tilde{b}_i}{i=1} = \left( \left[ \sqrt{1 - \prod_{j=1}^{x} (1 - (\mu_j^x)^2)}, \sqrt{1 - \prod_{j=1}^{x} (1 - (\mu_j^y)^2)} \right], \left[ \prod_{j=1}^{x} (\nu_j^x), \prod_{j=1}^{x} (\nu_j^y) \right] \right)
\]

(43)

\[
\frac{\tilde{b}_i}{i=1} = \left( \left[ \sqrt{1 - \prod_{j=1}^{x} (1 - (\mu_j^x)^2)}, \sqrt{1 - \prod_{j=1}^{x} (1 - (\mu_j^y)^2)} \right], \left[ \prod_{j=1}^{x} (\nu_j^x), \prod_{j=1}^{x} (\nu_j^y) \right] \right)
\]

(44)
Therefore,

\[
\left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( \frac{\sum b_{i_1} \cdots b_{i_k}}{\sum b_{i_1} \cdots b_{i_k}} \right) \right)^{\frac{1}{k}} = \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left[ 1 - \left( \prod_{j=1}^{k} \left( 1 - (\nu_j^2)^2 \right) \right) \right]^{\frac{1}{k}}, \right) \right)
\]

(45)

Then, we can get

\[
\left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left( \frac{\sum b_{i_1} \cdots b_{i_k}}{\sum b_{i_1} \cdots b_{i_k}} \right) \right)^{\frac{1}{k}} = \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left[ 1 - \left( \prod_{j=1}^{k} \left( 1 - (\nu_j^2)^2 \right) \right) \right]^{\frac{1}{k}}, \right) \right)
\]

(46)

From the aggregation result above, we prove the result of IVPFDHM aggregation is also an IVPFN in the following, then,

\[
0 \leq \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left[ 1 - \left( \prod_{j=1}^{k} \left( 1 - (\nu_j^2)^2 \right) \right) \right]^{\frac{1}{k}} \right) \leq 1
\]

(47)

\[
0 \leq \left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left[ 1 - \left( \prod_{j=1}^{k} \left( 1 - (\nu_j^2)^2 \right) \right) \right]^{\frac{1}{k}} \right) \leq 1
\]

(48)

And, we can prove

\[
\left( \prod_{1 \leq i_1 < \cdots < i_k \leq k} \left[ 1 - \left( \prod_{j=1}^{k} \left( 1 - (\nu_j^2)^2 \right) \right) \right]^{\frac{1}{k}} \right)^2 \leq 1
\]

(49)

Therefore, we proved that the aggregation result of IVPFDHM is also an IVPFN. Next, we will talk about some properties of the IVPFDHM operator. ∎

**Property 7.** (Idempotency.) If all \( \bar{b}_i = (\mu_i^Z, \mu_i^Y, [v_i^Z, v_i^Y]) \) are equal, i.e., \( \bar{b}_1 = \bar{b} = ([\mu_1^Z, \mu_1^Y], [v_1^Z, v_1^Y]) \), then,

\[
IVPFDHM^{(x)}(\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n) = \bar{b}
\]

(50)

**Property 8.** (Monotonicity.) Let \( \bar{b}_i = (\mu_i^Z, \mu_i^Y, [v_i^Z, v_i^Y]) \) and \( \bar{c}_i = (\mu_i^Z, \mu_i^Y, [v_i^Z, v_i^Y]) \) \( (i = 1, 2, \cdots, k) \) be two sets of IVPFNs. If \( (\mu_i^Z)^2 + (\mu_i^Y)^2 \leq (\mu_i^Z)^2 + (\mu_i^Y)^2 \) and \( (v_i^Z)^2 + (v_i^Y)^2 \), then,

\[
IVPFDHM^{(x)}(\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_k) \leq IVPFDHM^{(x)}(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_k)
\]

