Extensions of Móricz Classes and Convergence of Trigonometric Sine Series in $L^1$-Norm

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Received: 11 September 2018; Accepted: 21 November 2018; Published: 29 November 2018

Abstract: In this paper, the extensions of classes $\tilde{S}$, $\tilde{C}$ and $\tilde{BV}$ are made by defining the classes $\tilde{S}_r$, $\tilde{C}_r$ and $\tilde{BV}_r$, $r = 0, 1, 2, \ldots$ It is also shown that class $\tilde{S}_r$ is a subclass of $\tilde{C}_r \cap \tilde{BV}_r$. Moreover, the results on $L^1$-convergence of $r$ times differentiated trigonometric sine series have been obtained by considering the $r$th ($r = 0, 1, 2, \ldots$) derivative of modified sine sum under the new extended class $\tilde{C}_r \cap \tilde{BV}_r$.

Keywords: Dirichlet kernel; $L^1$-convergence; modified sine sum

1. Introduction

Consider the trigonometric sine series

$$\sum_{k=1}^{\infty} a_k \sin kx$$

where $a_0, a_1, a_2, \ldots$ are the real coefficients. The $n$th partial sum, $S_n$, of Series (1) is represented as

$$S_n(x) = \sum_{k=1}^{n} a_k \sin kx = -\sum_{k=1}^{n} b_k (\cos kx)'$$

where the prime denotes derivatives and $b_k = \frac{a_k}{k}$. Also, $f(x) = \lim_{n \to \infty} S_n(x)$.

Various conditions are given in the literature (see [1–9]), which guarantee that Series (1) is a Fourier series.

In 1984, Teljakovskii [9] introduced a class $\tilde{S}$, as follows:

Class $\tilde{S}$ [9]. A null sequence $\{a_k\}$ is said to belong to class $\tilde{S}$ if there exists a non-increasing sequence $\{B_k\}$ of numbers s.t.

$$|\Delta b_k| \leq B_k \quad \forall k = 1, 2, 3, \ldots$$

$$\sum_{k=1}^{\infty} k B_k < \infty.$$

where $b_k = \frac{a_k}{k}$, $\Delta b_k = b_k - b_{k+1}$ and proved the following result:

**Theorem 1** [9]. If $\{a_k\} \in \tilde{S}$, then Series (1) is the Fourier series of some function $f \in L^1(0, \pi)$.

In 1989, Móricz [5] introduced new classes $\tilde{BV}$ and $\tilde{C}$ of the coefficient sequences for the sine series.

Class $\tilde{BV}$ [5]. A null sequence $\{a_k\}$ belongs to $\tilde{BV}$ if

$$\sum_{k=1}^{\infty} k |\Delta b_k| < \infty$$

Class $\tilde{C} \, [5]$. A null sequence $\{a_k\}$ belongs to class $\tilde{C}$ if for every $\varepsilon > 0$ there exists $\delta > 0$, independent of $n$, and such that for all $n$, 
\[
\int_0^\delta \left| \sum_{k=n}^\infty \Delta b_k D_k'(x) \right| \, dx \leq \varepsilon. 
\]
(4)

where $D_k'(x)$ is the first derivative of Dirichlet kernel $D_k(x) = \frac{\sin((k+\frac{1}{2})x)}{2\sin \frac{x}{2}}$.
Equation (4) implies that, for $1 \leq n \leq N$,
\[
\int_0^\delta \left| \sum_{k=n}^N \Delta b_k D_k'(x) \right| \, dx \leq 2\varepsilon.
\]

The following result was proved by Móricz [7].

**Theorem 2** [5]. If $\{a_k\} \in \tilde{B}V$, then
\[
\|u_n - f\| \to 0 \quad (n \to \infty) \quad \text{if and only if} \quad \{a_k\} \in \tilde{C}.
\]
where $u_n(x) = S_n(x) + b_{n+1}D_k'(x)$.

The classes $\tilde{S}$, $\tilde{B}V$ and $\tilde{C}$ seem to be more appropriate for the sine series than the classes $S$ ([7,8]) $BV$ [10], and $C$ [3] in the ordinary sense. Also, Móricz [5] has proved that $\tilde{S} \subset \tilde{B}V \cap \tilde{C}$.

