Oriented Algebras and the Hochschild Cohomology Group

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Abstract: Koam and Pirashivili developed the equivariant version of Hochschild cohomology by mixing the standard chain complexes computing group with associative algebra cohomologies to obtain the bicomplex $\tilde{C}^\bullet_G(A, X)$. In this paper, we form a new bicomplex $\tilde{\mathcal{F}}^\bullet_G(A, X)$ by deleting the first column and the first row and reindexing. We show that $\tilde{H}^1_G(A, X)$ classifies the singular extensions of oriented algebras.

Keywords: group cohomology; hochschild cohomology; oriented algebras

1. Introduction

One of the main applications of homological algebra is the classical cohomology of associative algebras invented by Hochschild [1] in 1945. It is a particular case of general machinery developed by Cartan and Eilenberg. Let $A$ be an associative $k$-algebra and let $M$ be an $A$-$A$-bimodule. The low dimensional groups ($n \leq 2$) have well known interpretations of classical algebraic structures such as derivations and extensions [2–4].

Let $G$ be a group and $\epsilon : G \rightarrow \{\pm 1\}$ be a group homomorphism. An oriented algebra is an associative algebra $A$ equipped with a $G$-module structure $(g, a) \mapsto ga$, satisfying the condition

$$\gamma(ab) = \begin{cases} ga^b & \text{if } \epsilon(g) = +1 \\ g^b a & \text{if } \epsilon(g) = -1 \end{cases}. $$

Hence, oriented algebras are more general than $G$-algebras as well as algebras with involutions. The aim of this work is to prove that $\tilde{H}^1_G(A, X)$ classifies the singular extensions of oriented algebras. The construction is based on the possibility of mixing the standard chain complexes computing group with associative algebra cohomologies. We obtain a new bicomplex by deleting the first column and the first row and reindexing.

2. Preliminaries

In this section we fix some notations for the standard chain complexes associated to groups and associative algebras. We also introduce a bicomplex which we will use throughout the paper.

In what follows, $k$ denotes a ground commutative ring with the unit. All modules and algebras are considered over $k$. Moreover, we write $\otimes$ and $\text{Hom}$ instead of $\otimes_k$ and $\text{Hom}_k$. For a group $G$ and $G$-module $C$, we let $C^\bullet(G, C)$ denote the standard complex computing the group cohomology. Recall that

$$C^n(G, C) = \text{Maps}(G^n, C)$$
and the coboundary map $\partial : \text{Maps}(G^n, C) \to \text{Maps}(G^{n+1}, C)$ is given by

\[
\partial(a)(x_1, \cdots, x_{n+1}) = x_1a(x_2, \cdots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i a(x_1, \cdots, x_i x_{i+1}, \cdots, x_{n+1}) + (-1)^{n+1} a(x_1, \cdots, x_n).
\]

Therefore, by the definition

\[
H^n(G, C) = H^n(\mathcal{C}^*(G, C)),
\]

we will say that a cochain complex

\[
\mathcal{C}^* = C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \xrightarrow{\delta} \cdots
\]

is a $G$-complex if each module $C^n$ is endowed with a structure of $G$-module and each boundary is a $G$-homomorphism. If this is the case, we let $\mathcal{C}^*(G, C^*)$ be the total complex of the following bicomplex:

\[
\begin{array}{c}
\vdots \\
\vdots \\
C^0(G, C^2) \xrightarrow{\partial^0} C^1(G, C^2) \xrightarrow{\partial^1} C^2(G, C^2) \xrightarrow{\partial^2} \cdots \\
\vdots \\
\vdots \\
C^0(G, C^1) \xrightarrow{\partial^0} C^1(G, C^1) \xrightarrow{\partial^1} C^2(G, C^1) \xrightarrow{\partial^2} \cdots \\
\vdots \\
C^0(G, C^0) \xrightarrow{\partial^0} C^1(G, C^0) \xrightarrow{\partial^1} C^2(G, C^0) \xrightarrow{\partial^2} \cdots
\end{array}
\]

The cohomology of $\mathcal{C}^*(G, C^*)$ is denoted by $H^*(G, C^*)$ and is called the hyperco-homology of $G$ with coefficients in $C^*$.