(51)
**Property 9.** (Boundedness.) Let \( \tilde{b}_i = \left( \left[ \mu_i^Z, \nu_i^Z \right], \left[ \mu_i^Y, \nu_i^Y \right] \right) \) be a set of IVPFNs. If \( \tilde{b}^+ = \left( \left[ \mu_{\text{maxi}}^Z, \nu_{\text{maxi}}^Z \right], \left[ \mu_{\text{maxi}}^Y, \nu_{\text{maxi}}^Y \right] \right) \) and \( \tilde{b}^- = \left( \left[ \mu_{\text{mini}}^Z, \nu_{\text{mini}}^Z \right], \left[ \mu_{\text{mini}}^Y, \nu_{\text{mini}}^Y \right] \right) \) for some \( i \), then, because of Properties 7 and 8, we get

\[
\tilde{b}_i^+ < \text{IVPFDHM}^{(x)} \left( \tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k \right) < \tilde{b}_i^-
\]

(52)

**Example 6.** Let \( \tilde{b}_1 = (0.2, 0.5, 0.1, 0.2), \tilde{b}_2 = (0.1, 0.3, 0.3, 0.4), \tilde{b}_3 = (0.4, 0.6, 0.2, 0.5), \tilde{b}_4 = (0.5, 0.7, 0.4, 0.6) \) be four IVPFNs. Then we use the proposed IVPFDHM operator to aggregate four IVPFNs, suppose \( x = 2 \).

At last, we get IVPFDHM\(^{(x)}\left( \tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4 \right) = (0.3156, 0.5432) \) [0.2428, 0.4377].

### 3.4. IVPFWDHM Operator

It can be seen that the IVPFDHM operator doesn’t consider the importance of the aggregated arguments and that the weights of attributes play an important role in the process of aggregation. To overcome the limitation of IVPFDHM, we develop the interval-valued Pythagorean fuzzy weighted DHM (IVPFWDHM) operator.

**Definition 11.** Let \( \tilde{b}_i = \left( \left[ \mu_i^Z, \mu_i^1 \right], \left[ \nu_i^Z, \nu_i^1 \right] \right) \) be a set of IVPFNs with weight vector being \( \omega = (\omega_1, \omega_2, \cdots, \omega_k)^T, \sum_{i=1}^k \omega_i = 1 \), then the IVPFWDHM operator is given as:

\[
\text{IVPFWDHM}_{\omega}^{(x)} \left( \tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k \right) = \left\{ \begin{array}{ll}
\left( \prod_{1 \leq i_1 < \cdots < i_x \leq k} \left( 1 - \sum_{j=1}^x \omega_j \frac{\tilde{a}_{i_j}^Z \otimes \tilde{a}_{i_j}^1}{\tilde{a}_{i_j}^Z + \tilde{a}_{i_j}^1} \right) \right)^{\frac{1}{x-1}} & (1 \leq x < k) \\
\prod_{i=1}^{k-1} \frac{\tilde{a}_i^Z \otimes \tilde{a}_i^1}{\tilde{a}_i^Z + \tilde{a}_i^1} & (x = k) 
\end{array} \right.
\]

(53)

Based on the operations of the IVPFN described, we can obtain Theorem 5.
Theorem 5. Let $\tilde{b}_i = ([\mu^2_i, \mu^Y_i], [\nu^2_i, \nu^Y_i])$ ($i = 1, 2, \cdots, k$) be a collection of IVPFNs, then the aggregated result of IVPFWFHDM operator is also an IVPFN, and

$$
\text{IVPFWDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k) = \left( \prod_{1 \leq i < j \leq k} \left( 1 - \left( \tilde{b}_i \left( 1 - \mu^2_i \right) \right)^{\frac{x - 1}{x}} \right) \right)^{\frac{1}{x}}
$$

or

$$
\text{IVPFWDHM}^{(x)}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k) = \left( \prod_{1 \leq i < j \leq k} \left( 1 - \left( \tilde{b}_i \left( 1 - \mu^2_i \right) \right)^{\frac{x - 1}{x}} \right) \right)^{\frac{1}{x}}
$$

Proof.

$$
\frac{\tilde{b}_1 \cdots \tilde{b}_k}{x} = \left( \left( 1 - \left( \tilde{b}_1 \left( 1 - \mu^2_1 \right) \right)^{\frac{x - 1}{x}} \right) \left( 1 - \left( \tilde{b}_1 \left( 1 - \mu^2_1 \right) \right)^{\frac{x - 1}{x}} \right) \left( 1 - \left( \tilde{b}_1 \left( 1 - \mu^2_1 \right) \right) \right) \right)
$$

