Article

Certain Notions of Neutrosophic Topological $K$-Algebras

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Abstract: The concept of neutrosophic set from philosophical point of view was first considered by Smarandache. A single-valued neutrosophic set is a subclass of the neutrosophic set from a scientific and engineering point of view and an extension of intuitionistic fuzzy sets. In this research article, we apply the notion of single-valued neutrosophic sets to $K$-algebras. We introduce the notion of single-valued neutrosophic topological $K$-algebras and investigate some of their properties. Further, we study certain properties, including $C_5$-connected, super connected, compact and Hausdorff, of single-valued neutrosophic topological $K$-algebras. We also investigate the image and pre-image of single-valued neutrosophic topological $K$-algebras under homomorphism.

Keywords: $K$-algebras; single-valued neutrosophic sets; homomorphism; compactness; $C_5$-connectedness

MSC: 06F35; 03G25; 03B52

1. Introduction

A new kind of logical algebra, known as $K$-algebra, was introduced by Dar and Akram in [1]. A $K$-algebra is built on a group $G$ by adjoining the induced binary operation on $G$. The group $G$ is particularly of the type in which each non-identity element is not of order 2. This algebraic structure is, in general, non-commutative and non-associative with right identity element [1–3].

Akram et al. [4] introduced fuzzy $K$-algebras. They then developed fuzzy $K$-algebras with other researchers worldwide. The concepts and results of $K$-algebras have been broadened to the fuzzy setting frames by applying Zadeh’s fuzzy set theory and its generalizations, namely, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, bipolar fuzzy sets and vague sets [5]. In handling information regarding various aspects of uncertainty, non-classical logic is considered to be a more powerful tool than the classical logic. It has become a strong mathematical tool in computer science, medical, engineering, information technology, etc. In 1998, Smarandache [6] introduced neutrosophic set as a generalization of intuitionistic fuzzy set [7]. A neutrosophic set is identified by three functions called truth-membership ($T$), indeterminacy-membership ($I$) and falsity-membership ($F$) functions. To apply neutrosophic set in real-life problems more conveniently, Smarandache [6] and Wang et al. [8] defined single-valued neutrosophic sets which takes the value from the subset of $[0, 1]$. Thus, a single-valued neutrosophic set is an instance of neutrosophic set.

Algebraic structures have a vital place with vast applications in various areas of life. Algebraic structures provide a mathematical modeling of related study. Neutrosophic set theory has also been
applied to many algebraic structures. Agboola and Davazz introduced the concept of neutrosophic BCI/BCK-algebras and discuss elementary properties in [9]. Jun et al. introduced the notion of \((\phi, \psi)\) neutrosophic subalgebra of a BCK/BCI-algebra [10]. Jun et al. [11] defined interval neutrosophic sets on BCK/BCI-algebra [11]. Jun et al. [12] proposed neutrosophic positive implicative \(N\)-ideals and study their extension property [12] Several set theories and their topological structures have been introduced by many researchers to deal with uncertainties. Chang [13] was the first to introduce the notion of fuzzy topology. Later, Lowen [14], Pu and Liu [15], and Chattopadhyay and Samanta [16] introduced other concepts related to fuzzy topology. Coker [17] introduced the notion of intuitionistic fuzzy topology as a generalization of fuzzy topology. Salama and Alblowi [18] defined the topological structure of neutrosophic set theory. Akram and Dar [19] introduced the concept of fuzzy topological \(K\)-algebras. They extended their work on intuitionistic fuzzy topological \(K\)-algebras [20]. In this paper, we introduce the notion of single-valued neutrosophic topological \(K\)-algebras and investigate some of their properties. Further, we study certain properties, including \(C_5\)-connected, super connected, compact and Hausdorff, of single-valued neutrosophic topological \(K\)-algebras. We also investigate the image and pre-image of single-valued neutrosophic topological \(K\)-algebras under homomorphism.

2. Preliminaries

The notion of \(K\)-algebra was introduced by Dar and Akram in [1].

**Definition 1.** [1] Let \((G, \cdot, e)\) be a group in which each non-identity element is not of order 2. A \(K\)-algebra is a structure \(K = (G, \cdot, e)\) over a particular group \(G\), where \(\cdot\) is an induced binary operation \(\cdot : G \times G \to G\) is defined by \(\cdot(s, t) = s \cdot t = s.t^{-1}\), and satisfy the following conditions:

(i) \((s \cdot t) \cdot (s \cdot u) = (s \cdot ((e \cdot u) \cdot (e \cdot t))) \cdot s; \)
(ii) \(s \cdot(s \cdot t) = (s \cdot (e \cdot t)) \cdot s; \)
(iii) \(s \cdot e = e; \)
(iv) \(s \cdot e = s; \) and
(v) \(e \cdot s = s^{-1} \)

for all \(s, t, u \in G\). The homomorphism between two \(K\)-algebras \(K_1\) and \(K_2\) is a mapping \(f : K_1 \to K_2\) such that, for all \(u, v \in K_1\), \(f(u \cdot v) = f(u) \cdot f(v)\).

In [6], Smarandache initiated the idea of neutrosophic set theory which is a generalization of intuitionistic fuzzy set theory. Later, Smarandache and Wang et al. introduced a single-valued neutrosophic set (SNS) as an instance of neutrosophic set in [8].

**Definition 2.** [8] Let \(Z\) be a space of points with a general element \(s \in Z\). A SNS \(A\) in \(Z\) is equipped with three membership functions: truth membership function \((T_A)\), indeterminacy membership function \((I_A)\) and falsity membership function\((F_A)\), where \(\forall s \in Z, T_A(s), I_A(s), F_A(s) \in [0, 1]\). There is no restriction on the sum of these three components. Therefore, \(0 \leq T_A(s) + I_A(s) + F_A(s) \leq 3\).

**Definition 3.** [8] A single-valued neutrosophic empty set \((\emptyset_{SN})\) and single-valued neutrosophic whole set \((1_{SN})\) on \(Z\) is defined as:

- \(\emptyset_{SN}(u) = \{u \in Z : (u, 0, 0, 1)\}\).
- \(1_{SN}(u) = \{u \in Z : (u, 1, 1, 0)\}\).

**Definition 4.** [8] If \(f\) is a mapping from a set \(Z_1\) into a set \(Z_2\), then the following statements hold:

(i) Let \(A\) be a SNS in \(Z_1\) and \(B\) be a SNS in \(Z_2\), then the pre-image of \(B\) is a SNS in \(Z_1\), denoted by \(f^{-1}(B)\), defined as:

\[ f^{-1}(B) = \{z_1 \in Z_1 : f^{-1}(T_B)(z_1) = T_B(f(z_1)), f^{-1}(I_B)(z_1) = I_B(f(z_1)), f^{-1}(F_B)(z_1) = F_B(f(z_1))\}. \]
(ii) Let $A = \{ z_1 \in Z_1 : \mathcal{T}_A(z_1), \mathcal{I}_A(z_1), \mathcal{F}_A(z_1) \}$ be a SNS in $Z_1$ and $B = \{ z_2 \in Z_2 : \mathcal{T}_B(z_2), \mathcal{I}_B(z_2), \mathcal{F}_B(z_2) \}$ be a SNS in $Z_2$. Under the mapping $f$, the image of $A$ is a SNS in $Z_2$, denoted by $f(A)$, defined as: $f(A) = \{ z_2 \in Z_2 : f_{\sup}(\mathcal{T}_A)(z_2), f_{\sup}(\mathcal{I}_A)(z_2), f_{\inf}(\mathcal{F}_A)(z_2) \}$, where for all $z_2 \in Z_2$.

\[
\begin{align*}
f_{\sup}(\mathcal{T}_A)(z_2) &= \begin{cases} 
\sup_{z_1 \in f^{-1}(z_2)} \mathcal{T}_A(z_1), & \text{if } f^{-1}(z_2) \neq \emptyset, \\
0, & \text{otherwise},
\end{cases} \\
f_{\sup}(\mathcal{I}_A)(z_2) &= \begin{cases} 
\sup_{z_1 \in f^{-1}(z_2)} \mathcal{I}_A(z_1), & \text{if } f^{-1}(z_2) \neq \emptyset, \\
0, & \text{otherwise},
\end{cases} \\
f_{\inf}(\mathcal{F}_A)(z_2) &= \begin{cases} 
\inf_{z_1 \in f^{-1}(z_2)} \mathcal{F}_A(z_1), & \text{if } f^{-1}(z_2) \neq \emptyset, \\
0, & \text{otherwise}.
\end{cases}
\end{align*}
\]

We formulate the following proposition.