Let $A$ be an associative $k$-algebra. Recall that the Hochschild cohomology of $A$ with coefficients in a $A$-bimodule $M$ is the cohomology of the following cochain complex:

\[
0 \to M \xrightarrow{\delta^0} \text{Hom}(A, M) \xrightarrow{\delta^1} \text{Hom}(A \otimes^2, M) \xrightarrow{\delta^2} \cdots
\]

where the coboundary map

\[
\delta^n : \text{Hom}(A^\otimes n, M) \to \text{Hom}(A^\otimes {n+1}, M)
\]

is given by

\[
\delta(f)(a_1, \cdots, a_{n+1}) = a_1f(a_2, \cdots, a_{n+1}) + \sum_{0 < i < n+1} (-1)^i f(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) + (-1)^{n+1} f(a_1, \cdots, a_n a_{n+1}).
\]

Hence, $H^n(A, M) = H^n(C^n(A, M))$, where $C^n(A, M) = \text{Hom}(A^\otimes n, M)$.

3. Oriented Algebras

In this section, we define oriented algebras and provide some examples.
Definition 1. An orientation is a pair \((G, \varepsilon)\) [5], where \(G\) is a group, and \(\varepsilon\) is a group homomorphism
\[
\varepsilon : G \rightarrow \{\pm 1\}
\]
If such orientation is fixed, then we say that \(G\) is an oriented group.

Example 1.

1. Any group \(G\) can be equipped with a trivial orientation: \(\varepsilon(g) = 1\) for all \(g \in G\).
2. For more interesting examples, we can consider the following:
   
   a) \(G = \{\pm 1\}\) and \(\varepsilon = \text{id}\);
   
   b) more generally, we can consider \(G = S_n\) and \(\varepsilon(\sigma) = \text{sgn}(\sigma)\);
   
   c) \(G = \{1, -1, i, -i\}\) and \(\varepsilon(1) = 1, \varepsilon(-1) = -1, \varepsilon(i) = 1, \varepsilon(-i) = -1\);

Definition 2. Let \(G\) be an oriented group and \(A\) be an associative algebra [5]. An oriented action of \((G, \varepsilon)\) on \(A\) is given by a map
\[
G \times A \rightarrow A,
\]
\((g, a) \mapsto ga\)
such that under this action \(A\) is a \(G\)-module and
\[
\varepsilon(g) = \begin{cases} +1 & \text{if } \varepsilon(g) = +1 \\ -1 & \text{if } \varepsilon(g) = -1 \end{cases}
\]

An oriented algebra over \((G, \varepsilon)\) is an associative algebra equipped with an oriented action of \((G, \varepsilon)\) on \(A\).

Example 2.

1. Observe that if \(G\) is equipped with a trivial orientation, then \(G\) acts on \(A\) via algebra automorphisms; therefore, in this case oriented algebra is nothing but a \(G\)-algebra in the classical sense.
2. Another interesting example is obtained when \(G = \{\pm 1\}\) and \(\varepsilon = \text{id}\). In this case \(A\) is nothing but involutive algebra. Recall that an involutive algebra is an associative algebra \(A\) together with a \(k\)-linear map
\[
A \rightarrow A
\]
\(a \mapsto \overline{a}\)
such that
\[
\overline{a + b} = \overline{a} + \overline{b}
\]
\[
\overline{ab} = \overline{b} \overline{a}
\]
\[
\overline{\overline{a}} = a.
\]
3. Let \(M\) be a \(G\)-module. Consider the tensor algebra
\[
T^*(M) = k \oplus M \oplus M^\otimes 2 \oplus \cdots \oplus M^\otimes n \oplus \cdots.
\]
Define an action of \(G\) on \(T^*(M)\) by
\[
\varepsilon(m_1 \otimes \cdots \otimes m_n) = \begin{cases} \varepsilon(m_1) \otimes \cdots \otimes \varepsilon(m_n) & \text{if } \varepsilon(g) = +1 \\ \varepsilon(m_n) \otimes \cdots \otimes \varepsilon(m_1) & \text{if } \varepsilon(g) = -1 \end{cases}
\]
One checks that this action on the tensor algebra defines an oriented algebra structure.
Definition 3. Let $A$ and $B$ be oriented algebras over an oriented group $(G, \varepsilon)$ [5]. A homomorphism of $G$-modules $f : A \to B$ is called a homomorphism of oriented algebras provided $f$ is a homomorphism of algebras.

