Article

On a Length Problem for Univalent Functions

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Abstract: Let \( g \) be an analytic function with the normalization in the open unit disk. Let \( L(r) \) be the length of \( g(\{ z : |z| = r \}) \). In this paper we present a correspondence between \( g \) and \( L(r) \) for the case when \( g \) is not necessary univalent. Furthermore, some other results related to the length of analytic functions are also discussed.

Keywords: analytic functions; starlike functions; univalent functions; length problems

MSC: 30C45; 30C80

1. Introduction

Let \( A \) be the family of functions of the form

\[
g(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( S \) denote the subfamily of \( A \) consisting of all univalent functions in \( D \).

Let \( C(r) \) denote the image curve of the \( |z| = r < 1 \) under the function \( g \in A \) which bound the area \( A(r) \). Furthermore, let \( L(r) \) be the length of \( C(r) \) and \( M(r) = \max_{|z|=r<1} |g(z)| \).

If \( g \in A \) satisfies

\[
\Re \left\{ \frac{z g'(z)}{g(z)} \right\} > 0, \quad z \in \mathbb{D},
\]

then \( g \) is said to be starlike with respect to the origin in \( \mathbb{D} \) and we write \( g \in S^* \). It is known (for details, see [1,2]) that \( S^* \subset S \).

The aim of the present paper is to prove, using a modified methodology, that in the following implication

\[
g \in S^* \quad \Rightarrow \quad L(r) = \mathcal{O} \left( M(r) \log \frac{1}{1-r} \right) \quad \text{as} \quad r \to 1,
\]

where \( \mathcal{O} \) denotes the Landau’s symbol, the assumption that \( g \) is starlike univalent can be changed by a weaker one. Result (2) was proved by Keogh [3]. Moreover, some other length problems for analytic functions are investigated. Several interesting developments related to length problems for univalent functions were considered in [4–15].
2. Main Results

**Theorem 1.** Let \( g \) be of the form (1) and suppose that
\[
\left| \frac{zg'(z)}{g(z)} \right| \leq \frac{1+z}{1-z}, \quad z \in \mathbb{D}.
\] (3)

Then
\[
L(r) = \mathcal{O} \left( M(r) \log \frac{1}{1-r} \right) \quad \text{as} \quad r \to 1,
\]
where
\[
M(r) = \max_{|z|=r} |g(z)|
\]
and \( \mathcal{O} \) means Landau’s symbol.

**Proof.** Let \( z = re^{i\nu} \). We have \( g \neq 0 \) in \( \mathbb{D} \setminus \{0\} \). In fact, if \( g = 0 \) in \( \mathbb{D} \), it contradicts hypothesis (3). Applying [3] (Theorem 1) and the hypothesis of Theorem 1, we have
\[
L(r) = \int_0^{2\pi} |zg'(z)|\,dv = \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| |g(z)|\,dv
\]
\[
\leq M(r) \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| \, dv \leq M(r) \int_0^{2\pi} \left| \frac{1+re^{i\nu}}{1-re^{i\nu}} \right| \, dv
\]
\[
\leq M(r) \left( 2\pi + 4 \log \frac{1+r}{1-r} \right) \quad \text{as} \quad r \to 1.
\]

\[\Box\]

**Remark 1.** If \( g \) satisfies the condition of Theorem 1, then \( g \) is not necessary univalent in \( \mathbb{D} \). It is well known that if \( g \in \mathcal{S} \), then it follows that
\[
\frac{1-|z|}{1+|z|} \leq \left| \frac{zg'(z)}{g(z)} \right| \leq \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}
\]
(for details, see [1] (Vol. 1, p. 69)). If \( g \in \mathcal{A} \) satisfies
\[
\Re \left\{ \frac{zg'(z)}{g(z)h'(z)} \right\} > 0, \quad z \in \mathbb{D}
\]
for some \( h \in \mathcal{S}^* \) and some \( \gamma \in (0, \infty) \), then \( g \) is said to be a Bazilevič function of type \( \gamma \) [13]. The class of Bazilevič functions of type \( \gamma \) is denoted by \( g \in \mathcal{B}(\gamma) \). We note that Theorem 1 improves the implication (2) by Keogh [3] and it is also related to Theorem 3 given by Thomas [13].

We will need the following Tsuji’s result.