Thereafter,

$$
\left( 1 - \tilde{b}_1 \left( 1 - \mu^2_1 \right) \right) \left( 1 - \tilde{b}_2 \left( 1 - \mu^2_2 \right) \right) \left( 1 - \tilde{b}_3 \left( 1 - \mu^2_3 \right) \right) \cdots \left( 1 - \tilde{b}_k \left( 1 - \mu^2_k \right) \right)
$$

Moreover,

$$
\left( 1 - \tilde{b}_1 \left( 1 - \mu^2_1 \right) \right) \left( 1 - \tilde{b}_2 \left( 1 - \mu^2_2 \right) \right) \left( 1 - \tilde{b}_3 \left( 1 - \mu^2_3 \right) \right) \cdots \left( 1 - \tilde{b}_k \left( 1 - \mu^2_k \right) \right)
$$
Therefore,
\[
\left( \prod_{i=1}^{n} \left( \frac{1}{i^{\alpha}} \right) \right)^{\frac{1}{\alpha}} = \left( \frac{\prod_{i=1}^{n} \left( 1 - \left( \frac{1}{i^{\alpha}} \right)^{\frac{1}{\alpha}} \right)}{\prod_{i=1}^{n} \left( 1 - \left( \frac{1}{i^{\alpha}} \right)^{\frac{1}{\alpha}} \right)} \right)^{\frac{1}{\alpha}} \leq 1
\]

For the second case, when \(x = k\), we get:
\[
\prod_{i=1}^{n} \left( 1 - \left( \frac{1}{i^{\alpha}} \right)^{\frac{1}{\alpha}} \right) = \left( \prod_{i=1}^{n} \left( 1 - \left( \frac{1}{i^{\alpha}} \right)^{\frac{1}{\alpha}} \right) \right) \leq 1
\]

Then, we can obtain,
\[
0 \leq \left( \prod_{i=1}^{n} \left( \sqrt{1 - \left( \frac{1}{i^{\alpha}} \right)^{\frac{1}{\alpha}}} \right) \right)^{\frac{1}{\alpha}} \leq 1
\]

And,
\[
0 \leq \prod_{i=1}^{n} \left( \left( \frac{1}{i^{\alpha}} \right)^{\frac{1}{\alpha}} \right) \leq 1
\]
For the second case, \( x = k \), we can easily prove that it is kept. Therefore, the aggregation result produced by Definition 11 is still an IVPFN. Next, we shall deduce some desirable properties of the IVPFWDHM operator.

**Property 10.** (Idempotency.) Let \( \tilde{b}_i = \left( \left[ \mu_{i,i}^Z, \mu_{i,i}^Y \right], \left[ v_{i,i}^Z, v_{i,i}^Y \right] \right) \) and weight vector meets \( \omega = (\omega_1, \omega_2, \cdots, \omega_k)^T \), \( \omega_i \in [0,1] \) and \( \sum_{i=1}^k \omega_i = 1 \), then,

\[
\text{IVPFWDHM}^{(x)}_\omega(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k) = \tilde{b}
\]  

(67)

**Property 11.** (Monotonicity.) Let \( \tilde{b}_i = \left( \left[ \mu_{i,i}^Z, \mu_{i,i}^Y \right], \left[ v_{i,i}^Z, v_{i,i}^Y \right] \right) \) and \( \tilde{c}_i = \left( \left[ \mu_{i,i}^Z, \mu_{i,i}^Y \right], \left[ v_{i,i}^Z, v_{i,i}^Y \right] \right) \) (\( i = 1, 2, \cdots, k \)) be two sets of IVPFNs with weight vector being \( \omega = (\omega_1, \omega_2, \cdots, \omega_k)^T \), \( \omega_i \in [0,1] \), \( \sum_{i=1}^k \omega_i = 1 \). If \( (\mu_{i,i}^Z)^2 + (\mu_{i,i}^Y)^2 \leq (\mu_{i,i}^Z)^2 + (\mu_{i,i}^Y)^2 \) and \( (v_{i,i}^Z)^2 + (v_{i,i}^Y)^2 \geq (v_{i,i}^Z)^2 + (v_{i,i}^Y)^2 \), then,

\[
\text{IVPFWDHM}^{(x)}_\omega(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k) \leq \text{IVPFWDHM}^{(x)}_\omega(\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_k)
\]  

