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On the Generalization of a Class of Harmonic Univalent Functions Defined by Differential Operator

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Abstract: In this article, a new class of harmonic univalent functions, defined by the differential operator, is introduced. Some geometric properties, like, coefficient estimates, extreme points, convex combination and convolution (Hadamard product) are obtained.

Keywords: harmonic univalent function; coefficient inequality; extreme points; convex combination; Hadamard product

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic. In any simply connected domain $\mathcal{B} \subset \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in $\mathcal{B}$. We call h and g are analytic part and co-analytic part of f respectively. Clunie and Sheil-Small [1] observed that a necessary and sufficient condition for the harmonic functions $f = h + \overline{g}$ to be locally univalent and sense-preserving in $\mathcal{B}$ is that $|h'(z)| > |g'(z)|, (z \in \mathcal{B})$.

Denote by S_H the family of harmonic functions $f = h + \overline{g}$, which are univalent and sense-preserving in the open unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$ where h and g are analytic in $\mathcal{B}$ and f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, |b_1| < 1. \tag{1}$$

Note that S_H reduces to the class of normalized analytic univalent functions if the co-analytic part of its members equals to zero.

Also, denote by $S_{\overline{H}}$ the subclass of S_H consisting of all functions $f_k(z) = h(z) + \overline{g_k(z)}$, where h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \text{ and } g_k(z) = (-1)^k \sum_{n=1}^{\infty} |b_n| z^n, |b_1| < 1. \tag{2}$$

In 1984 Clunie and Sheil-Small [1] investigated the class S_H, as well as its geometric subclass and obtained some coefficient bounds. Many authors have studied the family of harmonic univalent function (see References [2,3–7]).

In 2016 Makinde [8] introduced the differential operator F^k such that

$$F^k f(z) = z + \sum_{n=2}^{\infty} C_{nk} z^n, \tag{3}$$

where

$$C_{nk} = \frac{n!}{|n-k|!}, F^k f(z) = z^k \left[z^{-(k-1)} + \sum_{n=2}^{\infty} C_{nk} z^n \right], k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

and

$$F^0 f(z) = f(z), F^1 f(z) = z + \sum_{n=2}^{\infty} C_{n1} z^n.$$

Thus, it implies that $F^k f(z)$ is identically the same as $f(z)$ when $k = 0$. Also, it reduced the first differential coefficient of the Salagean differential operator when $k = 1$.

For $f = h + \overline{g}$ given by Equation (1), Sharma and Ravindar [9] considered the differential operator which defined by Equation (3) of f as

$$F^k f(z) = F^k h(z) + (-1)^k \overline{F^k g(z)}, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{C}, \tag{4}$$

where

$$F^k h(z) = z + \sum_{n=2}^{\infty} C_{nk} a_n z^n, F^k g(z) = \sum_{n=1}^{\infty} C_{nk} b_n z^n \text{ and } C_{nk} = \frac{n!}{|n-k|!}.$$

In this paper, motivated by study in [9], a new class $A_H(k, \alpha, \gamma)$($k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, 0 \le \gamma \le 1, 0 \le \alpha < 1,$) of harmonic univalent functions in $U = \{z \in \mathbb{C}: |z| < 1\}$ is introduced and studied. Furthermore, coefficient conditions, distortion bounds, extreme points, convex combination and radii of convexity for this class are obtained.

2. Main Results

2.1. The Class $A_H(k, \alpha, \gamma)$

Definition 1. *Let* $f(z) = h(z) + \overline{g(z)}$ *be a harmonic function, where* $h(z)$ *and* $g(z)$ *are given by Equation (1). Then* $f(z) \in A_H(k, \alpha, \gamma)$ *it satisfies*

$$Re\left\{ \frac{F^{k+1} f(z)}{(1-\gamma)z + \gamma F^k f(z)} \right\} > \alpha, \tag{5}$$

for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, 0 \le \gamma \le 1, 0 \le \alpha < 1, z \in U,$ *and* $F^k f(z)$ *defined by Equation (4)*

Let $A_{\overline{H}}(k, \alpha, \gamma)$ be the subclass of $A_H(k, \alpha, \gamma)$, where $A_{\overline{H}}(k, \alpha, \gamma) = S_{\overline{H}} \cap A_H(k, \alpha, \gamma)$.