**Proposition 1.** Let $f : Z_1 \rightarrow Z_2$ and $A_i (A_j, j \in J)$ be a SNS in $Z_1$ and $B$ be a SNS in $Z_2$. Then, $f$ possesses the following properties:

(i) If $f$ is onto, then $f(1_{\text{SN}}) = 1_{\text{SN}}$.
(ii) $f(\emptyset_{\text{SN}}) = \emptyset_{\text{SN}}$.
(iii) $f^{-1}(1_{\text{SN}}) = 1_{\text{SN}}$.
(iv) $f^{-1}(\emptyset_{\text{SN}}) = \emptyset_{\text{SN}}$.
(v) If $f$ is onto, then $f(f^{-1}(B)) = B$.
(vi) $f^{-1}(\bigcup_{i=1}^n A_i) = \bigcup_{i=1}^n f^{-1}(A_i)$.

3. Neutrosophic Topological $K$-algebras

**Definition 5.** Let $Z$ be a nonempty set. A collection $\chi$ of single-valued neutrosophic sets (SNSs) in $Z$ is called a single-valued neutrosophic topology (SNT) on $Z$ if the following conditions hold:

(a) $\emptyset_{\text{SN}}, 1_{\text{SN}} \in \chi$
(b) If $A, B \in \chi$, then $A \cap B \in \chi$
(c) If $A_i \in \chi, \forall i \in I$, then $\bigcup_{i \in I} A_i \in \chi$

The pair $(Z, \chi)$ is called a single-valued neutrosophic topological space (SNTS). Each member of $\chi$ is said to be $\chi$-open or single-valued neutrosophic open set (SNOS) and compliment of each open single-valued neutrosophic set is a single-valued neutrosophic closed set (SNCS). A discrete topology is a topology which contains all single-valued neutrosophic subsets of $Z$ and indiscrete if its elements are only $\emptyset_{\text{SN}}, 1_{\text{SN}}$.

**Definition 6.** Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a single-valued neutrosophic set in $K$. Then, $A$ is called a single-valued neutrosophic $K$-subalgebra of $K$ if following conditions hold for $A$:

(i) $\mathcal{T}_A(e) \supseteq \mathcal{T}_A(s), \mathcal{I}_A(e) \supseteq \mathcal{I}_A(s), \mathcal{F}_A(e) \subseteq \mathcal{F}_A(s)$.
(ii) $\mathcal{T}_A(s \odot t) \geq \min\{\mathcal{T}_A(s), \mathcal{T}_A(t)\}$,
$\mathcal{I}_A(s \odot t) \geq \min\{\mathcal{I}_A(s), \mathcal{I}_A(t)\}$,
$\mathcal{F}_A(s \odot t) \leq \max\{\mathcal{F}_A(s), \mathcal{F}_A(t)\}$ \(\forall s, t \in K\).
Example 1. Consider a K-algebra $K = (G, \cdot, \circ, e)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ is the cyclic group of order 9 and Cayley’s table for $\circ$ is given as:

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If we define a single-valued neutrosophic set $A, B$ in $K$ such that:

$A = \{(e, 0.4, 0.5, 0.8), (s, 0.3, 0.4, 0.7)\}$,

$B = \{(e, 0.3, 0.4, 0.8), (s, 0.2, 0.3, 0.6)\}$

$\forall s \neq e \in G$.

According to Definition 5, the family $\{\emptyset_{SN}, 1_{SN}, A, B\}$ of SNSs of $K$-algebra is a SNT on $K$. We define a SNS $A = \{T_A, I_A, F_A\}$ in $K$ such that $T_A(e) = 0.7, I_A(e) = 0.5, F_A(e) = 0.2, T_A(s) = 0.2, I_A(s) = 0.4, F_A(s) = 0.6$. Clearly, $A = (T_A, I_A, F_A)$ is a SN $K$-subalgebra of $K$.

Definition 7. Let $K = (G, \cdot, \circ, e)$ be a K-algebra and let $\chi_K$ be a topology on $K$. Let $A$ be a SNS in $K$ and let $\chi_A$ be a topology on $K$. Then, an induced single-valued neutrosophic topology on $A$ is a collection or family of single-valued neutrosophic subsets of $A$ which are the intersection with $A$ and single-valued neutrosophic open sets in $K$ defined as $\chi_A = \{A \cap F : F \in \chi_K\}$. Then, $\chi_A$ is called a single-valued neutrosophic induced topology on $A$ or relative topology and the pair $(A, \chi_A)$ is called an induced topological space or single-valued neutrosophic subspace of $(K, \chi_K)$.

Definition 8. Let $(K_1, \chi_1)$ and $(K_2, \chi_2)$ be two SNTSs and let $f : (K_1, \chi_1) \to (K_2, \chi_2)$. Then, $f$ is called single-valued neutrosophic continuous if following conditions hold:

(i) For each SNS $A \in \chi_1$, $f^{-1}(A) \in \chi_1$.

(ii) For each SN $K$-subalgebra $A \in \chi_1$, $f^{-1}(A)$ is a SN $K$-subalgebra in $\chi_1$.

Definition 9. Let $(K_1, \chi_1)$ and $(K_2, \chi_2)$ be two SNTSs and let $(A, \chi_A)$ and $(B, \chi_B)$ be two single-valued neutrosophic subspaces over $(K_1, \chi_1)$ and $(K_2, \chi_2)$. Let $f$ be a mapping from $(K_1, \chi_1)$ into $(K_2, \chi_2)$, then $f$ is a mapping from $(A, \chi_A)$ to $(B, \chi_B)$ if $f(A) \subset B$.

Definition 10. Let $f$ be a mapping from $(A, \chi_A)$ to $(B, \chi_B)$. Then, $f$ is relatively single-valued neutrosophic continuous if for every SNOS $Y_B$ in $\chi_B$, $f^{-1}(Y_B) \cap A \in \chi_A$.

Definition 11. Let $f$ be a mapping from $(A, \chi_A)$ to $(B, \chi_B)$. Then, $f$ is relatively single-valued neutrosophic open if for every SNOS $X_A$ in $\chi_A$, the image $f(X_A) \in \chi_B$.

Proposition 2. Let $(A, \chi_A)$ and $(B, \chi_B)$ be single-valued neutrosophic subspaces of $(K_1, \chi_1)$ and $(K_2, \chi_2)$, where $K_1$ and $K_2$ are K-algebras. If $f$ is a single-valued neutrosophic continuous function from $K_1$ to $K_2$ and $f(A) \subset B$. Then, $f$ is relatively single-valued neutrosophic continuous function from $A$ into $B$.