(68)

**Property 12.** (Boundedness.) Let \( \tilde{b}_i = \left( \left[ \mu_{i,i}^Z, \mu_{i,i}^Y \right], \left[ v_{i,i}^Z, v_{i,i}^Y \right] \right), \tilde{b}_i^+ = \left( \left[ \mu_{i,i}^Z, \mu_{i,i}^Y \right], \left[ v_{i,i}^Z, v_{i,i}^Y \right] \right) \) and \( \tilde{b}_i^- = \left( \left[ \mu_{i,i}^Z, \mu_{i,i}^Y \right], \left[ v_{i,i}^Z, v_{i,i}^Y \right] \right) \) (\( i = 1, 2, \cdots, k \)) be a set of IVPFNs with weight vector being \( \omega = (\omega_1, \omega_2, \cdots, \omega_k)^T \), \( \omega_i \in [0,1] \), \( \sum_{i=1}^k \omega_i = 1 \), because of Property 11, then,

\[
\tilde{b}_i^- < \text{IVPFWDHM}^{(x)}_\omega(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_k) < \tilde{b}_i^+
\]  

(69)

**Example 7.** Let \( \tilde{b}_1 = ([0.3, 0.5], [0.2, 0.4]), \tilde{b}_2 = ([0.4, 0.6], [0.6, 0.7]), \tilde{b}_3 = ([0.5, 0.7], [0.2, 0.3]), \tilde{b}_4 = ([0.4, 0.8], [0.1, 0.2]) \) be four IVPFNs. The weighting vector of attributes be \( \omega = \{0.3, 0.2, 0.3, 0.2\} \). Then, we use the proposed IVPFWDHM operator to aggregate four IVPHNs, (suppose \( x = 2 \)).
Finally, we get $\text{IVPFWDM}_{(2)}^{(3)}(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3,\tilde{b}_4) = ([0.7306, 0.9099], [0.2547, 0.3905])$.

4. Models for MADM with IVPFNs

Based on the IVPFWHM and IVPFWDM operators, in this section, we shall propose the model for MADM with IVPFNs. Let $E = \{E_1, E_2, \cdots, E_m\}$ be a discrete set of alternatives, and $D = \{D_1, D_2, \cdots, D_n\}$ be the set of attributes, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weight of the attribute $D_j(j = 1, 2, \cdots, n)$, where $\omega_j \in [0, 1], \sum_{j=1}^{n} \omega_j = 1$. Suppose that $\tilde{B}_i = (\tilde{b}_{ij})_{m \times n}$ is the interval-valued Pythagorean fuzzy decision matrix, where $[\mu_{ij}, \nu_{ij}]$ indicates the degree that the alternative $E_i$ satisfies the attribute $D_j$ given by the decision maker, $[v_{ij}^Z, v_{ij}^Y]$ depicts the degree that the alternative $E_i$ doesn’t satisfy the attribute $D_j$ given by the decision maker. $[\mu_{ij}, \nu_{ij}] \subset [0, 1], [v_{ij}^Z, v_{ij}^Y] \subset [0, 1]$, $(\mu_{ij})^2 + (v_{ij}^Y)^2 \leq 1, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$.

In the following, we use the IVPFWHM (IVPFWDM) operator to solve the MADM problems with IVPFNs.