Motivated by the aforesaid authors, new extended classes $\tilde{S}_r$, $\tilde{B}V_r$, and $\tilde{C}_r \, (r = 0, 1, 2, \ldots)$ are defined in this paper as follows:

**Class $\tilde{S}_r$**. A sequence $\{a_k\}$ is said to belong to class $\tilde{S}_r \, (r = 0, 1, 2, \ldots)$ if $a_k \to 0$ as $k \to \infty$, and there exists a non-increasing sequence $\{B_k\}$ of numbers s.t.
\[
|\Delta b_k| \leq B_k \quad \forall k = 1, 2, 3, \ldots
\]
\[
\sum_{k=1}^\infty k^{r+1} B_k < \infty, \quad r = 0, 1, 2, 3, \ldots
\]

where $b_k = \frac{a_k}{k^r}$, $r = 0, 1, 2, 3, \ldots$

\[
B_k \downarrow 0 \quad \text{and} \quad \sum_{k=1}^\infty k^{r+1} B_k < \infty, \quad \text{implies that} \quad k^{r+2} B_k = o(1) \quad \text{as} \quad k \to \infty \quad (r = 0, 1, 2, \ldots).
\]

**Remark 1**. For $r = 0$, $\tilde{S}_r = \tilde{S}$.

**Remark 2**. Obviously, $\tilde{S}_{r+1} \subset \tilde{S}_r$, but the converse need not be true.

**Example 1**. Consider a sequence $\Delta b_n = \frac{1}{n^{r+1}}$, $r = 0, 1, 2, \ldots$ and $n \in N$.
\[
a_n = nb_n = n \sum_{k=n}^\infty \Delta b_k \leq \sum_{k=n}^\infty \frac{k}{k^{r+3}} = \sum_{k=n}^\infty \frac{1}{k^{r+2}} \to 0 \quad \text{as} \quad n \to \infty.
\]

Choose $B_n = \frac{1}{n^{r+1}}$, $r = 0, 1, 2, \ldots$ \quad $\forall n$. Clearly, $B_n \downarrow 0$ as $n \to \infty$ and $|\Delta b_n| \leq B_n \forall n$.

Consider the series
\[
\sum_{n=1}^\infty n^{r+1} B_n = \sum_{n=1}^\infty n^{r+1} \frac{1}{n^{r+3}} \approx \sum_{n=1}^\infty \frac{1}{n^2} \quad \text{which is convergent}.
\]
This implies \( \{a_n\} \in \tilde{\mathcal{S}}_r \).

But the series \( \sum_{n=1}^{\infty} n^{r+2} B_n \approx \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent.

This implies that \( \{a_n\} \) does not belong to class \( \tilde{\mathcal{S}}_{r+1} \).

**Class \( \tilde{B}V_r \).** A null sequence \( \{a_k\} \) belongs to \( \tilde{B}V_r \) \( (r = 0, 1, 2, \ldots) \) if

\[
\sum_{k=1}^{\infty} k^{r+1} |\Delta b_k| < \infty
\]

**Remark 3.** For \( r = 0 \), \( \tilde{B}V_r = \tilde{B}V \).

**Remark 4.** Clearly, \( \tilde{B}V_{r+1} \subset \tilde{B}V_r \) \( (r = 0, 1, 2, \ldots) \), but the converse may not be true.

**Class \( \tilde{C}_r \).** A null sequence \( \{a_k\} \) belongs to class \( \tilde{C}_r \) \( (r = 0, 1, 2, \ldots) \), if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \), independent of \( n \), and such that for all \( n \),

\[
\int_{0}^{\delta} \left| \sum_{k=n}^{\infty} \Delta b_k D_{k+1}^r(x) \right| dx \leq \varepsilon
\]

Here, \( D_{k+1}^r(x) \) is the \( (r+1) \)th derivative of Dirichlet kernel.