Definition 12. Let $(K_1, \chi_1)$ and $(K_2, \chi_2)$ be two SNTSs. A mapping $f : (K_1, \chi_1) \to (K_2, \chi_2)$ is called a single-valued neutrosophic homomorphism if following conditions hold:

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*Note: The document contains definitions and theorems related to single-valued neutrosophic spaces, and the text above is a excerpt from a mathematical paper.*
(i) \( f \) is a one-one and onto function.

(ii) \( f \) is a single-valued neutrosophic continuous function from \( K_1 \) to \( K_2 \).

(iii) \( f^{-1} \) is a single-valued neutrosophic continuous function from \( K_2 \) to \( K_1 \).

Theorem 1. Let \((K_1, \chi_1)\) be a SNTS and \((K_2, \chi_2)\) be an indiscrete SNTS on \( K \)-algebras \( K_1 \) and \( K_2 \), respectively. Then, each function \( f \) defined as \( f: (K_1, \chi_1) \rightarrow (K_2, \chi_2) \) is a single-valued neutrosophic continuous function from \( K_1 \) to \( K_2 \). If \((K_1, \chi_1)\) and \((K_2, \chi_2)\) be two discrete SNTSs \( K_1 \) and \( K_2 \), respectively, then each homomorphism \( f: (K_1, \chi_1) \rightarrow (K_2, \chi_2) \) is a single valued neutrosophic continuous function from \( K_1 \) to \( K_2 \).

Proof. Let \( f \) be a mapping defined as \( f: K_1 \rightarrow K_2 \). Let \( \chi_1 \) be SNT on \( K_1 \) and \( \chi_2 \) be SNT on \( K_2 \), where \( \chi_2 = \{0_{SN}, 1_{SN}\} \). We show that \( f^{-1}(A) \) is a single-valued neutrosophic \( K \)-subalgebra of \( K_1 \), i.e., for each \( A \in \chi_2 \), \( f^{-1}(A) \in \chi_1 \). Since \( \chi_2 = \{0_{SN}, 1_{SN}\} \), then for any \( u \in \chi_1 \), consider \( 0_{SN} \in \chi_2 \) such that \( f^{-1}(0_{SN})(u) = 0_{SN} = f(u) = 0_{SN}(u) \).

Hence, \( f^{-1}(0_{SN}) = 0_{SN} \in \chi_1 \). Likewise, \( f^{-1}(1_{SN}) = 1_{SN} \in \chi_1 \). Hence, \( f \) is a SN continuous function from \( K_1 \) to \( K_2 \).

Now, for the second part of the theorem, where both \( \chi_1 \) and \( \chi_2 \) are SNTSs on \( K_1 \) and \( K_2 \), respectively, and \( f: (K_1, \chi_1) \rightarrow (K_2, \chi_2) \) is a homomorphism. Therefore, for all \( A \in \chi_2 \) and \( f^{-1}A \in \chi_1 \), where \( f \) is not a usual inverse homomorphism. To prove that \( f^{-1}(A) \) is a single-valued neutrosophic \( K \)-subalgebra in of \( K_1 \). Let for \( u, v \in K_1 \),

\[
\begin{align*}
f^{-1}(T_A)(u \circ v) &= T_A(f(u \circ v)) \\
&= T_A(f(u) \circ f(v)) \\
&\geq \min\{T_A(f(u)) \circ T(f(v))\} \\
&= \min\{f^{-1}(T_A(u)), f^{-1}(T_A)(v)\},
\end{align*}
\]

\[
\begin{align*}
f^{-1}(I_A)(u \circ v) &= I_A(f(u \circ v)) \\
&= I_A(f(u) \circ f(v)) \\
&\geq \min\{I_A(f(u)) \circ I(f(v))\} \\
&= \min\{f^{-1}(I_A(u)), f^{-1}(I_A)(v)\},
\end{align*}
\]

\[
\begin{align*}
f^{-1}(F_A)(u \circ v) &= F_A(f(u \circ v)) \\
&= F_A(f(u) \circ f(v)) \\
&\leq \max\{F_A(f(u)) \circ F(f(v))\} \\
&= \max\{f^{-1}(F_A(u)), f^{-1}(F_A)(v)\}.
\end{align*}
\]

Hence, \( f \) is a single-valued neutrosophic continuous function from \( K_1 \) to \( K_2 \). \( \square \)

Proposition 3. Let \( \chi_1 \) and \( \chi_2 \) be two SNTSs on \( K \). Then, each homomorphism \( f: (K, \chi_1) \rightarrow (K, \chi_2) \) is a single-valued neutrosophic continuous function.

Proof. Let \((K, \chi_1)\) and \((K, \chi_2)\) be two SNTSs, where \( K \) is a \( K \)-algebra. To prove the above result, it is enough to show that result is false for a particular topology. Let \( A = (T_A, I_A, F_A) \) and \( B = (T_B, I_B, F_B) \) be two SNSs in \( K \). Take \( \chi_1 = \{0_{SN}, 1_{SN}, A\} \) and \( \chi_2 = \{0_{SN}, 1_{SN}, B\} \). If \( f: (K, \chi_1) \rightarrow (K, \chi_2) \), defined by \( f(u) = e \circ u \), for all \( u \in K \), then \( f \) is a homomorphism. Now, for \( u \in A, v \in \chi_2 \), \( (f^{-1}(B))(u) = B(f(u)) = B(e \circ u) = B(u), \forall u \in K \), i.e., \( f^{-1}(B) = B \). Therefore, \( (f^{-1}(B)) \notin \chi_1 \). Hence, \( f \) is not a single-valued neutrosophic continuous mapping. \( \square \)

Definition 13. Let \( K = (G, \cdot, \circ, e) \) be a \( K \)-algebra and \( \chi \) be a SNT on \( K \). Let \( A \) be a single-valued neutrosophic \( K \)-algebra (\( K \)-subalgebra) of \( K \) and \( \chi_A \) be a SNT on \( A \). Then, \( A \) is said to be a single-valued neutrosophic topological \( K \)-algebra (\( K \)-subalgebra) on \( K \) if the self mapping \( \rho_a: (A, \chi_A) \rightarrow (A, \chi_A) \) defined as \( \rho_a(u) = u \circ a, \forall a \in K \), is a relatively single-valued neutrosophic continuous mapping.
Theorem 2. Let $\chi_1$ and $\chi_2$ be two SNTSs on $K_1$ and $K_2$, respectively, and $f : K_1 \to K_2$ be a homomorphism such that $f^{-1}(\chi_2) = \chi_1$. If $A = \{T_A, I_A, F_A\}$ is a single-valued neutrosophic topological $K$-algebra of $K_2$, then $f^{-1}(A)$ is a single-valued neutrosophic topological $K$-algebra of $K_1$.

Proof. Let $A = \{T_A, I_A, F_A\}$ be a single-valued neutrosophic topological $K$-algebra of $K_2$. To prove that $f^{-1}(A)$ is a single-valued neutrosophic topological $K$-algebra of $K_1$. Let for any $u, v \in K_1$,

$$T_{f^{-1}(A)}(u \circ v) = T_A(f(u \circ v)) \geq \min\{T_A(f(u)), T_A(f(v))\} = \min\{T_{f^{-1}(A)}(u), T_{f^{-1}(A)}(v)\},$$

$$I_{f^{-1}(A)}(u \circ v) = I_A(f(u \circ v)) \geq \min\{I_A(f(u)), I_A(f(v))\} = \min\{I_{f^{-1}(A)}(u), I_{f^{-1}(A)}(v)\},$$

$$F_{f^{-1}(A)}(u \circ v) = F_A(f(u \circ v)) \geq \max\{F_A(f(u)), F_A(f(v))\} = \max\{F_{f^{-1}(A)}(u), F_{f^{-1}(A)}(v)\}.$$

Hence, $f^{-1}(A)$ is a single-valued neutrosophic $K$-algebra of $K_1$.