Remark 1. *The class* $A_{\overline{H}}(k, \alpha, \gamma)$ *reduces to the class* $B_{\overline{H}}(k, \alpha)$ *[9], when* $\gamma = 1$.

Here, we give a sufficient condition for a function f to be in the class $A_H(k, \alpha, \gamma)$.

Theorem 1. *Let* $f(z) = h(z) + \overline{g(z)}$ *where* $h(z)$ *and* $g(z)$ *were given by (1.1). If*

$$\sum_{n=2}^{\infty} \emptyset(n,k,\alpha,\gamma)\,|a_n| + \sum_{n=1}^{\infty} \psi(n,k,\alpha,\gamma)\,|b_n| \le 1, \quad (6)$$

where

$$\emptyset(n,k,\alpha,\gamma) = \frac{(|n-k| - \alpha\gamma)C_{nk}}{(1-\alpha)}$$

$$\psi(n,k,\alpha,\gamma) = \frac{(|n-k| + \alpha\gamma)C_{nk}}{(1-\alpha)}$$

$$(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, 0 \le \gamma \le 1, 0 \le \alpha < 1, n \in \mathbb{N}),$$

then $f(z)$ is harmonic univalent and sense-preserving in U and $f(z) \in A_H(k,\alpha,\gamma)$.

Proof. Firstly, to show that $f(z)$ is harmonic univalent in U, suppose that $z_1, z_2 \in U$ for $\lfloor z_1 \rfloor \le \lfloor z_2 \rfloor < 1$, we have by inequality so that $z_1 \ne z_2$, then

$$\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right|$$

$$\ge 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| = 1 - \left|\frac{\sum_{n=1}^{\infty} b_n (z_1^n - z_2^n)}{(z_1 - z_2) - \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n)}\right|$$

$$\ge 1 - \frac{\sum_{n=1}^{\infty} n|b_n|}{1 - \sum_{n=2}^{\infty} n|a_n|} \ge 1 - \frac{\sum_{n=1}^{\infty} \frac{(|n-k| + \alpha\gamma)C_{nk}}{(1-\alpha)} |b_n|}{1 - \sum_{n=2}^{\infty} \frac{(|n-k| - \alpha\gamma)C_{nk}}{(1-\alpha)} |a_n|} \ge 0.$$

Thus f is a univalent function in U.

Note that f is sense-preserving in U. This is because

$$|h'(z)| \ge 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n| \ge 1 - \sum_{n=2}^{\infty} \frac{(|n-k| - \alpha\gamma)C_{nk}}{(1-\alpha)} |a_n|$$

$$\ge \sum_{n=1}^{\infty} \frac{(|n-k| + \alpha\gamma)C_{nk}}{(1-\alpha)} |b_n| \ge \sum_{n=1}^{\infty} n|b_n| \ge \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \ge |g'(z)|.$$

According to the condition of Equation (5), we only need to show that if Equation (6) holds, then

$$Re\left\{\frac{F^{k+1} f(z)}{(1-\gamma)z + \gamma F^k f(z)}\right\} = Re\left(w = \frac{A(z)}{B(z)}\right) > \alpha$$

where $z = re^{i\theta}, 0 \le \theta \le 2\pi, 0 \le r < 1$ and $0 \le \alpha < 1$.

Note that $A(z) = F^{k+1} f(z)$ and $B(z) = (1-\gamma)z + \gamma F^k f(z)$.