Step 1. We use the IVPFNs given in matrix $\tilde{B}$, and the IVPFWHM operator,

$$
\text{IVPFWHM}_{(2)}^{(3)}(\tilde{b}_1,\tilde{b}_2,\ldots,\tilde{b}_k) = \frac{1}{\sum_{i=1}^{k} \frac{1}{\tilde{b}_{ij}}^{\text{IVFWHM}(2)}} = \left( \frac{1}{\sum_{i=1}^{k} \frac{1}{\tilde{b}_{ij}}^{\text{IVFWHM}(2)}} \right) \left( \frac{1}{\sum_{i=1}^{k} \frac{1}{\tilde{b}_{ij}}^{\text{IVFWHM}(2)}} \right) = \left( \frac{1}{\sum_{i=1}^{k} \frac{1}{\tilde{b}_{ij}}^{\text{IVFWHM}(2)}} \right) \left( \frac{1}{\sum_{i=1}^{k} \frac{1}{\tilde{b}_{ij}}^{\text{IVFWHM}(2)}} \right) = (1 \leq x \leq k)
$$

(70)

$$
\text{IVPFWHM}_{(2)}^{(3)}(\tilde{b}_1,\tilde{b}_2,\ldots,\tilde{b}_k) = \frac{1}{\sum_{i=1}^{k} \frac{1}{\tilde{b}_{ij}}^{\text{IVFWHM}(2)}} = \left( \frac{1}{\sum_{i=1}^{k} \frac{1}{\tilde{b}_{ij}}^{\text{IVFWHM}(2)}} \right) \left( \frac{1}{\sum_{i=1}^{k} \frac{1}{\tilde{b}_{ij}}^{\text{IVFWHM}(2)}} \right) = (x = k)
$$

(71)
Step 1. According to Table 1, aggregate all IVPFNs by utilizing the IVPFHM, IVPFWHM, IVPFDHM, and IVPFWDHM operators to solve the MADM problem with IVPFNs, which includes the following calculating steps:

1. **Problems.** A company wants to select an air-conditioning product and knows there are four possible suppliers as candidates: (1) D1 quality of product; (2) D2 price of product; (3) D3 product life cycle; (4) D4 after-sale service. The weight vector of attributes is \( \omega = (0.3, 0.2, 0, 1, 0.4)^T \); (suppose \( x = 2 \)) as shown in Table 1.

2. **With the rapid development of economic globalization and the growing competitive enterprise environment, the competition between modern enterprises has become the competition between supply chain and supply chain. Owing to the diversity of the people’s consumption concept, the new product life cycles are getting shorter, volatility of the demand market and those from external factors drives enterprises to achieve effective supply chain integration and management, and strategic alliance with other enterprises in order to enhance core competitiveness and resist external risk. The key measure to achieving this goal is the supplier selection. Therefore, supplier selection has gained great attention whether in supply chain management theory or in actual production management problems. A company wants to select an air-conditioning and know there are four possible suppliers as candidates \( E_i \) \( (i = 1, 2, 3, 4) \) to select. The experts select four attributes to assess the four possible suppliers: (1) D1 quality of product; (2) D2 price of product; (3) D3 product life cycle; (4) D4 after-sale service. The weight vector of attributes is \( \omega = (0.3, 0.2, 0, 1, 0.4)^T \); (suppose \( x = 2 \)) as shown in Table 1.

3. **In the following, in order to select the best suppliers in supply chain management (SCM), we utilize the IVPFHM, IVPFWHM, IVPFDHM, and IVPFWDHM operators to solve the MADM problem with IVPFNs, which includes the following calculating steps:**

4. **Step 1.** According to Table 1, aggregate all IVPFNs \( b_{ij} (i = 1, 2, 3, 4, j = 1, 2, \cdots, n) \) by using the IVPFHM, IVPFWHM, IVPFDHM, and IVPFWDHM operators to derive the overall IVPFNs \( b_{ij} (i = 1, 2, 3, 4, j = 1, 2, \cdots, n) \) of the alternative \( E_i \). The aggregation results are shown in Table 3; (suppose \( x = 2 \)).
Step 2. According to the results listed in Table 3, the score functions of the suppliers are shown in Table 2.