4. Oriented Bimodules and Cohomology

Definition 4. Let $A$ be an oriented algebra over an oriented group $(G, \varepsilon)$ [5]. An oriented bimodule over $A$ is a usual bimodule $X$ together with a $G$-module structure on $X$ such that

$$g(ax) = \begin{cases} ga_x, & \text{if } \varepsilon(g) = +1 \\ a_xg, & \text{if } \varepsilon(g) = -1 \end{cases}$$

$$g(xa) = \begin{cases} gxg, & \text{if } \varepsilon(g) = +1 \\ xga, & \text{if } \varepsilon(g) = -1 \end{cases}.$$

If $X$ and $Y$ are oriented bimodules over an oriented algebra $A$, then a linear map $f : X \to Y$ is a homomorphism of oriented bimodules if

$$g(f(x)) = f(gx), \quad f(ax) = af(x), \quad \text{and} \quad f(xa) = f(x)a$$

for all $a \in A$, $x \in X$, and $g \in G$.

Let $A$ be an oriented algebra over an oriented group $(G, \varepsilon)$ and let $X$ be an oriented bimodule. For any $n \geq 0$ one defines an action of $G$ on $\text{Hom}(A \otimes^n, X)$ by

$$(g^f)(a_1, \ldots, a_n) = \begin{cases} g^f(s^{-1}a_1, \ldots, s^{-1}a_n), & \text{if } \varepsilon(g) = +1, \\ (-1)^{(n-1)(n-2)/2}g^f(s^{-1}a_n, \ldots, s^{-1}a_1), & \text{if } \varepsilon(g) = -1 \end{cases} \quad \text{(2)}$$

In particular, for $n = 1$ the action is independent on the parity of $\varepsilon(g)$.

Lemma 1. With the action of Equation (2) [5], the Hochschild complex

$$0 \to X \overset{\delta}{\to} \text{Hom}(A, X) \overset{\delta^1}{\to} \text{Hom}(A \otimes^2, X) \overset{\delta^2}{\to} \cdots$$

is a $G$-complex.

Thus, one can form the following bicomplex $C_{G}^*(A, X)$:

$$\cdots \to \text{Maps}(G^2, X) \overset{\delta''}{\to} \text{Maps}(G^2, \text{Hom}(A, X)) \overset{\delta'}{\to} \text{Maps}(G^2, \text{Hom}(A \otimes^2, X)) \overset{\delta''}{\to} \cdots$$

$$\cdots \to \text{Maps}(G, X) \overset{\delta''}{\to} \text{Maps}(G, \text{Hom}(A, X)) \overset{\delta'}{\to} \text{Maps}(G, \text{Hom}(A \otimes^2, X)) \overset{\delta''}{\to} \cdots$$

$$\cdots \to X \overset{\delta'}{\to} \text{Hom}(A, X) \overset{\delta^1}{\to} \text{Hom}(A \otimes^2, X) \overset{\delta'}{\to} \cdots$$

where the coboundary maps are given as follows:
The coboundary of every horizontal maps $\delta^i$ is given by

$$(\delta^i a)(g_1, \cdots, g_n, a_1, \cdots, a_{n+1}) = a_1 a(g_1, \cdots, g_n, a_2, \cdots, a_{n+1}) + \sum_{0<i<n+1} (-1)^i a(g_1, \cdots, g_i, a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) + (-1)^{n+1} a(g_1, \cdots, g_n, a_1, \cdots, a_n) a_{n+1}$$