Using the fact that $Re(w) > \alpha$ if and only if $|w - (1+\alpha)| \le |w + (1-\alpha)|$, it suffices to show that

$$|A(z) - (1+\alpha)B(z)| - |A(z) + (1-\alpha)B(z)| \le 0 \quad (7)$$

Substituting for $A(z)$ and $B(z)$ in $|A(z) - (1+\alpha)B(z)|$, we obtain

$$|A(z)-(1+\alpha)B(z)| = |F^{k+1}f(z)-(1+\alpha)[(1-\gamma)z+\gamma F^k f(z)]|$$
$$= \left|\left[z+\sum_{n=2}^{\infty} C_{n(k+1)}a_n z^n + (-1)^{(k+1)}\sum_{n=1}^{\infty} C_{n(k+1)}\overline{b_n z^n}\right]\right.$$
$$\left. -(1+\alpha)\left[(1-\gamma)z+\gamma z+\gamma\sum_{n=2}^{\infty} C_{nk}a_n z^n+\gamma(-1)^k\sum_{n=1}^{\infty} C_{nk}\overline{b_n z^n}\right]\right| \quad (8)$$
$$\leq \alpha|z| + \sum_{n=2}^{\infty}\left|\left(\gamma(1+\alpha)\right)-|n-k|\right| C_{nk}|a_n||z|^n$$
$$+\sum_{n=1}^{\infty}\left|\left(\gamma(1+\alpha)\right)+|n-k|\right| C_{nk}|a_n||\overline{z}|^n.$$

Now, substituting for $A(z)$ and $B(z)$ in $|A(z)+(1-\alpha)B(z)|$, we obtain

$$|A(z)+(1-\alpha)B(z)| = |F^{k+1}f(z)+(1-\alpha)[(1-\gamma)z+\gamma F^k f(z)]|$$
$$= \left|\left[z+\sum_{n=2}^{\infty} C_{n(k+1)}a_n z^n + (-1)^{(k+1)}\sum_{n=1}^{\infty} C_{n(k+1)}\overline{b_n z^n}\right]\right.$$
$$\left. +(1-\alpha)\left[(1-\gamma)z+\gamma z+\gamma\sum_{n=2}^{\infty} C_{nk}a_n z^n+\gamma(-1)^k\sum_{n=1}^{\infty} C_{nk}\overline{b_n z^n}\right]\right| \quad (9)$$
$$\geq (2-\alpha)|z| - \sum_{n=2}^{\infty}\left|\left(\gamma(\alpha-1\,)\right)-|n-k|\right| C_{nk}|a_n||z|^n$$
$$-\sum_{n=1}^{\infty}\left||n-k|-\left(\gamma(1-\alpha)\right)\right| C_{nk}|a_n||\overline{z}|^n.$$

Substituting for Equation (8) and Equation (9) in the inequality we obtain

$$|A(z)-(1+\alpha)B(z)|-|A(z)+(1-\alpha)B(z)|$$
$$\leq \alpha|z| + \sum_{n=2}^{\infty}\left|\left(\gamma(1+\alpha)\right)-|n-k|\right| C_{nk}|a_n||z|^n$$
$$+\sum_{n=1}^{\infty}\left|\left(\gamma(1+\alpha)\right)+|n-k|\right| C_{nk}|b_n||\overline{z}|^n$$
$$+(\alpha-2)|Z| + \sum_{n=2}^{\infty}\left|\left(\gamma(\alpha-1\,)\right)-|n-k|\right| C_{nk}|a_n||z|^n$$
$$+\sum_{n=1}^{\infty}\left||n-k|-\left(\gamma(1-\alpha)\right)\right| C_{nk}|b_n||\overline{z}|^n.$$
$$= 2\sum_{n=2}^{\infty}(|n-k|-\alpha\gamma)C_{nk}|a_n| + 2\sum_{n=1}^{\infty}(|n-k|+\alpha\gamma)C_{nk}|b_n| - 2(1-\alpha)$$
$\leq 0.$ (by hypothesis).