**Table 1. Decision matrix.**

<table>
<thead>
<tr>
<th></th>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
<th>D₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>E₁</td>
<td>([0.10,0.40], [0.20,0.50])</td>
<td>([0.50,0.60], [0.20,0.30])</td>
<td>([0.30,0.40], [0.30,0.50])</td>
<td>([0.40,0.50], [0.20,0.30])</td>
</tr>
<tr>
<td>E₂</td>
<td>([0.50,0.60], [0.10,0.30])</td>
<td>([0.50,0.60], [0.20,0.30])</td>
<td>([0.50,0.70], [0.40,0.60])</td>
<td>([0.60,0.70], [0.40,0.60])</td>
</tr>
<tr>
<td>E₃</td>
<td>([0.60,0.70], [0.20,0.30])</td>
<td>([0.50,0.60], [0.10,0.20])</td>
<td>([0.20,0.50], [0.40,0.60])</td>
<td>([0.50,0.80], [0.20,0.30])</td>
</tr>
<tr>
<td>E₄</td>
<td>([0.30,0.70], [0.20,0.30])</td>
<td>([0.30,0.70], [0.10,0.20])</td>
<td>([0.20,0.50], [0.40,0.60])</td>
<td>([0.30,0.70], [0.20,0.30])</td>
</tr>
</tbody>
</table>

**Table 2. The rank and score of the suppliers by using IVPFH, IVPFW, IVPFDH, and IVPFWDH operators.**

<table>
<thead>
<tr>
<th></th>
<th>E₁</th>
<th>E₂</th>
<th>E₃</th>
<th>E₄</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>IVPFHM</td>
<td>0.5603</td>
<td>0.6654</td>
<td>0.5623</td>
<td>0.6033</td>
<td>E₂ &gt; E₄ &gt; E₃ &gt; E₁</td>
</tr>
<tr>
<td>IVPFWHM</td>
<td>0.6436</td>
<td>0.7352</td>
<td>0.6514</td>
<td>0.6862</td>
<td>E₂ &gt; E₄ &gt; E₃ &gt; E₁</td>
</tr>
<tr>
<td>IVPFDHM</td>
<td>0.4891</td>
<td>0.5980</td>
<td>0.5042</td>
<td>0.5428</td>
<td>E₂ &gt; E₄ &gt; E₃ &gt; E₁</td>
</tr>
<tr>
<td>IVPFWDHM</td>
<td>0.3972</td>
<td>0.5155</td>
<td>0.4156</td>
<td>0.4470</td>
<td>E₂ &gt; E₄ &gt; E₃ &gt; E₁</td>
</tr>
</tbody>
</table>

**Table 3. The aggregation result of IVPFH, IVPFW, IVPFDH, and IVPFWDH operators.**

<table>
<thead>
<tr>
<th></th>
<th>E₁</th>
<th>E₂</th>
<th>E₃</th>
<th>E₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>IVPFHM</td>
<td>([0.2428,0.6315], [0.2261,0.4068])</td>
<td>([0.4490,0.7434], [0.2259,0.2769])</td>
<td>([0.2655,0.6007], [0.2310,0.3588])</td>
<td>([0.3201,0.6608], [0.1788,0.3062])</td>
</tr>
<tr>
<td>IVPFWHM</td>
<td>([0.2421,0.8608], [0.2317,0.4138])</td>
<td>([0.4599,0.9083], [0.1318,0.2813])</td>
<td>([0.2765,0.8403], [0.2317,0.3578])</td>
<td>([0.3056,0.8760], [0.1726,0.2937])</td>
</tr>
<tr>
<td>IVPFDHM</td>
<td>([0.2598,0.4798], [0.2237,0.5935])</td>
<td>([0.4566,0.5856],[0.1225,0.3802])</td>
<td>([0.2790,0.4590],[0.2168,0.4742])</td>
<td>([0.3315,0.5039],[0.1687,0.4051])</td>
</tr>
<tr>
<td>IVPFWDHM</td>
<td>([0.2542,0.4787], [0.2281,0.8080])</td>
<td>([0.4650,0.3997],[0.1265,0.7055])</td>
<td>([0.2982,0.4644],[0.2121,0.7729])</td>
<td>([0.3156,0.4967],[0.1657,0.7286])</td>
</tr>
</tbody>
</table>

According to the result of the company order, we know that the best choice is supplier 2. We obtain the same result by different aggregation, proving the effectiveness of result.