Now, we prove that $f^{-1}(A)$ is single-valued neutrosophic topological $K$-algebra of $K_1$. Since $f$ is a single-valued neutrosophic continuous function, then by proposition 3.1, $f$ is also a relatively single-valued neutrosophic continuous function which maps $(f^{-1}(A), \chi_{f^{-1}(A)})$ to $(A, \chi_A)$.

Let $a \in K_1$ and $Y$ be a SNS in $\chi_A$, and let $X$ be a SNS in $\chi_{f^{-1}(A)}$ such that

$$f^{-1}(Y) = X. \quad (1)$$

We are to prove that $\rho_a : (f^{-1}(A), \chi_{f^{-1}(A)}) \to (f^{-1}(A), \chi_{f^{-1}(A)})$ is relatively single-valued neutrosophic continuous mapping, then for any $a \in K_1$, we have

$$T_{\rho_a^{-1}(X)}(u) = T_X(\rho_a(u)) = T_X(u \circ a) = T_{f^{-1}(Y)}(u \circ a) = T_{\chi}(f(u \circ a)) = T_{\chi}(f(u)) \circ f(a) = T_{\rho_f(a)}(f(a)) = T_{\chi}(\rho_f(a)) = T_{\rho_a^{-1}(Y)}(u).$$

$$I_{\rho_a^{-1}(X)}(u) = I_X(\rho_a(u)) = I_X(u \circ a) = I_{f^{-1}(Y)}(u \circ a) = I_{\chi}(f(u \circ a)) = I_{\chi}(f(u)) \circ f(a) = I_{\rho_f(a)}(f(a)) = I_{\chi}(\rho_f(a)) = I_{\rho_a^{-1}(Y)}(u),$$

$$F_{\rho_a^{-1}(X)}(u) = F_X(\rho_a(u)) = F_X(u \circ a) = F_{f^{-1}(Y)}(u \circ a) = F_{\chi}(f(u \circ a)) = F_{\chi}(f(u)) \circ f(a) = F_{\rho_f(a)}(f(a)) = F_{\chi}(\rho_f(a)) = F_{\rho_a^{-1}(Y)}(u).$$

It concludes that $\rho_a^{-1}(X) = f^{-1}(\rho_{f(a)}^{-1}(Y))$. Thus, $\rho_a^{-1}(X) \cap f^{-1}(A) = f^{-1}(\rho_{f(a)}^{-1}(Y)) \cap f^{-1}(A)$ is a SNS in $f^{-1}(A)$ and a SNS in $\chi_{f^{-1}(A)}$. Hence, $f^{-1}(A)$ and a single-valued neutrosophic topological $K$-algebra of $K$. Hence, the proof. 

Theorem 3. Let $(K_1, \chi_1)$ and $(K_2, \chi_2)$ be two SNTSs on $K_1$ and $K_2$, respectively, and let $f$ be a bijective homomorphism of $K_1$ into $K_2$ such that $f^{-1}(\chi_2) = \chi_1$. If $A$ is a single-valued neutrosophic topological $K$-algebra of $K_1$, then $f(A)$ is a single-valued neutrosophic topological $K$-algebra of $K_2$.

Proof. Suppose that $A = \{T_A, I_A, F_A\}$ is a SN topological $K$-algebra of $K_1$. To prove that $f(A)$ is a single-valued neutrosophic topological $K$-algebra of $K_2$, let, for $u, v \in K_2,$
\[ f(A) = (f_{\sup}(T_A)(v), f_{\sup}(I_A)(v), f_{\inf}(F_A)(v)). \]

Let \( a_0 \in f^{-1}(u), b_0 \in f^{-1}(v) \) such that
\[
\begin{align*}
\sup_{x \in f^{-1}(u)} T_A(x) &= T_A(a_0), \quad \sup_{x \in f^{-1}(v)} T_A(x) = T_A(b_0), \\
\sup_{x \in f^{-1}(u)} I_A(x) &= I_A(a_0), \quad \sup_{x \in f^{-1}(v)} I_A(x) = I_A(b_0), \\
\inf_{x \in f^{-1}(u)} F_A(x) &= F_A(a_0), \quad \inf_{x \in f^{-1}(v)} F_A(x) = F_A(b_0).
\end{align*}
\]

Now,
\[
\begin{align*}
T_{f(A)}(u \odot v) &= \sup_{x \in f^{-1}(u \odot v)} T_A(x) \\
&\geq T_A(a_0, b_0) \\
&\geq \min\{T_A(a_0), T_A(b_0)\} \\
&= \min\{\sup_{x \in f^{-1}(u)} T_A(x), \sup_{x \in f^{-1}(v)} T_A(x)\} \\
&= \min\{T_{f(A)}(u), T_{f(A)}(v)\},
\end{align*}
\]
\[
\begin{align*}
I_{f(A)}(u \odot v) &= \sup_{x \in f^{-1}(u \odot v)} I_A(x) \\
&\geq I_A(a_0, b_0) \\
&\geq \min\{I_A(a_0), I_A(b_0)\} \\
&= \min\{\sup_{x \in f^{-1}(u)} I_A(x), \sup_{x \in f^{-1}(v)} I_A(x)\} \\
&= \min\{I_{f(A)}(u), I_{f(A)}(v)\},
\end{align*}
\]
\[
\begin{align*}
F_{f(A)}(u \odot v) &= \inf_{x \in f^{-1}(u \odot v)} F_A(x) \\
&\leq F_A(a_0, b_0) \\
&\leq \max\{F_A(a_0), F_A(b_0)\} \\
&= \max\{\inf_{x \in f^{-1}(u)} F_A(x), \inf_{x \in f^{-1}(v)} F_A(x)\} \\
&= \max\{F_{f(A)}(u), F_{f(A)}(v)\}.
\end{align*}
\]

Hence, \( f(A) \) is a single-valued neutrosophic K-subalgebra of \( K_2 \). Now, we prove that the self mapping \( \rho_b : (f(A), \chi_{f(A)}) \to (f(A), \chi_{f(A)}) \), defined by \( \rho_b(v) = v \odot b \), for all \( b \in K_2 \), is a relatively single-valued neutrosophic continuous mapping. Let \( Y_A \) be a SNS in \( \chi_A \), there exists a SNS “\( Y \)” in \( \chi_1 \) such that \( Y_A = Y \cap A \). We show that for a SNS in \( \chi_{f(A)} \),
\[
\rho^{-1}_b(Y_{f(A)}) \cap f(A) \in \chi_{f(A)}
\]