Therefore, we have

$$\sum_{n=2}^{\infty}(|n-k|-\alpha\gamma)C_{nk}|a_n| + \sum_{n=1}^{\infty}(|n-k|+\alpha\gamma)C_{nk}|b_n| \leq (1-\alpha).$$

The harmonic univalent function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1}{\emptyset(n,k,\alpha,\gamma)} \mathcal{X}_n z^n + \sum_{n=1}^{\infty} \frac{1}{\psi(n,k,\alpha,\gamma)} \overline{\mathcal{Y}_n z^n}, \tag{10}$$

where $k \in \mathbb{N}_0$ and $\sum_{k=2}^{\infty}|\mathcal{X}_n| + \sum_{k=1}^{\infty}|\mathcal{Y}_n| = 1$, shows that the coefficient bound given by Equation (6) is sharp. Since

$$\sum_{n=2}^{\infty} \emptyset(n,k,\alpha,\gamma)\,|a_n| + \sum_{n=1}^{\infty} \psi(n,k,\alpha,\gamma)\,|b_n|$$

$$= \sum_{n=2}^{\infty} \emptyset(n,k,\alpha,\gamma) \frac{1}{\emptyset(n,k,\alpha,\gamma)} |\mathcal{X}_n| + \sum_{n=1}^{\infty} \psi(n,k,\alpha,\gamma) \frac{1}{\psi(n,k,\alpha,\gamma)} |\mathcal{Y}_n|$$

$$= \sum_{n=2}^{\infty} |\mathcal{X}_n| + \sum_{n=1}^{\infty} |\mathcal{Y}_n| = 1.$$

Now, we show that the condition of Equation (6) is also necessary for functions $f_k = h + \overline{g_k}$, where h and g_n are given by Equation (6).

Theorem 2. *Let* $f_k = h + \overline{g_k}$ *be given by Equation (6). Then* $f_k(z) \in A_{\overline{H}}(k,\alpha,\gamma)$ *if and only if the coefficient in condition of Equation (6) holds.*

Proof. We only need to prove the "only if" part of the theorem because of $A_{\overline{H}}(k,\alpha,\gamma) \subset A_H(k,\alpha,\gamma)$. Then by Equation (5), we have

$$Re\left\{\frac{F^{k+1}f(z)}{(1-\gamma)z + \gamma F^k f(z)}\right\} > \alpha$$

or, equivalently

$$Re\left[\frac{\begin{array}{c} z - \sum_{n=2}^{\infty} C_{n(k+1)}|a_n|z^n + (-1)^{2k+1}\sum_{n=1}^{\infty} C_{n(k+1)}|b_n|\overline{z}^n \\ -\alpha\left\{(1-\gamma)z + \gamma z + \gamma\sum_{n=2}^{\infty} C_{nk}|a_n|z^n + \gamma(-1)^{2k}\sum_{n=1}^{\infty} C_{nk}|b_n|\overline{z}^n\right\}\end{array}}{(1-\gamma)z + \gamma z - \gamma\sum_{n=2}^{\infty} C_{nk}|a_n|z^n + \gamma(-1)^{2k}\sum_{n=1}^{\infty} C_{nk}|b_n|\overline{z}^n}\right] \geq 0 \tag{11}$$

We observe that the above-required condition of Equation (11) must behold for all values of $z\ in\ U$. If we choose z to be real and $z \dashrightarrow 1^-$, we get

$$\frac{\begin{array}{c}(1-\alpha) - \sum_{n=2}^{\infty}(|n-k| - \alpha\gamma)C_{nk}|a_n| \\ + \sum_{n=1}^{\infty}(|n-k| + \alpha\gamma)C_{nk}|b_n|\end{array}}{1 - \gamma\sum_{n=2}^{\infty} C_{nk}|a_n|z^{n-1} + \gamma\sum_{n=1}^{\infty} C_{nk}|b_n|\overline{z}^{n-1}} \geq 0 \tag{12}$$

If the condition (6) does not hold, then the numerator in Equation (12) is negative for r sufficiently closed to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in Equation (12) is negative, therefore there is a contradicts the required condition for $f_k \in A_{\overline{H}}(k,\alpha,\gamma)$. □

2.2. Extreme Points

Here, we determine the extreme points of the closed convex hull of $A_{\overline{H}}(k,\alpha,\gamma)$, denoted by $clcoA_{\overline{H}}(k,\alpha,\gamma)$.