Equation (4) implies, for \( 1 \leq n \leq N \),

\[
\int_{0}^{\delta} \left| \sum_{k=n}^{N} \Delta b_k D_{k+1}^r(x) \right| dx \leq 2\varepsilon
\]

**Remark 5.** For \( r = 0 \), \( \tilde{C}_r = \tilde{C} \).

**Remark 6.** It is obvious that \( \tilde{C}_{r+1} \subset \tilde{C}_r \) but the converse need not be true.

**Example 2.** Define \( \Delta b_n = \frac{1}{n^{r+3}} \), \( r = 0, 1, 2, \ldots \) and \( n = 1, 2, 3, \ldots \)

\[
a_n = nb_n = n \sum_{k=n}^{\infty} \Delta b_k \leq \sum_{k=n}^{\infty} \frac{k}{k^{r+2}} = \sum_{k=n}^{\infty} \frac{1}{k^{r+2}} \to 0 \text{ as } n \to \infty.
\]

Consider, the integral

\[
\int_{0}^{\pi} \left| \sum_{k=n}^{\infty} \Delta b_k D_{k+2}^r(x) \right| dx = \sum_{k=n}^{\infty} \frac{1}{n^{r+2}} \int_{0}^{\pi} |D_{k+2}^r(x)| dx = O \left( \sum_{k=n}^{\infty} \frac{1}{n^{r+2}} \left( n^{r+2} \log n \right) \right)
\]

\[
= O \left( \sum_{k=n}^{\infty} \frac{\log n}{n} \right)
\]

which is divergent.

However,

\[
\int_{0}^{\pi} \left| \sum_{k=n}^{\infty} \Delta b_k D_{k+1}^r(x) \right| dx = \sum_{k=n}^{\infty} \frac{1}{n^{r+2}} \int_{0}^{\pi} |D_{k+1}^r(x)| dx = O \left( \sum_{k=n}^{\infty} \frac{1}{n^{r+2}} \left( n^{r+1} \log n \right) \right)
\]

\[
= O \left( \sum_{k=n}^{\infty} \frac{\log n}{n^{r+1}} \right) \text{ which is convergent.}
\]
Therefore \( \{a_n\} \in \tilde{C}_r \).

Lemmas related to the main results are given in Section 2. The Section 3 comprises the main results of this paper. Firstly, in this section, we have shown that the new extended class \( \tilde{S}_r \) is a subclass of \( \tilde{C}_r \cap \tilde{BV}_r \) \( (r = 0, 1, 2, \ldots) \). Moreover, the theorems are presented concerning the \( L^1 \) convergence of trigonometric sine series using modified sine sum \( \{11\} \), defined as

\[
\beta_n(x) = \sum_{k=1}^{n} \left( \frac{a_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2 \left( \frac{a_j}{j} \right) \right) k \sin kx
\]

(5)

under the extended classes of numerical sequences.

2. Lemmas

Lemma 1. [6] Let \( n \geq 1 \) and \( r \) be a nonnegative integer \( x \in [\varepsilon, \pi] \). Then, \( |D_r'(x)| \leq \frac{C}{x} \), where \( C \) denotes a positive absolute constant.

Lemma 2. [6] \( \|D_r'(x)\|_{L^1} = O(n^r \log n) \), \( r = 0, 1, 2, \ldots \) where \( D_r'(x) \) represents the \( r^{th} \) derivative of the Dirichlet kernel.

3. Main Results

Theorem 3. The following relation holds \( \tilde{S}_r \subset \tilde{C}_r \cap \tilde{BV}_r \) for each \( r \in \{0, 1, 2, \ldots\} \).

Proof. It is plain that \( \tilde{S}_r \subset \tilde{C}_r \).

In order to prove that \( \tilde{S}_r \subset \tilde{C}_r \), we take a sequence \( \{a_k\} \) in \( \tilde{S}_r \), and consider

\[
\int_0^\pi \left| \sum_{k=1}^{n} \Delta b_k D_r^{k+1}(x) \right| dx; \text{ where } b_k = \frac{a_k}{k}
\]

If we apply summation by parts, we obtain

\[
\frac{n}{n} \sum_{k=1}^{n} \Delta b_k D_r^{k+1}(x) \right| dx \leq \lim_{N \to \infty} \left[ \sum_{k=1}^{n} \Delta b_k \int_0^\pi \sum_{j=0}^{k} \frac{\Delta b_j}{B_j} D_r^{j+1}(x) \right] dx + B_N \int_0^\pi \sum_{k=0}^{N} \frac{\Delta b_k}{B_k} D_r^{k+1}(x) \right| dx

+ B_n \int_0^\pi \left| \sum_{k=0}^{n-1} \frac{\Delta b_k}{B_k} D_r^{k+1}(x) \right| dx
\]