**Lemma 1** ([16] (p. 226)). (Theorem 3) If \( 0 \leq r < R \) and \( z = e^{i\nu} \), then
\[
\frac{R-r}{R+r} \leq \Re \left\{ \frac{Re^{i\nu} + z}{Re^{i\nu} - z} \right\} = \frac{R^2 - r^2}{R^2 - 2Rr \cos(\phi - \nu) + r^2} \leq \frac{R+r}{R-r}.
\] (4)

Moreover,
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\phi - \nu) + r^2} \, d\nu = 1.
\] (5)
Theorem 2. Let \( g \) be of the form (1) and suppose that
\[
\left| \frac{zg'(z)}{g(z)} \right| \leq \left| \frac{1 + z}{1 - z} \right|, \quad z \in \mathbb{D}
\]
and
\[
M(r, \beta) = \max_{|z|=r<1} |g(z)| \leq \left| \frac{1 + z}{1 - z} \right|^{\beta},
\]
where \( 1 < \beta \). Then
\[
L(r) = \mathcal{O} \left( \frac{1}{(1 - r)^{\beta}} \right) \quad \text{as} \quad r \to 1,
\]
where \( \mathcal{O} \) means Landau’s symbol.

Proof. From the hypotheses (6) and (7), it follows that
\[
L(r) = \int_0^{2\pi} |zg'(z)| \, dv = \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| |g(z)| \, dv
\]
\[
\leq \int_0^{2\pi} \left| \frac{1 + z}{1 - z} \right| \left| \frac{1 + z}{1 - z} \right|^{\beta} \, dv \leq 2^{1+\beta} \int_0^{2\pi} \frac{1}{|1 - z|^{1+\beta}} \, dv
\]
\[
= \frac{2^{1+\beta}}{(1 - r)^{\beta-1}} \int_0^{2\pi} \frac{1}{1 - 2r \cos \nu + r^2} \, dv.
\]
From (5), we have
\[
\int_0^{2\pi} \frac{1}{1 - 2r \cos \nu + r^2} \, dv = \frac{2\pi}{1 - r^2}.
\]
Hence, we obtain
\[
L(r) \leq \frac{2^{1+\beta}}{(1 - r)^{\beta-1}} \frac{2\pi}{1 - r^2}
\]
\[
= \mathcal{O} \left( \frac{1}{(1 - r)^{\beta}} \right) \quad \text{as} \quad r \to 1.
\]

Therefore, we complete the proof of Theorem 2. \( \square \)

Let us recall the following Fejér-Riesz’s result.

Lemma 2 ([16]). Let \( h \) be analytic in \( \mathbb{D} \) and continuous on \( \overline{\mathbb{D}} \). Then
\[
\int_{-1}^{1} |h(z)|^p \, |dz| \leq \frac{1}{2} \int_{|z|=1} |h(z)|^p \, |dz|,
\]
where \( p > 0 \).

Theorem 3. Let \( g \) be of the form (1) and suppose that
\[
\left| \frac{1 - z}{1 + z} \right| \leq \left| \frac{zg'(z)}{g(z)} \right| \leq \left| \frac{1 + z}{1 - z} \right|, \quad z \in \mathbb{D}
\]
and
\[
M(r, \beta) = \max_{|z|=r<1} |g(z)| \leq \left| \frac{1 + z}{1 - z} \right|^{\beta},
\]
where \( 1 < \beta \). Then
\[
\mathcal{O} \left( m(r) \log \frac{1}{1 - r} \right) \leq L(r) \leq \mathcal{O} \left( \frac{M(r)}{1 - r} \right) \quad \text{as} \quad r \to 1,
\]
where
\[
m(r) = \min_{|z|=r<1} |g(z)|, \quad M(r) = \max_{|z|=r<1} |g(z)|
\]
and \( O \) means Landau’s symbol.

**Proof.** From the assumption, we have

\[
L(r) = \int_0^{2\pi} |zg'(z)| \nu = \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| |g(z)| \nu,
\]

\[
\geq m(r) \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| \nu
\]

because \( g(z) \neq 0 \) in \( D \setminus \{0\} \). In fact, if \( g(z) = 0 \) in \( D \setminus \{0\} \), it contradicts hypothesis (8).

Applying Fejér-Riesz’s Lemma 2, we have

\[
L(r) \geq m(r) \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| \nu \geq m(r) \int_r^{1-\rho} \frac{1}{1+\rho} d\rho
\]

\[
= O \left( m(r) \log \frac{1}{1-r} \right) \quad \text{as} \quad r \to 1.
\]

While, we obtain

\[
L(r) = \int_0^{2\pi} |zg'(z)| \nu = \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| |g(z)| \nu
\]

\[
= M(r) \int_0^{2\pi} \frac{1+|z|}{1-|z|} \nu = 2\pi M(r) \frac{1+r}{1-r}
\]

\[
= O \left( \frac{M(r)}{1-r} \right) \quad \text{as} \quad r \to 1.
\]

Therefore, we complete the proof of Theorem 3. \( \square \)

From Theorem 3, we have the following result.