Since \( f \) is an injective mapping, then \( f(Y_A) = f(Y \cap A) = f(Y) \cap f(A) \) is a SNS in \( \chi_{f(A)} \) which shows that \( f \) is relatively single-valued neutrosophic open. In addition, \( f \) is surjective, then for all \( b \in K_2, a = f(b) \), where \( a \in K_1 \).
Now,

\[ T_{f^{-1}(\rho^{-1}_b(Y_{f(A)}))}(u) = T_{f^{-1}(\rho^{-1}_a(Y_{f(A)}))}(u) 
= T_{\rho^{-1}_a(Y_{f(A)})}(f(u)) 
= T_{\rho^{-1}_a(Y_{f(A)})}(f(u)) 
= T_{\rho^{-1}_a(Y_{f(A)})}(f(u) \circ f(a)) 
= T_{f^{-1}(Y_{f(A)})}(u \circ a) 
= T_{f^{-1}(Y_{f(A)})}(\rho_a(u)) 
= T_{f^{-1}(Y_{f(A)})}(f^{-1}(Y_{f(A)}))(u), \]

\[ I_{f^{-1}(\rho^{-1}_b(Y_{f(A)}))}(u) = I_{f^{-1}(\rho^{-1}_a(Y_{f(A)}))}(u) 
= I_{\rho^{-1}_a(Y_{f(A)})}(f(u)) 
= I_{\rho^{-1}_a(Y_{f(A)})}(f(u)) 
= I_{\rho^{-1}_a(Y_{f(A)})}(f(u) \circ f(a)) 
= I_{f^{-1}(Y_{f(A)})}(u \circ a) 
= I_{f^{-1}(Y_{f(A)})}(\rho_a(u)) 
= I_{f^{-1}(Y_{f(A)})}(f^{-1}(Y_{f(A)}))(u), \]

\[ F_{f^{-1}(\rho^{-1}_b(Y_{f(A)}))}(u) = F_{f^{-1}(\rho^{-1}_a(Y_{f(A)}))}(u) 
= F_{\rho^{-1}_a(Y_{f(A)})}(f(u)) 
= F_{\rho^{-1}_a(Y_{f(A)})}(f(u)) 
= F_{\rho^{-1}_a(Y_{f(A)})}(f(u) \circ f(a)) 
= F_{f^{-1}(Y_{f(A)})}(u \circ a) 
= F_{f^{-1}(Y_{f(A)})}(\rho_a(u)) 
= F_{f^{-1}(Y_{f(A)})}(f^{-1}(Y_{f(A)}))(u). \]

This implies that \( f^{-1}(\rho^{-1}_b((Y_{f(A)}))) = \rho^{-1}_a(f^{-1}(Y_{f(A)})) \). Since \( \rho_a : (\mathcal{A}, \chi_{\mathcal{A}}) \to (\mathcal{A}, \chi_{\mathcal{A}}) \) is relatively single-valued neutrosophic continuous mapping and \( f \) is relatively single-valued neutrosophic continues mapping from \( (\mathcal{A}, \chi_{\mathcal{A}}) \) into \( (f(\mathcal{A}), \chi_{f(\mathcal{A})}) \), \( f^{-1}(\rho^{-1}_b((Y_{f(A)}))) \cap \mathcal{A} = \rho^{-1}_a(f^{-1}(Y_{f(A)})) \cap \mathcal{A} \) is a SNS in \( \chi_{\mathcal{A}} \). Hence, \( f(f^{-1}(\rho^{-1}_b((Y_{f(A)}))) \cap \mathcal{A}) = \rho^{-1}_a((Y_{f(A)})) \cap f(\mathcal{A}) \) is a SNS in \( \chi_{\mathcal{A}} \), which completes the proof. \( \square \)

**Example 2.** Let \( K = (G, \cdot, \ominus, e) \) be a K-algebra, where \( G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\} \) is the cyclic group of order 9 and Caley’s table for \( \circ \) is given in Example 1. We define a SNS as:

\[ \mathcal{A} = \{(e, 0.4, 0.5, 0.8), (s, 0.3, 0.4, 0.6)\}, \]

\[ \mathcal{B} = \{(e, 0.3, 0.4, 0.8), (s, 0.2, 0.3, 0.6)\}, \]

for all \( s \neq e \in G \), where \( \mathcal{A}, \mathcal{B} \in [0, 1] \). The collection \( \chi_K = \{\ominus_{SN}, 1_{SN}, \mathcal{A}, \mathcal{B}\} \) of SNSs of \( K \) is a SNT on \( K \) and \((K, \chi_K)\) is a SNTS. Let \( \mathcal{C} \) be a SNS in \( K \), defined as:

\[ \mathcal{C} = \{(e, 0.7, 0.5, 0.2), (s, 0.5, 0.4, 0.6)\}, \forall s \neq e \in G. \]
Clearly, $C$ is a single-valued neutrosophic $K$-subalgebra of $C$. By direct calculations relative topology $\chi_C$ is obtained as $\chi_C = \{(O_A, 1_A, A)\}$. Then, the pair $(C, \chi_C)$ is a single-valued neutrosophic subspace of $(K, \chi_K)$. We show that $C$ is a single-valued neutrosophic topological $K$-subalgebra of $K$, i.e., the self mapping $\rho_a : (C, \chi_C) \rightarrow (C, \chi_C)$ defined by $\rho_a(u) = u \cup a, \forall a \in K$ is relatively single-valued neutrosophic continuous mapping, i.e., for a SNOS $A$ in $(C, \chi_C)$, $\rho_a^{-1}(A) \cap C \in \chi_C$. Since $\rho_a$ is homomorphism, then $\rho_a^{-1}(A) \cap C = A \in \chi_C$. Therefore, $\rho_a : (C, \chi_C) \rightarrow (C, \chi_C)$ is relatively single-valued neutrosophic continuous mapping. Hence, $C$ is a single-valued neutrosophic topological $K$-algebra of $K$.

**Example 3.** Let $K = (G, \cdot, \circ, e)$ be a $K$-algebra, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ is the cyclic group of order 9 and Cayley’s table for $\circ$ is given in Example 3.1. We define a SNS as:

$$A = \{(e, 0, 4, 0, 5, 0, 8), (s, 0, 3, 0, 4, 0, 6)\},$$

$$B = \{(e, 0, 3, 0, 4, 0, 8), (s, 0, 2, 0, 3, 0, 6)\},$$

$$D = \{(e, 0, 2, 0, 1, 0, 3), (s, 0, 1, 0, 1, 0, 5)\},$$

for all $s \neq e \in G$, where $A, B \in [0, 1]$. The collection $\chi_1 = \{O_{SN}, 1_{SN}, D\}$ and $\chi_2 = \{O_{SN}, 1_{SN}, A, B\}$ of SNSs of $K$ are SNTSs on $K$ and $(K, \chi_1), (K, \chi_2)$ be two SNTSs. Let $C$ be a SNS in $(K, \chi_2)$, defined as:

$$\mathcal{C} = \{(e, 0, 7, 0, 5, 0, 2), (s, 0, 5, 0, 4, 0, 6)\}, \forall s \neq e \in G.$$

Now, Let $f : (K, \chi_1) \rightarrow (K, \chi_2)$ be a homomorphism such that $f^{-1}(\chi_2) = \chi_1$ (we have not consider $K$ to be distinct), then, by Proposition 3, $f$ is a single-valued neutrosophic continuous function and $f$ is also relatively single-valued neutrosophic continues mapping from $(K, \chi_1)$ into $(K, \chi_2)$. Since $C$ is a SNS in $(K, \chi_2)$ and with relative topology $\chi_C = \{O_A, 1_A, A\}$ is also a single-valued neutrosophic topological $K$-algebra of $(K, \chi_2)$. We prove that $f^{-1}(\mathcal{C})$ is a single-valued neutrosophic topological $K$-algebra in $(K, \chi_1)$. Since $f$ is a continuous function, then, by Definition 3, $f^{-1}(\mathcal{C})$ is a single-valued neutrosophic $K$-subalgebra in $(K, \chi_1)$. To prove that $f^{-1}(\mathcal{C})$ is a single-valued neutrosophic topological $K$-algebra, then for $b \in K_1$ take

$$\rho_b : (f^{-1}(\mathcal{C}), \chi_{f^{-1}(\mathcal{C})}) \rightarrow (f^{-1}(\mathcal{C}), \chi_{f^{-1}(\mathcal{C})}),$$

for $A \in \chi_{f^{-1}(\mathcal{C})}, \rho_b^{-1}(A) \cap f^{-1}(\mathcal{C}) \in \chi_{f^{-1}(\mathcal{C})}$ which shows that $f^{-1}(\mathcal{C})$ is a single-valued neutrosophic topological $K$-algebra in $(K, \chi_1)$. Similarly, we can show that $f(\mathcal{C})$ is a a single-valued neutrosophic topological $K$-algebra in $(K, \chi_2)$ by considering a bijective homomorphism.