The coboundary of the first vertical maps is given by

$$(\delta^0 f)(g_1, \cdots, g_{n+1}) = g_1 f(g_2, \cdots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}). + (-1)^{n+1} f(g_1, \cdots, g_n)$$

The coboundary of the second vertical maps when $\varepsilon(g) = \pm 1$ is given by

$$(\delta^\varepsilon \beta)(g_1, \cdots, g_{n+1}, a) = g_1 \beta(g_2, \cdots, g_{n+1}, \varepsilon^{-1} a) + \sum_{i=1}^{n} (-1)^i \beta(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}, a) + (-1)^{n+1} \beta(g_1, \cdots, g_n, a)$$

The coboundary of the third vertical maps when $\varepsilon(g) = +1$ is given by

$$(\delta^+ \gamma)(g_1, \cdots, g_{n+1}, a, b) = g_1 \gamma(g_2, \cdots, g_{n+1}, \varepsilon^{-1} a, \varepsilon^{-1} b) + \sum_{i=1}^{n} (-1)^i \gamma(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}, a, b) + (-1)^{n+1} \gamma(g_1, \cdots, g_n, a, b)$$

The coboundary of the third vertical maps when $\varepsilon(g) = -1$ is given by

$$(\delta^- \gamma)(g_1, \cdots, g_{n+1}, a, b) = g_1 \gamma(g_2, \cdots, g_{n+1}, \varepsilon^{-1} b, \varepsilon^{-1} a) + \sum_{i=1}^{n} (-1)^i \gamma(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}, a, b) + (-1)^{n+1} \gamma(g_1, \cdots, g_n, a, b)$$

**Definition 5.** The homologies of the total complex is denoted by $H^n_C(A, X)$ where $n \geq 0$ [5].

**Example 3.**

1. Let $A$ be separable $k$-algebra. Since $H^q(A, X) = 0$ for $q > 0$, then we obtain $H^n_C(A, X) = H^n(C, H^0(A, X))$.

2. Let $G$ be a cyclic group of order 2, $k = \mathbb{Z}$, and $d$ be an integer which is square-free:

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 1 \pmod{4} \\ -1 + \sqrt{d} & \text{if } d \not\equiv 1 \pmod{4} \\ 2 & \text{if } d \equiv 0 \pmod{4} \end{cases}$$
so that
\[
\begin{align*}
\omega^2 - d &= 0 \quad \text{if } d \equiv 1 \pmod{4} \\
\omega^2 + \omega - \frac{d - 1}{4} &= 0 \quad \text{if } d \not\equiv 1 \pmod{4}.
\end{align*}
\]

Now, let \( A = \mathbb{Z}[\omega] = \{ m + n\omega \mid m, n \in \mathbb{Z} \} \). We define an action of \( G \) on \( A \) by
\[
t(\omega) = \omega
\]
where
\[
\omega = \begin{cases} 
-\omega & \text{if } d \equiv 1 \pmod{4} \\
\frac{-1 - \sqrt{d}}{2} & \text{if } d \not\equiv 1 \pmod{4}.
\end{cases}
\]

Hence, we obtain the following bicomplex:

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
1-t & t-1 & 1-t & 1-t \\
X & X^{tw} & X^{tw} & X^{tw} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1+t & 1+t & 1+t & 1-t \\
X & X^{tw} & X^{tw} & X^{tw} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1-t & t-1 & 1-t & 1-t \\
X & X^{tw} & X^{tw} & X^{tw} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1+t & 1+t & 1+t & 1-t \\
X & X^{tw} & X^{tw} & X^{tw} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
& & & \\
\end{array}
\]

where \( X^{tw} \) as an abelian group is \( X \), but to make difference between \( X \) and \( X^{tw} \) for an element \( x \in X \), we write \( \tau(x) \in X^{tw} \), the corresponding element in \( X^{tw} \). The action of \( G \) on \( X^{tw} \) is given by
\[
t(\tau(x)) = -\tau(t(x)).
\]