**Corollary 1.** Let \( g \) be of the form (1) and suppose that \( g \) is univalent in \( D \). Then we have

\[
O \left( m(r) \log \frac{1}{1-r} \right) \leq L(r) \leq O \left( \frac{M(r)}{1-r} \right) \quad \text{as} \quad r \to 1,
\]

where \( m(r) \) and \( M(r) \) are given by (9), respectively.

**Proof.** From the hypothesis, we have

\[
\frac{1-|z|}{1+|z|} \leq \left| \frac{zg'(z)}{g(z)} \right| \leq \frac{1+|z|}{1-|z|}, \quad z \in D,
\]

which completes the proof. \( \square \)

**Lemma 3** ([17] (p. 280) and [18] (p. 491)).

\[
\int_0^{2\pi} \frac{\nu}{|1-re^{i\nu}|^\beta} = \begin{cases} 
O \left( (1-r)^{1-\beta} \right) & \text{for the case } 1 < \beta, \\
O \left( \log \frac{1}{1-r} \right) & \text{for the case } \beta = 1, \\
O (1) & \text{for the case } 0 \leq \beta < 1,
\end{cases}
\]
Theorem 4. Let \( g \) be of the form (1) and suppose that
\[
\left| \frac{2g'(z)}{g(z)} \right| \leq \frac{1}{1-|z|}, \quad z \in \mathbb{D}
\] (10)
and
\[
|g(z)| \leq \frac{1}{|1-z|^\nu}, \quad z \in \mathbb{D}.
\] (11)

Then
\[
L(r) \leq \begin{cases} 
O \left( (1-r)^{-3/2} \right) & \text{for } 1 < \beta \leq 3/2, \\
O \left( (1-r)^{-3/2} \log \frac{1}{1-r} \right) & \text{for the case } \beta = 3/2, \\
O \left( (1-r)^{-\beta} \right) & \text{for the case } 3/2 < \beta,
\end{cases}
\]
where \( 0 < |z| = r < 1 \) and \( O \) means Landau’s symbol.

Proof. From the hypothesis (10), it follows that \( g(z) \neq 0 \) in \( \mathbb{D} \setminus \{0\} \). Then we have
\[
L(r) = \int_0^{2\pi} \left| re^{iv} g'(re^{i\phi}) \right| dv = \int_0^{2\pi} \left| \frac{2g'(z)}{g(z)} \right| |g(z)| dv
\]
\[
\leq \int_0^{2\pi} \left( \frac{1}{1-|z|} \right) \left( \frac{1}{|1-z|^\nu} \right) dv
\]
\[
= \int_0^{2\pi} \left( \frac{1}{1-|z|} \right) \left( \frac{1}{|1-z|^\beta -1} \right) \left( \frac{1}{|1-z|^\nu} \right) dv
\]
\[
\leq \left( \int_0^{2\pi} \frac{1}{|1-z|^\nu} dv \right)^{1/2} \left( \int_0^{2\pi} \frac{1}{|1-z|^\beta -1} dv \right)^{1/2} \left( \frac{1}{1-|z|^\nu} \right)^{1/2}.
\]

Applying Hayman’s Lemma 3, we have
\[
L(r) \leq \left( \frac{1}{1-r^2} \right)^{1/2} \left( \frac{1}{1-r} \right) O(1)
\]
\[
= O \left( \frac{1}{(1-r)^{3/2}} \right) \quad \text{as } r \to 1
\]
for the case \( 1 < \beta < 3/2 \),
\[
L(r) \leq \left( \frac{1}{1-r^2} \right)^{1/2} \left( \frac{1}{1-r} \right) O \left( \log \frac{1}{1-r} \right)
\]
\[
= O \left( \frac{1}{(1-r)^{3/2} \log \frac{1}{1-r}} \right) \quad \text{as } r \to 1
\]
for the case \( \beta = 3/2 \) and
\[
L(r) = \left( \frac{1}{1-r^2} \right)^{1/2} \left( \frac{1}{1-r} \right) \left( \frac{1}{1-r} \right)^{(2\beta-3)/2} \quad \text{as } r \to 1
\]
for the case \( 3/2 < \beta \). \( \Box \)