Clearly \( \left| \frac{\Delta b_k}{B_k} \right| \leq 1 \). Now, if we first apply Bernstein’s inequality [12] and then Sidon Fomin’s inequality ([1], [7]), we get

\[
\int_0^\pi \left| \sum_{k=0}^{n} \frac{\Delta b_k}{B_k} D_r^{k}(x) \right| dx \leq M(n+1)^{r+2}, \quad r = 0, 1, 2, \ldots
\]

\[
\int_0^\pi \left| \sum_{k=0}^{n} \frac{\Delta b_k}{B_k} D_r^{k+1}(x) \right| dx \leq \lim_{N \to \infty} \left\{ \sum_{k=1}^{n} \Delta b_k (k+1)^{r+2} + B_N (N+1)^{r+2} \right\} + n^{r+2}B_n
\]

\[
= \sum_{k=n}^{\infty} \left[ (k+1)^{r+2} - k^{r+2} \right] B_k + n^{r+2}B_n
\]

\[
= O \left( \sum_{k=n}^{\infty} k^{r+1}B_k \right) + n^{r+2}B_n
\]
Theorem 4. Let \( \delta \) provided

\[
\frac{\|f \|}{\| \|} \leq \frac{\varepsilon}{2} \text{ if } n \text{ is large enough say } n \geq n_0.
\] (6)

For any \( 1 \leq n \leq N \), we can estimate as follows:

\[
\frac{\delta}{0} \sum_{k=n}^{N} \Delta b_k D_k^{r+1}(x) \, dx \leq \int_{0}^{\delta} \sum_{k=n_0}^{N} \Delta b_k D_k^{r+1}(x) \, dx + \int_{0}^{\delta} \sum_{k=n_0}^{n} \Delta b_k D_k^{r+1}(x) \, dx \\
\leq \frac{1}{2} \delta \sum_{k=1}^{n_0} k(k+1)^{r+1} \, |\Delta b_k| + \frac{\varepsilon}{2} < \varepsilon
\]

provided \( \delta \) is small enough. This proves that \( \{a_k\} \in \mathcal{C}_r. \) □

Theorem 4. Let \( \{a_k\} \) be a sequence of numbers belonging to the class \( \mathcal{C}_r \cap BV \) and if \( \lim_{n \to \infty} a_n \log n = 0 \), then

\[ \| \beta_n - f \| = o(1), \quad n \to \infty. \]

Proof. The modified trigonometric sine sum is given by

\[
\beta_n(x) = \sum_{k=1}^{n} \left( a_{k+1} + \sum_{j=k}^{n} \Delta^2 \left( \frac{a_j}{j} \right) \right) k \sin kx
\]

\[
= \sum_{k=1}^{n} a_k \sin kx + \left( \frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1} \right) \sum_{k=1}^{n} k \sin kx
\]

\[
= - \sum_{k=1}^{n} b_k (\cos kx)' - (b_{n+2} - b_{n+1}) D_n'(x)
\]

By using the summation by parts, we get

\[
\beta_n = - \sum_{k=1}^{n} \Delta b_k D_k'(x) - b_n D_n'(x) - (b_{n+2} - b_{n+1}) D_n'(x)
\]

Under the given hypothesis and Lemma 1, series \( \sum_{k=1}^{n} \Delta b_k D_k'(x) \) converges absolutely and \( b_n D_n'(x) \to 0 \) as \( n \to \infty \).

Hence \( \lim_{n \to \infty} \beta_n(x) = f(x) \) exists in \((0, \pi)\).

Next, consider

\[
\| f(x) - \beta_n(x) \| = \| \sum_{k=n+1}^{\infty} a_k \sin kx - \left( \frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1} \right) \sum_{k=1}^{n} k \sin kx \|
\]

\[
= \int_{0}^{\pi} \sum_{k=n+1}^{\infty} b_k (\cos kx)' - (b_{n+1} - b_{n+2}) D_n'(x) \, dx
\] (7)

By using Abel’s transformation, we have

\[
= \int_{0}^{\pi} \sum_{k=n+1}^{\infty} \Delta b_k D_k'(x) + b_{n+2} D_n'(x) \, dx
\]

\[
= \int_{0}^{\pi} \sum_{k=n+1}^{\infty} \Delta b_k D_k'(x) \, dx + \frac{n}{n+2} a_{n+2} \log n
\] (8)
The second term of the above equation is of $o(1)$ as $a_n \log n = 0$ as $n \to \infty$. For the remaining part, let $\varepsilon > 0$, then there exists $\delta > 0$, such that
\[
\int_0^\varepsilon \left| \sum_{k=n+1}^\infty \Delta b_k D_k'(x) \right| \, dx < \frac{\varepsilon}{2} \text{ for all } n \geq 0. \tag{9}
\]