The coboundary maps of \( \delta' \) and \( \delta'' \) are given as follows:

- For \( \omega = \sqrt{d} \), one has
  \[
  \delta'(x) = \omega \tau(x) - \tau(x) \omega \\
  \delta''(\tau(x)) = \omega x + x \omega.
  \]

- For \( \omega = \frac{-1 + \sqrt{d}}{2} \) one has
  \[
  \delta'(x) = \omega \tau(x) - \tau(x) \omega \\
  \delta''(\tau(x)) = \omega x + x \omega + x.
  \]

Now, let \( X = \mathbb{Z}[\omega] \) and by computing explicitly the cohomology of the total complex when \( \omega = \sqrt{d} \) we obtain the following:
\[
\begin{align*}
H^0 &= \mathbb{Z} \\
H^1 &= \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\
H^2 &= \mathbb{Z}/10 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H^3 &= \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\end{align*}
\]
Similarly, by computing explicitly the cohomology of the total complex when \( \omega = \frac{-1 + \sqrt{d}}{2} \) we obtain the following:

\[
\begin{align*}
H^0 &= \mathbb{Z} \\
H^1 &= \mathbb{Z}/2 \oplus \mathbb{Z} \\
H^2 &= \mathbb{Z}/10 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
H^3 &= \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/10.
\end{align*}
\]

Now, we will form a new double complex \( \tilde{F}_G^* (A, X) \), which is obtained by deleting the first column and the first row and reindexing.

**Definition 6.** The homologies of the total complex of \( \tilde{F}_G^* (A, X) \) is denoted by \( \tilde{H}_1^* G (A, X) \) where \( n \geq 0 \) [5].

**5. Classification of Singular Extensions of Oriented Algebras**

In [5], we proved that \( \tilde{H}_1^* G (A, X) \) classifies the singular extensions of oriented algebras. Here we obtain a similar result.

**Definition 7.** Let \( A \) be an oriented algebra over an oriented group \( (G, \varepsilon) \) [5]. Moreover, let \( X \) be an oriented bimodule over \( A \). A singular extension of \( A \) by \( X \) is a \( k \)-split short exact sequence of \( G \)-modules

\[
0 \rightarrow X \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0
\]

where \( B \) is also an oriented algebra over \( (G, \varepsilon) \). Furthermore, \( p \) is a homomorphism of oriented algebras and \( i \) is homomorphism of \( G \)-modules such that

\[
\begin{align*}
i(x_1)i(x_2) &= 0 \\
i(x)b &= i(xp(b)) \\
b_i(x) &= i(p(b)x)
\end{align*}
\]

for all \( x, x_1, x_2 \in X \) and \( b \in B \).

**Theorem 1.** Let \( A \) be an oriented algebra over an oriented group \( (G, \varepsilon) \). Moreover, let \( X \) be an oriented bimodule over \( A \). Then there is one-to-one correspondence between equivalence classes of extensions of \( A \) by \( X \) and \( \tilde{H}_1^* G (A, X) \).
Before giving the proof, observe that $\mathcal{H}^1_G(A, X) = \mathcal{Z}^1_G(A, X) / \mathcal{B}^1_G(A, X)$, where $\mathcal{Z}^1_G(A, X)$ is the collection of pairs $(\alpha, \beta)$. Here $\alpha \in \mathcal{Maps}(G^2, \text{Hom}(A, X))$ and $\beta \in \mathcal{Maps}(G, \text{Hom}(A \otimes^2 X))$, satisfying the following conditions:

$$\begin{align*}
\alpha(gh, k, a) &= \delta \alpha(h, k, s^{-1}a) + \alpha(g, hk, a) - \alpha(g, h, a) \\
\alpha_1 a_1 a_2 - \alpha(g_1, g_2, a_1 a_2) + \alpha(g_1, g_2, a_1) a_2 &= \\
&= \begin{cases} 
\beta(g, a_1, a_2) - \beta(gh, a_1, a_2) - \delta \beta(h, s^{-1}a_1, s^{-1}a_2), & \text{if } \epsilon(g) = +1 \\
\beta(g, a_1, a_2) - \beta(gh, a_1, a_2) - \delta \beta(h, s^{-1}a_2, s^{-1}a_1), & \text{if } \epsilon(g) = -1 
\end{cases} \\
\beta(g, a_1, a_2) &= \beta(g, a_1 a_2, a_3) + \beta(g, a_1, a_2 a_3) - \beta(g, a_1, a_2) a_3 = 0.
\end{align*}$$