5.2. Influence of the Parameter on the Final Result

The aggregation method to extend IVPFS with HM has two advantages, one is that it can reduce the bad effects of the unduly high and low assessments on the final result, the other is that it can capture the interrelationship between IVPFNs. These aggregation operators have a parameter vector, which makes the extended operator more flexible, causing a different vector to lead to different aggregation results, scores, and ranking results. In order to illustrate the influence of the parameter vector $x$ on the ranking result, we discuss the influence with several parameter vectors. The results of which are given in Tables 4 and 5.

We can see that the different parameters lead to different results and different ranking orders. The more attributes we consider, the bigger the scores and the bigger the attribute value, the lower the scores. Therefore, the parameter vector can be considered as the decision maker’s risk preference.
Table 4. Ranking results by utilizing a different parameter vector \( x \) in the IVPFWHM operator.

<table>
<thead>
<tr>
<th>( x = 1, 2, 3, 4 )</th>
<th>Scores</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 1 )</td>
<td>0.5259</td>
<td>0.6225</td>
</tr>
<tr>
<td>( x = 2 )</td>
<td>0.6436</td>
<td>0.7382</td>
</tr>
<tr>
<td>( x = 3 )</td>
<td>0.5115</td>
<td>0.6006</td>
</tr>
<tr>
<td>( x = 4 )</td>
<td>0.5079</td>
<td>0.6101</td>
</tr>
</tbody>
</table>

Table 5. Ranking results by utilizing a different parameter vector \( x \) in the IVPFWDHM operator.

<table>
<thead>
<tr>
<th>( x = 1, 2, 3, 4 )</th>
<th>Scores</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 1 )</td>
<td>0.5079</td>
<td>0.6101</td>
</tr>
<tr>
<td>( x = 2 )</td>
<td>0.3972</td>
<td>0.5155</td>
</tr>
<tr>
<td>( x = 3 )</td>
<td>0.5236</td>
<td>0.6226</td>
</tr>
<tr>
<td>( x = 4 )</td>
<td>0.5259</td>
<td>0.6244</td>
</tr>
</tbody>
</table>

5.3. Comparative Analysis

Then, we compare the proposed method with the IVPFWA operator and IVPFWG operator proposed by Garg [28].

Definition 12. ([28]). Suppose that \( \tilde{B} = \left( \tilde{b}_{ij} \right)_{m \times n} = \left( \left[ \mu_{ij}^{\omega}, \mu_{ij}^{\omega} \right], \left[ \nu_{ij}^{\omega}, \nu_{ij}^{\omega} \right] \right)_{m \times n} \) be an IVPFN matrix, \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \) be the weight of \( w_j \), \( 0 \leq \omega_i \leq 1 \), \( \sum_{i=1}^{n} \omega_i = 1 \). Then,

\[
\begin{align*}
\bar{b}_j = & \text{IVPFWA}_\omega \left( \tilde{b}_{11}, \tilde{b}_{12}, \cdots, \tilde{b}_{1n} \right) = \frac{1}{n} \sum_{i=1}^{n} (\omega_i \tilde{b}_{ij}) \\
= & \left( \prod_{i=1}^{n} \left( 1 - \left( \mu_{ij}^{\omega} \right)^2 \right)^{\omega_i} \right)^{\frac{1}{n}} \left( \prod_{i=1}^{n} \left( 1 - \left( \nu_{ij}^{\omega} \right)^2 \right)^{\omega_i} \right)^{\frac{1}{n}} \\
\end{align*}
\]

(74)

\[
\begin{align*}
\tilde{b}_j = & \text{IVPFWG}_\omega \left( \tilde{b}_{11}, \tilde{b}_{12}, \cdots, \tilde{b}_{1n} \right) = \frac{1}{n} \sum_{i=1}^{n} (\tilde{b}_{ij})^{\omega_i} \\
= & \left( \prod_{i=1}^{n} \left( \mu_{ij}^{\omega} \right)^{\omega_i} \right)^{\frac{1}{n}} \left( \prod_{i=1}^{n} \left( \nu_{ij}^{\omega} \right)^{\omega_i} \right)^{\frac{1}{n}} \\
\end{align*}
\]

(75)

By utilizing \( \tilde{B} \) and the IVPFWA and IVPFWG operators, the aggregation results were derived and are shown in Table 6.