Then
\[
\left| \sum_{k=n+1}^\infty \Delta b_k D_k'(x) \right| \, dx = \left| \sum_{k=n+1}^\infty \Delta b_k D_k'(x) \right| \, dx + \left| \sum_{k=n+1}^\infty \Delta b_k D_k'(x) \right| \, dx \leq \frac{\varepsilon}{2} + \sum_{k=n+1}^\infty |\Delta b_k| \int_0^\pi |D_k'(x)| \, dx \leq \frac{\varepsilon}{2} + C\delta^{-1} \sum_{k=n+1}^\infty k |\Delta b_k| \leq \varepsilon \tag{10}
\]

This proves that $\|f(x) - \beta_n(x)\| = o(1)$ as $n \to \infty$. □

**Theorem 5.** Let $\{a_k\}$ be a sequence of numbers belonging to the class $\tilde{C}\cap\tilde{BV}$, and if $\lim_{n \to \infty} a_n \log n = 0$, then
\[
\|S_n - f\| = o(1), \quad n \to \infty.
\]

**Proof.** $\|S_n - f\| \leq \|S_n - \beta_n\| + \|\beta_n - f\|$ \[
\leq |b_{n+1}| \int_0^\pi |D_n'(x)| \, dx + |b_{n+2}| \int_0^\pi |D_n'(x)| \, dx + o(1) \leq |a_{n+1}| \log n + |a_{n+2}| \log n (\text{by Lemma 2}) = o(1), \quad n \to \infty
\]

□

**Theorem 6.** Let $\{a_k\}$ be a sequence of numbers belonging to the class $\tilde{C}\cap\tilde{BV}$, and if $n^r a_n \log n = 0$, as $n \to \infty$, for each $r = 0, 1, 2, \ldots$ Then
\[
\|\beta_n^r(x) - f^r(x)\| = o(1), \quad n \to \infty
\]

Here, $f^r(x)$ is the $r$th derivative of $f(x)$, where $r = 0, 1, 2, \ldots$

**Proof.** Consider the modified trigonometric sine sum as
\[
\beta_n(x) = \sum_{k=1}^n \left( d_k + \frac{\Delta^2}{7} a_j \right) k \sin kx
\]
\[
= n \sum_{k=1}^n d_k \sin kx + \left( \frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1} \right) \sum_{k=1}^n k \sin kx \tag{11}
\]
Taking $r$-times differentiation of $\beta_n(x)$, we get
\( \beta_n'(x) = S_n'(x) + \left( \frac{d_n+1}{n+2} - \frac{d_n+1}{n+4} \right) \sum_{k=1}^{n} k^{r+1} \sin \left( kx + \frac{\pi}{2} \right) \)

\[
= \sum_{k=1}^{n} k' a_k \sin \left( kx + \frac{\pi}{2} \right) + \left( \frac{d_n+1}{n+2} - \frac{d_n+1}{n+4} \right) \sum_{k=1}^{n} k^{r+1} \cos \left( kx + \frac{(r+1)\pi}{2} \right)
\]

If we apply Abel's transformation on the first term of above equation, we get

\[
\beta_n'(x) = -\sum_{k=1}^{n-1} \Delta b_k D_{k+1}^r(x) - b_n D_{n+1}^r(x) + (b_{n+1} - b_{n+2}) D_{n+1}^r(x)
\]

The series \( \sum_{k=1}^{\infty} \Delta b_k D_{k+1}^r(x) \) converges absolutely and \( b_n D_{n+1}^r(x) \to 0 \) as \( n \to \infty \) using Lemma 1 and given hypothesis.

Therefore \( \lim_{n \to \infty} \beta_n'(x) = f'(x) \) exists in \((0, \pi)\).