Observe that the last equality simply states that $\beta$ is a $G$-Hochschild 2-cocycle. Moreover, $(\alpha, \beta) \in \mathcal{B}^1_G(A, X)$ if and only if there exists $\gamma \in \mathcal{Maps}(G, \text{Hom}(A, X))$ such that

$$\beta(g, a_1, a_2) = a_1 \gamma(g, a_2) - \gamma(g, a_1 a_2) + \gamma(g, a_1) a_2$$

and

$$\alpha(g, h, a) = \delta \gamma(h, s^{-1}a) - \gamma(gh, a) + \gamma(g, a).$$

**Proof.** Let us start with a singular extension as above. To simplify the notation we will assume that $X$ is a submodule of $B$ and $i(x) = x$. Choose a linear map $s : A \rightarrow B$ such that $ps = id_A$. One defines

$$\alpha \in \mathcal{Maps}(G^2, \text{Hom}(A, X))$$

and

$$\beta \in \mathcal{Maps}(G, \text{Hom}(A \otimes^2 X))$$

by

$$\alpha(g, h, a) = s(gh, a) - \delta s(h, s^{-1}a) - s(g, a)$$

and

$$\beta(g, a_1, a_2) = s(g, a_1) s(g, a_2) - s(g, a_1 a_2).$$

We claim that $(\alpha, \beta) \in \mathcal{Z}^1_G(A, X)$. By the classical argument, $\beta$ is a $G$-Hochschild 2-cocycle. Next, we have

$$\delta \alpha(h, k, s^{-1}a) + \alpha(gh, k, a) - \alpha(g, h, a) = \delta s(hk, s^{-1}a) - \delta s(k, h^{-1} s^{-1}a) - \delta s(h, s^{-1}a)$$

$$+ s(ghk, a) - s(hk, s^{-1}a) - s(g, a)$$

$$- s(gh, a) + \delta s(h, s^{-1}a) + s(g, a)$$

$$= \delta s(hk, s^{-1}a) - \delta s(k, h^{-1} s^{-1}a) - \delta s(h, s^{-1}a)$$

$$+ s(ghk, a) - s(hk, s^{-1}a) - s(g, a)$$

$$- s(gh, a) + \delta s(h, s^{-1}a) + s(g, a)$$

$$= s(ghk, a) - \delta s(k, h^{-1} s^{-1}a) - s(gh, a)$$

$$= a(gh, k, a)$$

(5)
To obtain the remaining equations, we have to consider two cases. If $\varepsilon(g) = +1$ we have, from Equation (3),

$$
\begin{align*}
  s(g, a_1 a_2) &= s(gh, a_1 a_2) - \delta s(h, s^{-1} a_1 s^{-1} a_2) - a(g, h, a_1 a_2) \\
  &= s(gh, a_1) s(gh, a_2) - \beta(gh, a_1, a_2) \\
  &\quad + \delta (s(h, s^{-1} a_1) s(h, s^{-1} a_2) - \beta(h, s^{-1} a_1, s^{-1} a_2)) + a(g, h, a_1 a_2) \\
  &= s(gh, a_1) s(gh, a_2) - \beta(gh, a_1, a_2) + \delta s(h, s^{-1} a_1) s(s, h, s^{-1} a_2) \\
  &\quad - \delta \beta(h, s^{-1} a_1, s^{-1} a_2) + a(g, h, a_1 a_2)
\end{align*}
$$