**Definition 14.** Let $\chi$ be a SNT on $K$ and $(K, \chi)$ be a SNTS. Then, $(K, \chi)$ is called single-valued neutrosophic $C_5$-disconnected topological space if there exist a SNOS and SNCS $\mathcal{H}$ such that $\mathcal{H} = (\mathcal{T}_H, I_H, F_H) \neq 1_{SN}$ and $\mathcal{H} = (\mathcal{T}_H, I_H, F_H) \neq O_{SN}$, otherwise $(K, \chi)$ is called single-valued neutrosophic $C_5$-connected.

**Example 4.** Every indiscrete SNT space on $K$ is $C_5$-connected.

**Proposition 4.** Let $(K_1, \chi_1)$ and $(K_2, \chi_2)$ be two SNTSs and $f : (K_1, \chi_1) \rightarrow (K_2, \chi_2)$ be a surjective single-valued neutrosophic continuous mapping. If $(K_1, \chi_1)$ is a single-valued neutrosophic $C_5$-connected space, then $(K_2, \chi_2)$ is also a single-valued neutrosophic $C_5$-connected space.

**Proof.** Suppose on contrary that $(K_2, \chi_2)$ is a single-valued neutrosophic $C_5$-disconnected space. Then, by Definition 14, there exist both SNOS and SNCS $\mathcal{H}$ be such that $\mathcal{H} \neq 1_{SN}$ and $\mathcal{H} \neq O_{SN}$. Since $f$ is a single-valued neutrosophic continuous and onto function, so $f^{-1}(\mathcal{H}) = 1_{SN}$ or $f^{-1}(\mathcal{H}) = O_{SN}$, where $f^{-1}(\mathcal{H})$ is both SNOS and SNCS. Therefore,

$$\mathcal{H} = f(f^{-1}(\mathcal{H})) = f(1_{SN}) = 1_{SN} \tag{2}$$

and

$$\mathcal{H} = f(f^{-1}(\mathcal{H})) = f(\mathcal{O}_{SN}) = \mathcal{O}_{SN}, \tag{3}$$
a contradiction. Hence, \((K_2, \chi_2)\) is a single-valued neutrosophic \(C_3\)-connected space. \(\square\)

**Corollary 1.** Let \(\chi\) be a SNT on \(K\). Then, \((K, \chi)\) is called a single-valued neutrosophic \(C_3\)-connected space if and only if there does not exist a single-valued neutrosophic continuous map \(f : (K, \chi) \rightarrow (\mathcal{F}_T, \chi_T)\) such that \(f \neq 1_{SN}\) and \(f \neq \emptyset_{SN}\).

**Definition 15.** Let \(A = \{T_A, I_A, F_A\}\) be a SNS in \(K\). Let \(\chi\) be a SNT on \(K\). The interior and closure of \(A\) in \(K\) is defined as:

\[ A^{Int}: \text{The union of SNOSs which contained in } A. \]

\[ A^{Clo}: \text{The intersection of SNCSs for which } A \text{ is a subset of these SNCSs.} \]

**Remark 1.** Being union of SNOS \(A^{Int}\) is a SNO and \(A^{Clo}\) being intersection of SNCS is SNC.

**Theorem 4.** Let \(A\) be a SNS in a SNTS \((K, \chi)\). Then, \(A^{Int}\) is such an open set which is the largest open set of \(K\) contained in \(A\).

**Corollary 2.** \(A = \{T_A, I_A, F_A\}\) is a SNS in \(K\) if and only if \(A^{Int} = A\) and \(A = (T_A, I_A, F_A)\) is a SNCS in \(K\) if and only if \(A^{Clo} = A\).

**Proposition 5.** Let \(A\) be a SNS in \(K\). Then, following results hold for \(A\):

(i) \((1_{SN})^{Int} = 1_{SN}\).

(ii) \((1_{SN})^{Clo} = \emptyset_{SN}\).

(iii) \((\emptyset_{SN})^{Int} = (\emptyset_{SN})^{Clo}\).

(iv) \((\emptyset_{SN})^{Int} = (\emptyset_{SN})^{Clo}\).

**Definition 16.** Let \(K\) be a \(K\)-algebra and \(\chi\) be a SNT on \(K\). A SNOS \(A\) in \(K\) is said to be single-valued neutrosophic regular open if

\[ A = (A^{Clo})^{Int}. \] (4)

**Remark 2.** Every SNOS which is regular is single-valued neutrosophic open and every single-valued neutrosophic closed and open set is a single-valued neutrosophic regular open.

**Definition 17.** A single-valued neutrosophic super connected \(K\)-algebra is such a \(K\)-algebra in which there does not exist a single-valued neutrosophic regular open set \(A = \{T_A, I_A, F_A\}\) such that \(A \neq \emptyset_{SN}\) and \(A \neq 1_{SN}\). If there exists such a single-valued neutrosophic regular open set \(A = \{T_A, I_A, F_A\}\) such that \(A \neq \emptyset_{SN}\) and \(A \neq 1_{SN}\), then \(K\)-algebra is said to be a single-valued neutrosophic super disconnected.

**Example 5.** Let \(K = (G, \cdot, \circ, e)\) be a \(K\)-algebra, where \(G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}\) is the cyclic group of order 9 and Caley’s table for \(\circ\) is given in Example 1. We define a SNS as:

\[ A = \{(e, 0.2, 0.3, 0.8), (s, 0.1, 0.2, 0.6)\}. \]

Let \(\chi_K = \{0_{SN}, 1_{SN}, A\}\) be a SNT on \(K\) and let \(B = \{(e, 0.3, 0.3, 0.8), (s, 0.2, 0.2, 0.6)\}\) be a SNS in \(K\). Here

\begin{align*}
\text{SNOSs: } & 0_{SN} = \{0, 0, 1\}, 1_{SN} = \{1, 1, 0\}, A = \{(e, 0.2, 0.3, 0.8), (s, 0.1, 0.2, 0.6)\}. \\
\text{SNCSs: } & (0_{SN})^c = \{(0, 0, 1)\}^c = \{1, 1, 0\}, (1_{SN})^c = \{(1, 1, 0)\}^c = \{(0, 0, 1)\} = 0_{SN}, \\
\text{A}^c = \{(e, 0.2, 0.3, 0.8), (s, 0.1, 0.2, 0.6)\}^c = \{(e, 0.8, 0.3, 0.2), (s, 0.6, 0.2, 0.1)\} = A\end{align*}

(say).
Then, closure of $\mathcal{B}$ is the intersection of closed sets which contain $\mathcal{B}$. Therefore,

$$A' = \mathcal{B}^{\text{Clo}}.$$  \hspace{1cm} (5)