Table 6. The aggregation results of green suppliers by the IVPFWA (IVPFWG) operators.

<table>
<thead>
<tr>
<th></th>
<th>IVPFWA</th>
<th>IVPFWG</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_1 )</td>
<td>([0.2939, 0.4896], [0.2083, 0.3680])</td>
<td>([0.2232, 0.4732], [0.2124, 0.3971])</td>
</tr>
<tr>
<td>( E_2 )</td>
<td>([0.4355, 0.5500], [0.1072, 0.2551])</td>
<td>([0.4076, 0.5181], [0.1142, 0.2652])</td>
</tr>
<tr>
<td>( E_3 )</td>
<td>([0.2518, 0.4540], [0.2144, 0.3470])</td>
<td>([0.3192, 0.4300], [0.2498, 0.3751])</td>
</tr>
<tr>
<td>( E_4 )</td>
<td>([0.3914, 0.5291], [0.1663, 0.3085])</td>
<td>([0.3534, 0.5144], [0.2139, 0.3673])</td>
</tr>
</tbody>
</table>

According to Table 6, the score values were derived and are shown in Table 7.
Table 7. The score functions of the suppliers.

<table>
<thead>
<tr>
<th>Supplier</th>
<th>IVPFWA</th>
<th>IVPFWG</th>
</tr>
</thead>
<tbody>
<tr>
<td>E_1</td>
<td>0.5368</td>
<td>0.5180</td>
</tr>
<tr>
<td>E_2</td>
<td>0.6039</td>
<td>0.5878</td>
</tr>
<tr>
<td>E_3</td>
<td>0.5257</td>
<td>0.5074</td>
</tr>
<tr>
<td>E_4</td>
<td>0.5788</td>
<td>0.5522</td>
</tr>
</tbody>
</table>

Following from Table 7, the order of the suppliers is given in Table 8.

Table 8. Order of the green suppliers.

<table>
<thead>
<tr>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>IVPFWA</td>
</tr>
<tr>
<td>E_2 &gt; E_4 &gt; E_1 &gt; E_3</td>
</tr>
<tr>
<td>IVPFWG</td>
</tr>
<tr>
<td>E_2 &gt; E_4 &gt; E_1 &gt; E_3</td>
</tr>
</tbody>
</table>

From above, we can observe that we consistently identify the same best suppliers, showing the effectiveness of our approaches. However, the existing aggregation operators, such as IVPFWA operator and IVPFWG operator, don’t consider interrelationship among arguments being aggregated, and thus can’t eliminate the influence of unfair arguments. Our proposed IVPFWHM and IVPFWDHM operators consider the information concerning the interrelationship among arguments being aggregated.

In a word, we have verified the effectiveness of the proposed method and shown the advantages of solving the MADM problem with incomplete and indeterminate information.

6. Conclusions

IVPFNs can easily describe incomplete and indeterminate information by degrees of membership and non-membership, and the HM operator and dual HM (DMM) operators are good tools to deal with multiple attribute decision making (MADM) problems because they can capture the interrelationship among the multi-input arguments. Motivated by the studies regarding the HM operator and dual HM operator, in this paper, we proposed some new HM and DHM operators to cope with MADM with IVPFNs, including the IVPFHM operator, IVPFWHM operator, IVPFDMHM operator, and IVPFWDHM operator. Moreover, these proposed operators are utilized to solve the MADM problems with IVPFNs. Finally, we used an illustrative example for supplier selections in SCM to show the feasibility and validity of the proposed operators by comparison with the other existing methods. In subsequent studies, we shall extend the proposed operators to the different fields [56–72] as well as propose some new aggregation operators under the uncertain environment [73–80]. On the other hand, we can develop the potential applications of the proposed method to different domains [81–87].

Author Contributions: Z.L., G.W. and H.G. conceived and worked together to achieve this work, Z.L. compiled the computing program by Matlab and analyzed the data, Z.L. and G.W. wrote the paper. Finally, all the authors have read and approved the final manuscript.

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Conflicts of Interest: The authors declare no conflict of interest.

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