Next, consider

\[
\|f'(x) - \beta_n'(x)\| = \left\| \sum_{k=n+1}^{\infty} a_k \sin kx - \left( \frac{d_n+1}{n+2} - \frac{d_n+1}{n+4} \right) \sum_{k=1}^{n} k \sin kx \right\|
\]

\[
\|f'(x) - \beta_n'(x)\| = \left\| \sum_{k=n+1}^{\infty} k' a_k \sin \left( kx + \frac{\pi}{2} \right) - \left( \frac{d_n+1}{n+2} - \frac{d_n+1}{n+4} \right) \sum_{k=1}^{n} k^{r+1} \sin \left( kx + \frac{\pi}{2} \right) \right\|
\]

\[
= \left\| \sum_{k=n+1}^{\infty} k' a_k \sin \left( kx + \frac{\pi}{2} \right) + \left( \frac{d_n+1}{n+2} - \frac{d_n+1}{n+4} \right) \sum_{k=1}^{n} k^{r+1} \cos \left( kx + \frac{(r+1)\pi}{2} \right) \right\|
\]

\[
= \int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} k^{r+1} b_k \cos \left( kx + \frac{(r+1)\pi}{2} \right) + (b_{n+1} - b_{n+2}) D_{n+1}^r(x) \right| dx
\]

If we apply Abel's transformation, we obtain

\[
= \int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta b_k D_{k+1}^r(x) + b_{n+1} D_{n+1}^r(x) - b_{n+1} D_{n+1}^r(x) + b_{n+2} D_{n+1}^r(x) \right| dx
\]

\[
\leq \int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta b_k D_{k+1}^r(x) \right| dx + \left| b_{n+2} \right| \int_{0}^{\pi} \left| D_{n+1}^r(x) \right| dx
\]

\[
\leq \int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta b_k D_{k+1}^r(x) \right| dx + \delta \sum_{k=n+1}^{\infty} \Delta b_k \left| D_{k+1}^r(x) \right| dx
\]

\[
\leq \frac{\xi}{2} + \sum_{k=n+1}^{\infty} k^{r+1} \left| \Delta b_k \right| \int_{0}^{\pi} \left| D_{k+1}^r(x) \right| dx
\]

\[
\leq \frac{\xi}{2} + C \sum_{k=n+1}^{\infty} k^{r+1} \left| \Delta b_k \right| \int_{0}^{\pi} \left| D_{k+1}^r(x) \right| dx
\]

\[
\leq \frac{\xi}{2} + C \delta^{-(r+1)} \sum_{k=n+1}^{\infty} k^{r+1} \left| \Delta b_k \right| \leq \varepsilon \quad \text{by given hypothesis}
\]

Therefore, \( \|f'(x) - \beta_n'(x)\|_{L^1} = o(1) \) as \( n \to \infty \). \( \square \)
Remark 7. For \( r = 0 \), Theorem 6 reduces to Theorem 4.

Theorem 7. Let \( \{a_k\} \) be a sequence of numbers belonging to the class \( \tilde{C}_r \cap \tilde{BV}_r \), and if \( n'a_n \log n = o(1) \) as \( n \to \infty \). Then

\[ \|S'_n(x) - f'(x)\| = o(1), \quad n \to \infty. \]

where \( r = 0, 1, 2, \ldots \)

Proof. \( \|S'_n - f'\| \leq \|S'_n - \beta'_n\| + \|\beta'_n - f'\| \)

\[ \leq |b_{n+2}| \int_0^\pi |D_{n+1}^r(x)| dx + |b_{n+1}| \int_0^\pi |D_{n+1}^{r+1}(x)| dx + o(1) \]

\[ \leq \frac{|a_{n+2}|}{n+2} n^{r+1} \log n + \frac{|a_{n+1}|}{n+1} n^{r+1} \log n \quad \text{(by Lemma 2)} (14) \]

\[ = o(1) \text{ as } n \to \infty \]

\[ \square \]

Remark 8. For \( r = 0 \), Theorem 7 reduces to Theorem 5.

Remark 9. Combining Theorem 6 and Theorem 7 with Theorem 3, the following result holds:

Corollary 1. If \( \{a_k\} \in \tilde{S}_r \) \((r = 0, 1, 2, 3, \ldots)\) and if \( n'a_n \log n = o(1) \) as \( n \to \infty \). Then

(i) \( \|\beta'_n(x) - f'(x)\| = o(1), \quad n \to \infty. \)

(ii) \( \|S'_n(x) - f'(x)\| = o(1), \quad n \to \infty. \)

Author Contributions: All authors have contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript. Investigation, S.K.C.; Supervision, J. K. and S.S.B.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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