Now, interior of $\mathcal{B}$ is the union of open sets which contain in $\mathcal{B}$. Therefore,

$$\emptyset_{SN} \bigcup \mathcal{A} = \mathcal{A}$$

$$\mathcal{A} = \mathcal{B}^{\text{Int}}.$$  \hspace{1cm} (6)

Note that $(B^{\text{Clo}})^{\text{Clo}} = B^{\text{Clo}}$. Now, if we consider a SNS $\mathcal{A} = \{(e,0.2,0.3,0.8), (s,0.1,0.2,0.6)\}$ in a K-algebra $\mathcal{K}$ and if $\chi_{\mathcal{K}} = \{\emptyset_{SN}, 1_{SN}, \mathcal{A} \}$ is a SNT on $\mathcal{K}$. Then, $(\mathcal{A})^{\text{Clo}} = \mathcal{A}$ and $(\mathcal{A})^{\text{Int}} = \mathcal{A}$. Consequently,

$$\mathcal{A} = (\mathcal{A}^{\text{Clo}})^{\text{Int}},$$  \hspace{1cm} (7)

which shows that $\mathcal{A}$ is a SN regular open set in K-algebra $\mathcal{K}$. Since $\mathcal{A}$ is a SN regular open set in $\mathcal{K}$ and $\mathcal{A} \neq \emptyset_{SN}, \mathcal{A} \neq 1_{SN}$, then, by Definition 17, K-algebra $\mathcal{K}$ is a single-valued neutrosophic supper disconnected K-algebra.

**Proposition 6.** Let $\mathcal{K}$ be a K-algebra and let $\mathcal{A}$ be a SNOS. Then, the following statements are equivalent:

(i) A K-algebra is single-valued neutrosophic super connected.

(ii) $(\mathcal{A})^{\text{Clo}} = 1_{SN}$, for each SNOS $\mathcal{A} \neq \emptyset_{SN}$.

(iii) $(\mathcal{A})^{\text{Int}} = \emptyset_{SN}$, for each SNCS $\mathcal{A} \neq 1_{SN}$.

(iv) There do not exist SNOSs $\mathcal{A}, \mathcal{F}$ such that $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{A} \neq \emptyset_{SN} \neq \mathcal{F}$ in K-algebra $\mathcal{K}$.

**Definition 18.** Let $(\mathcal{K}, \chi)$ be a SNTS, where $\mathcal{K}$ is a K-algebra. Let $S$ be a collection of SNOSs in $\mathcal{K}$ denoted by $S = \{(T_{\mathcal{A}}, I_{\mathcal{A}}, F_{\mathcal{A}}) : j \in I\}$. Let $\mathcal{A}$ be a SNOS in $\mathcal{K}$. Then, $S$ is called a single-valued neutrosophic open covering of $\mathcal{A}$ if $\mathcal{A} \subseteq \bigcup S$.

**Definition 19.** Let $\mathcal{K}$ be a K-algebra and $(\mathcal{K}, \chi)$ be a SNTS. Let $L$ be a finite sub-collection of $S$. If $L$ is also a single-valued neutrosophic open covering of $\mathcal{A}$, then it is called a finite sub-covering of $S$ and $\mathcal{A}$ is called single-valued neutrosophic compact if each single-valued neutrosophic open covering $S$ of $\mathcal{A}$ has a finite sub-cover. Then, $(\mathcal{K}, \chi)$ is called compact K-algebra.

**Remark 3.** If either $\mathcal{K}$ is a finite K-algebra or $\chi$ is a finite topology on $\mathcal{K}$, i.e., consists of finite number of single-valued neutrosophic subsets of $\mathcal{K}$, then the SNT $(\mathcal{K}, \chi)$ is a single-valued neutrosophic compact topological space.

**Proposition 7.** Let $(\mathcal{K}_{1}, \chi_{1})$ and $(\mathcal{K}_{2}, \chi_{2})$ be two SNTSs and $f$ be a single-valued neutrosophic continuous mapping from $\mathcal{K}_{1}$ into $\mathcal{K}_{2}$. Let $\mathcal{A}$ be a SNOS in $(\mathcal{K}_{1}, \chi_{1})$. If $\mathcal{A}$ is single-valued neutrosophic compact in $(\mathcal{K}_{1}, \chi_{1})$, then $f(\mathcal{A})$ is single-valued neutrosophic compact in $(\mathcal{K}_{2}, \chi_{2})$.

**Proof.** Let $f : (\mathcal{K}_{1}, \chi_{1}) \rightarrow (\mathcal{K}_{2}, \chi_{2})$ be a single-valued neutrosophic continuous function. Let $S = \{f^{-1}(A_{j}) : j \in I\}$ be a single-valued neutrosophic open covering of $\mathcal{A}$ since $\mathcal{A}$ is a SNOS in $(\mathcal{K}_{1}, \chi_{1})$. Let $\mathcal{L} = (A_{j} : j \in I)$ be a single-valued neutrosophic open covering of $f(\mathcal{A})$. Since $\mathcal{A}$ is compact, there exists a single-valued neutrosophic finite sub-cover $\bigcup_{j=1}^{n} f^{-1}(A_{j})$ such that

$$\mathcal{A} \subseteq \bigcup_{j=1}^{n} f^{-1}(A_{j})$$

We have to prove that there also exists a finite sub-cover of $\mathcal{L}$ for $f(\mathcal{A})$ such that
\begin{equation*}
f(A) \subseteq \bigcup_{j=1}^{n} (A_j)
\end{equation*}

Now,
\begin{equation*}
A \subseteq \bigcup_{j=1}^{n} f^{-1}(A_j)
\end{equation*}
\begin{equation*}
f(A) \subseteq f(\bigcup_{j=1}^{n} f^{-1}(A_j))
\end{equation*}
\begin{equation*}
f(A) \subseteq \bigcup_{j=1}^{n} (f(f^{-1}(A_j)))
\end{equation*}
\begin{equation*}
f(A) \subseteq \bigcup_{j=1}^{n} (A_j).
\end{equation*}

Hence, \(f(A)\) is single-valued neutrosophic compact in \((\mathcal{K}_2, \chi_2)\). \(\square\)

**Definition 20.** A single-valued neutrosophic set \(A\) in a \(K\)-algebra \(K\) is called a single-valued neutrosophic point if
\[
\mathcal{T}_A(v) = \begin{cases} 
\alpha \in [0, 1], & \text{if } v = u \\
0, & \text{otherwise,}
\end{cases}
\]
\[
\mathcal{I}_A(v) = \begin{cases} 
\beta \in [0, 1], & \text{if } v = u \\
0, & \text{otherwise,}
\end{cases}
\]
\[
\mathcal{F}_A(v) = \begin{cases} 
\gamma \in [0, 1], & \text{if } v = u \\
0, & \text{otherwise,}
\end{cases}
\]
with support \(u\) and value \((\alpha, \beta, \gamma)\), denoted by \(u(\alpha, \beta, \gamma)\). This single-valued neutrosophic point is said to “belong to” a SNS \(A\), written as \(u(\alpha, \beta, \gamma) \in A\) if \(\mathcal{T}_A(u) \geq \alpha, \mathcal{I}_A(u) \geq \beta, \mathcal{F}_A(u) \leq \gamma\) and said to be “quasi-coincident with” a SNS \(A\), written as \(u(\alpha, \beta, \gamma) \in A\) if \(\mathcal{T}_A(u) + \alpha > 1, \mathcal{I}_A(u) + \beta > 1, \mathcal{F}_A(u) + \gamma < 1\).