and from Equation (4) we have

$$
\begin{align*}
  s(g, a_1 a_2) &= s(g, a_1) s(g, a_2) - \beta(g, a_1, a_2) \\
  &= (s(gh, a_1) - \delta s(h, s^{-1} a_1) - i(a(g, h, a_1))) (s(gh, a_2) - \delta s(h, s^{-1} a_2) \\
  &\quad + i(a(g, h, a_2))) - \beta(g, a_1, a_2) \\
  &= s(gh, a_1) s(gh, a_2) + \delta s(h, s^{-1} a_1) s(h, s^{-1} a_2) + a_1 a(g, h, a_2) \\
  &\quad + a(g, h, a_1) a_2 - \beta(g, a_1, a_2)
\end{align*}
$$

Comparing these expressions, we see that

$$
a_1 a(g, h, a_2) - a(g, h, a_1 a_2) + a(g, h, a_1) a_2 = \beta(g, a_1, a_2) - \beta(gh, a_1, a_2) - \delta \beta(h, s^{-1} a_1, s^{-1} a_2). \quad (6)
$$

By replacing $\delta^{-1} a_1 = b_1$ and $\delta^{-1} a_2 = b_2$ in Equation (6), we have

$$
\delta b_1 a(g, h, s b_2) - a(g, h, s b_1 s b_2) + a(g, h, s b_1) s b_2 = \beta(g, s b_1, s b_2) - \beta(gh, s b_1, s b_2) - \delta \beta(h, b_1, b_2). \quad (7)
$$

Similarly, if $\varepsilon(g) = -1$, from Equation (3), we have

$$
\begin{align*}
  s(g, a_1 a_2) &= s(g, a_1) s(g, a_2) - \delta s(h, s^{-1} a_2 s^{-1} a_1) - a(g, h, a_1 a_2) \\
  &= s(gh, a_1) s(gh, a_2) - \beta(gh, a_1, a_2) \\
  &\quad + \delta (s(h, s^{-1} a_2) s(h, s^{-1} a_1) - \beta(h, s^{-1} a_2, s^{-1} a_1)) + a(g, h, a_1 a_2) \\
  &= s(gh, a_1) s(gh, a_2) - \beta(gh, a_1, a_2) + \delta s(h, s^{-1} a_2) s(h, s^{-1} a_1) \\
  &\quad - \delta \beta(h, s^{-1} a_2, s^{-1} a_1) + a(g, h, a_1 a_2)
\end{align*}
$$

and from Equation (4) we have

$$
\begin{align*}
  s(g, a_1 a_2) &= s(g, a_1) s(g, a_2) - \beta(g, a_1, a_2) \\
  &= (s(gh, a_1) - \delta s(h, s^{-1} a_1) - i(a(g, h, a_1))) (s(gh, a_2) - \delta s(h, s^{-1} a_2) \\
  &\quad + i(a(g, h, a_2))) - \beta(g, a_1, a_2) \\
  &= s(gh, a_1) s(gh, a_2) + \delta s(h, s^{-1} a_1) s(h, s^{-1} a_2) + a_1 a(g, h, a_2) \\
  &\quad + a(g, h, a_1) a_2 - \beta(g, a_1, a_2)
\end{align*}
$$

Comparing these expressions, we see that

$$
a_1 a(g, h, a_2) - a(g, h, a_1 a_2) + a(g, h, a_1) a_2 = \beta(g, a_1, a_2) - \beta(gh, a_1, a_2) - \delta \beta(h, s^{-1} a_2, s^{-1} a_1). \quad (8)
$$
By replacing \( s^{-1}a_1 = b_2 \) and \( s^{-1}a_2 = b_1 \) in Equation (8), we have
\[
\begin{align*}
\hat{s}b_2a(g,h,sb_1) - a(g,h,\hat{s}b_2\hat{s}b_1) + a(g,h,\hat{s}b_2)sb_1 = \beta(g,\hat{s}b_2,\hat{s}b_1) - \beta(gh,\hat{s}b_2,\hat{s}b_1) - \hat{s}\beta(h, b_1, b_2). \\
\end{align*}
\]  
(9)

Hence, we show that in fact \((a, \beta) \in Z^1_G(A, X)\).