Lemma 4 ([16] (p. 227)). If \( g(z) = u(z) + iv(z) \) is analytic in \( |z| \leq R \), then
\[
g(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\phi}) \frac{Re^{i\phi} + z}{Re^{i\phi} - z} d\phi + iv(0).
\] (12)
Moreover, if \(|z| < R\) and \(v(0) = 0\), then

\[
|g(z)| = \frac{1}{2\pi} \int_0^{2\pi} |u(Re^{i\phi})| \left| \frac{Re^{i\phi} + z}{Re^{i\phi} - z} \right| d\phi.
\]

**Theorem 5.** Let \(g\) be of the form (1). Then

\[
M(r) = \mathcal{O}\left(A(r) \log \frac{1}{1-r}\right) \text{ as } r \to 1,
\]

where \(0 < |z| = r < 1\) and \(\mathcal{O}\) means Landau’s symbol.

**Proof.** It follows that

\[
M(r) = \max_{|z| = r < 1} \left| \int_0^r g'(s) ds \right| = \max_{|z| = r < 1} \left| \int_0^r g'(pe^{i\nu}) d\rho \right|.
\]

Applying (12), we have

\[
M(r) = \max_{|z| = r < 1} \left| \frac{1}{2\pi} \int_0^{2\pi} \Re g'(te^{i\nu}) \left| \frac{te^{i\phi} + \rho e^{i\nu}}{te^{i\phi} - \rho e^{i\nu}} \right| d\phi d\rho \right|,
\]

where \(0 \leq \rho \leq r < t < 1\). Then, applying Schwarz’s lemma, we have

\[
M(r) \leq \max_{|z| = r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| g'(te^{i\nu}) \right|^2 d\phi d\rho \right)^{1/2} \left( \int_0^{2\pi} \left| \frac{te^{i\phi} + \rho e^{i\nu}}{te^{i\phi} - \rho e^{i\nu}} \right|^2 d\phi d\rho \right)^{1/2}
\]

\[
\leq \max_{|z| = r < 1} (I_1)^{1/2}(I_2)^{1/2}, \text{ say.}
\]

Putting \(0 < r_1 < r\) and \(t = \sqrt{(1 + \rho^2)/2}\), we have

\[
\rho d\rho = 2\sqrt{\frac{1 + \rho^2}{2}} dt < 2 dt.
\]

Then we have

\[
I_1 = \frac{1}{2\pi} \int_0^{r_1} \int_0^{2\pi} \left| g'(te^{i\nu}) \right|^2 d\phi d\rho + \frac{1}{2\pi r_1^2} \int_{\sqrt{(1 + r_1^2)/2}}^{2\pi} \int_0^{2\pi} t \left| g'(te^{i\nu}) \right|^2 d\phi dt
\]

\[
\leq C + \frac{1}{2\pi r_1^2} A \left( \sqrt{\frac{1 + r_1^2}{2}} \right)
\]

\[
= C + \frac{1}{2\pi r_1^2} A \left( \frac{1 + r_1^2}{2r^2} \right)
\]

\[
= \mathcal{O}(A(r)) \text{ as } r \to 1,
\]
where $C$ is a bounded positive constant. On the other hand, putting $t \to 1^−$, we have

\[
I_2 = \int_{0}^{r} \int_{0}^{2\pi} \frac{|te^{i\phi} + \rho e^{iv}|^2}{|te^{i\phi} - \rho e^{iv}|^2} \, d\phi d\rho \\
\leq \int_{0}^{r} \int_{0}^{2\pi} \frac{4}{|te^{i\phi} - \rho e^{iv}|^2} \, d\phi d\rho \\
= \int_{0}^{r} \int_{0}^{2\pi} \frac{4}{t^2 - 2t\rho \cos(\phi - \nu) + \rho^2} \, d\phi d\rho.
\]

Using (5), we have

\[
I_2 \leq 8\pi \int_{0}^{r} \frac{1}{t^2 - \rho^2} \, d\rho \\
= \frac{4\pi}{t} \int_{0}^{r} \left( \frac{1}{t + \rho} + \frac{1}{t - \rho} \right) \, d\rho \\
= \frac{4\pi}{t} \log \frac{t + r}{t - r} \to O \left( \log \frac{1}{1 - r} \right) \text{ as } r \to 1.
\]

Therefore we complete the proof of (13). \hfill \Box

Remark 2. In Theorem 5, we do not suppose that $g$ is univalent in $|z| < 1$ and therefore, it improves the result by Pommerenke [2].

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References


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