**Definition 21.** Let \(K\) be a \(K\)-algebra and let \((\mathcal{K}, \chi)\) be a SNTS. Then, \((\mathcal{K}, \chi)\) is called a single-valued neutrosophic Hausdorff space if and only if, for any two distinct single-valued neutrosophic points \(u_1, u_2 \in \mathcal{K}\), there exist SNOSs \(B_1 = (T_{B_1}, I_{B_1}, F_{B_1}), B_2 = (T_{B_2}, I_{B_2}, F_{B_2})\) such that \(u_1 \in B_1, u_2 \in B_2\), i.e.,
\[
T_{B_1}(u_1) = 1, I_{B_1}(u_1) = 1, F_{B_1}(u_1) = 0,
\]
\[
T_{B_2}(u_2) = 1, I_{B_2}(u_2) = 1, F_{B_2}(u_2) = 0
\]
and satisfy the condition that \(B_1 \cap B_2 = \emptyset_{SN}\). Then, \((\mathcal{K}, \chi)\) is called single-valued neutrosophic Hausdorff space and \(K\)-algebra is said to be a Hausdorff \(K\)-algebra. In fact, \((\mathcal{K}, \chi)\) is a Hausdorff \(K\)-algebra.

**Example 6.** Let \(K = (G, \cdot, \circ, e)\) be a \(K\)-algebra and let \((\mathcal{K}, \chi_K)\) be a SNTS on \(K\), where
\[G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}\] is the cyclic group of order 9 and Cayley’s table for \(\circ\) is given in Example 1. We define two SNSs as \(A = \{(e, 1, 1, 0), (s, 0, 0, 1)\}\), \(B = \{(e, 0, 0, 1), (s, 1, 1, 0)\}\). Consider a single-valued neutrosophic point for \(e \in K\) such that
\[
T_A(e) = \begin{cases} 
0.3, & \text{if } e = u \\
0, & \text{otherwise,}
\end{cases}
\]
\[
I_A(e) = \begin{cases} 
0.2, & \text{if } e = u \\
0, & \text{otherwise,}
\end{cases}
\]
This single-valued neutrosophic point belongs to SNS “A” but not SNS “B”.

Now, for all \( s \neq e \in K \):

\[
\mathcal{F}_A(e) = \begin{cases} 
0.4, & \text{if } e = u \\
0, & \text{otherwise}
\end{cases}
\]

Then, \( e(0.3, 0.2, 0.4) \) is a single-valued neutrosophic point with support \( e \) and value \( (0.3, 0.2, 0.4) \). This single-valued neutrosophic point belongs to SNS “A” but not SNS “B”. Thus, \( e(0.3, 0.2, 0.4) \in A \) and \( e(0.3, 0.2, 0.4) \notin B \).

**Theorem 5.** Let \((K_1, \chi_1)\), \((K_2, \chi_2)\) be two SNTSs. Let \( f \) be a single-valued neutrosophic homomorphism from \((K_1, \chi_1)\) into \((K_2, \chi_2)\). Then, \((K_1, \chi_1)\) is a single-valued neutrosophic Hausdorff space if and only if \((K_2, \chi_2)\) is a single-valued neutrosophic Hausdorff K-algebra.

**Proof.** Let \((K_1, \chi_1)\), \((K_2, \chi_2)\) be two SNTSs. Let \( K_1 \) be a single-valued neutrosophic Hausdorff space, then, according to the Definition 21, there exist two SNOSs \( X \) and \( Y \) for two distinct single-valued neutrosophic points \( u_1, u_2 \in \chi_2 \) also \( a, b \in K_1 (a \neq b) \) such that \( X \cap Y = \emptyset_{SN} \).

Now, for \( w \in K_1 \), consider \((f^{-1}(u_1))(w) = u_1(f^{-1}(w))\), where \( u_1(f^{-1}(w)) = s \in (0, 1) \) if \( w = f^{-1}(a) \), otherwise 0. That is, \((f^{-1}(u_1))(w) = ((f^{-1}(u))(w))_1\). Therefore, we have \( f^{-1}(u_1) = (f^{-1}(u))(1) \).

Similarly, \( f^{-1}(u_2) = (f^{-1}(u))(2) \). Now, since \( f^{-1} \) is a single-valued neutrosophic continuous mapping from \( K_2 \) into \( K_1 \), there exist two SNOSs \( f(X) \) and \( f(Y) \) of \( u_1 \) and \( u_2 \), respectively, such that \( f(X) \cap f(Y) = f(\emptyset_{SN}) = \emptyset_{SN} \). This implies that \( K_2 \) is a single-valued neutrosophic Hausdorff K-algebra. The converse part can be proved similarly. \( \square \)

**Theorem 6.** Let \( f \) be a single-valued neutrosophic continuous function which is both one-one and onto, where \( f \) is a mapping from a single-valued neutrosophic compact K-algebra \( K_1 \) into a single-valued neutrosophic Hausdorff K-algebra \( K_2 \). Then, \( f \) is a homomorphism.

**Proof.** Let \( f : K_1 \rightarrow K_2 \) be a single-valued neutrosophic continuous bijective function from single-valued neutrosophic compact K-algebra \( K_1 \) into a single-valued neutrosophic Hausdorff K-algebra \( K_2 \). Since \( f \) is a single-valued neutrosophic continuous mapping from \( K_1 \) into \( K_2 \), \( f \) is a homomorphism. Since \( f \) is bijective, we only prove that \( f \) is single-valued neutrosophic closed. Let \( D = (T_D, I_D, F_D) \) be a single-valued neutrosophic closed in \( K_1 \). If \( D = \emptyset_{SN} \) is single-valued neutrosophic closed in \( K_1 \), then \( f(D) = \emptyset_{SN} \) is single-valued neutrosophic closed in \( K_2 \). However, if \( D \neq \emptyset_{SN} \), then \( D \) will be a single-valued neutrosophic compact, being subset of a single-valued neutrosophic compact K-algebra. Then, \( f(D) \), being single-valued neutrosophic continuous image of a single-valued neutrosophic compact K-algebra, is also single-valued neutrosophic compact. Therefore, \( K_2 \) is closed, which implies that mapping \( f \) is closed. Thus, \( f \) is a homomorphism. \( \square \)
4. Conclusions

Non-classical logic is considered as a powerful tool for inspecting uncertainty and indeterminacy found in real world problems. Being a great extension of classical logic, neutrosophic set theory is considered as a useful mathematical tool to cope up with uncertainties in science, technology, and computer science. We have used this mathematical model with a topological structure to investigate the uncertainty in \( K \)-algebras. We have introduced the notion of single-valued neutrosophic topological \( K \)-algebras and presented certain concepts, including continuous function between two topological on \( K \)-algebras, relatively continuous function and homomorphism. We have investigated the image and pre-image of single-valued neutrosophic topological \( K \)-algebras under this homomorphism. We have proposed some conclusive concepts, including single-valued neutrosophic compact \( K \)-algebras and single-valued neutrosophic Hausdorff \( K \)-algebras. We plan to extend our study to: (i) single-valued neutrosophic soft topological \( K \)-algebras; and (ii) bipolar neutrosophic soft topological \( K \)-algebras.

For other notations and terminologies, readers are referred to [21–26].

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References
11. June, Y.B.; Kim, S.J.; Smarandache, F. Interval neutrosophic sets with applications in \( BCK/BCI \)-algebra. Axioms 2018, 7, 23. [CrossRef]
12. Jun, Y.B.; Smarandache, F.; Song, S.Z.; Khan, M. Neutrosophic positive implicative \( N \)-ideals in \( BCK \)-algebras. Axioms 2018, 7, 3. [CrossRef]


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