Conversely, starting with \((a, \beta) \in Z^1_G(A, X)\), one can define \(B = X \oplus A\) where the multiplication is given by
\[
(x_1, a_1)(x_2, a_2) = (x_1a_2 + a_1x_2 + \beta(g, a_1, a_2) - \beta(gh, a_1, a_2), a_1a_2)
\]
and
\[
\hat{s}(x, a) = (\hat{s}x - a(g,h,sa), \hat{s}a).
\]

We claim that \(B\) satisfies all properties of oriented algebra and defines an extension. Since \(\beta\) is a \(G\)-Hochschild 2-cocycle, then \(B\) is clearly an associative algebra. Therefore, we only check Equation (1).

There are two cases to consider: \(\epsilon(g) = +1\) and \(\epsilon(g) = -1\). Firstly, we deal with the first case, when \(\epsilon(g) = +1\). We have
\[
\hat{s}((x_1, a_1)(x_2, a_2)) = \hat{s}(x_1a_2 + a_1x_2 + \beta(g, a_1, a_2) - \beta(gh, a_1, a_2), a_1a_2)
\]
\[
= (\hat{s}x_1\hat{s}a_2 + \hat{s}a_1\hat{s}x_2 + \hat{s}\beta(g, a_1, a_2) - a(g,h,\hat{s}a_1\hat{s}a_2), \hat{s}a_1\hat{s}a_2)
\]
and
\[
\hat{s}(x_1, a_1)\hat{s}(x_2, a_2) = (\hat{s}x_1 - a(g,h,\hat{s}a_1), \hat{s}a_1)(\hat{s}x_2 - a(g,h,\hat{s}a_2), \hat{s}a_2)
\]
\[
= (\hat{s}x_1\hat{s}a_2 - a(g,h,\hat{s}a_1)\hat{s}a_2 + \hat{s}a_1\hat{s}x_2)
\]
\[
- \hat{s}a_1a(g,h,\hat{s}a_2) + \beta(g, \hat{s}a_1, \hat{s}a_2) - \beta(gh, \hat{s}a_1, \hat{s}a_2), \hat{s}a_1\hat{s}a_2)
\]

Therefore, from Equation (7), it follows that
\[
\hat{s}((x_1, a_1)(x_2, a_2)) = \hat{s}(x_1, a_1)\hat{s}(x_2, a_2).
\]

Next, we deal with the second case, when \(\epsilon(g) = -1\). We have
\[
\hat{s}((x_1, a_1)(x_2, a_2)) = \hat{s}(x_1a_2 + a_1x_2 + \beta(g, a_1, a_2) - \beta(gh, a_1, a_2), a_1a_2)
\]
\[
= (\hat{s}a_2\hat{s}x_1 + \hat{s}x_2\hat{s}a_1 + \hat{s}\beta(g, a_1, a_2) - a(g,h,\hat{s}a_2\hat{s}a_1), \hat{s}a_2\hat{s}a_1)
\]
and
\[
\hat{s}(x_2, a_2)\hat{s}(x_1, a_1) = (\hat{s}x_2 - a(g,h,\hat{s}a_2), \hat{s}a_2)(\hat{s}x_1 - a(g,h,\hat{s}a_1), \hat{s}a_1)
\]
\[
= (\hat{s}x_2\hat{s}a_1 - a(g,h,\hat{s}a_2)\hat{s}a_1 + \hat{s}a_2\hat{s}x_1)
\]
\[
- \hat{s}a_2a(g,h,\hat{s}a_1) + \beta(g, \hat{s}a_2, \hat{s}a_1) - \beta(gh, \hat{s}a_2, \hat{s}a_1), \hat{s}a_2\hat{s}a_1)
\]

Therefore, from Equation (9), it follows that
\[
\hat{s}((x_1, a_1)(x_2, a_2)) = \hat{s}(x_2, a_2)\hat{s}(x_1, a_1).
\]

Thus one obtains an inverse map from the cohomology to extensions. \(\square\)

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References