Theorem 3. $Let\ f_k\ given\ by\ (1.2). Then\ f_k \in A_{\overline{H}}(k,\alpha,\gamma)\ if\ and\ only\ if$

$$f_k(z) = \sum_{n=1}^{\infty}(\mathcal{X}_n h_n + \mathcal{Y}_n g_{kn})$$

where

$$h_1(z) = z,\ h_n(z) = z - \frac{1}{\emptyset(n,k,\alpha,\gamma)} z^n, n = 2,3,\ldots\ ,$$

$$g_{kn}(z) = z + (-1)^k \frac{1}{\psi(n,k,\alpha,\gamma)} \overline{z}^n, n = 1,2,\ldots$$

and

$$\mathcal{X}_n \geq 0, \mathcal{Y}_n \geq 0, \mathcal{X}_1 = 1 - \sum_{n=2}^{\infty} (\mathcal{X}_n + \mathcal{Y}_n) \geq 0$$

In particular the extreme points of $A_{\overline{H}}(k,\alpha,\gamma)$ are $\{h_n\}$ *and* $\{g_{kn}\}$.

Proof. Suppose

$$f_k(z) = \sum_{n=1}^{\infty} (\mathcal{X}_n h_n + \mathcal{Y}_n g_{kn})$$

$$= \sum_{n=1}^{\infty} (\mathcal{X}_n h_n + \mathcal{Y}_n g_{kn})\, z - \sum_{n=2}^{\infty} \frac{1}{\emptyset(n,k,\alpha,\gamma)} \mathcal{X}_n z^n + (-1)^k \sum_{n=1}^{\infty} \frac{1}{\psi(n,k,\alpha,\gamma)} \mathcal{Y}_n \overline{z}^n$$

$$= z - \sum_{n=2}^{\infty} \frac{1}{\emptyset(n,k,\alpha,\gamma)} \mathcal{X}_n z^n + (-1)^{k-1} \sum_{n=1}^{\infty} \frac{1}{\psi(n,k,\alpha,\gamma)} \mathcal{Y}_n \overline{z}^n$$

Then

$$\sum_{n=2}^{\infty} \emptyset(n,k,\alpha,\gamma)|a_n| + \sum_{n=1}^{\infty} \psi(n,k,\alpha,\gamma)|b_n|$$

$$= \sum_{k=2}^{\infty} \emptyset(n,k,\alpha,\gamma) \left(\frac{1}{\emptyset(n,k,\alpha,\gamma)} \mathcal{X}_n \right) + \sum_{k=1}^{\infty} \psi(n,k,\alpha,\gamma) \left(\frac{1}{\psi(n,k,\alpha,\gamma)} \mathcal{Y}_n \right)$$

$$= \sum_{n=2}^{\infty} \mathcal{X}_n + \sum_{n=1}^{\infty} \mathcal{Y}_n = 1 - \mathcal{X}_1 \leq 1\,.$$

Therefore $f_k(z) \in clcoA_{\overline{H}}(k,\alpha,\gamma)$.
Conversely, if $f_k(z) \in clcoA_{\overline{H}}(k,\alpha,\gamma)$. Then

Set $\mathcal{X}_n = \emptyset(n,k,\alpha,\gamma)|a_n|$, $(n = 2,3,\ldots)$ and $\mathcal{Y}_n = \psi(n,k,\alpha,\gamma)|b_n|$,

$(n = 1,2,\ldots)$ and $\mathcal{X}_1 = 1 - \sum_{n=2}^{\infty} \mathcal{X}_n + \sum_{n=1}^{\infty} \mathcal{Y}_n$

The required representation is obtained as

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{z}^n$$

$$= z - \sum_{n=2}^{\infty} \frac{1}{\emptyset(n,k,\alpha,\gamma)} \mathcal{X}_n z^n + (-1)^k \sum_{n=1}^{\infty} \frac{1}{\psi(n,k,\alpha,\gamma)} \mathcal{Y}_n \overline{z}^n$$

$$= z - \sum_{n=2}^{\infty} [z - h_n(z)] \mathcal{X}_n + \sum_{n=1}^{\infty} [z - g_{kn}(z)] \mathcal{Y}_n$$

$$= \left[1 - \sum_{n=2}^{\infty} \mathcal{X}_n - \sum_{n=1}^{\infty} \mathcal{Y}_n \right] z + \sum_{n=2}^{\infty} h_n(z)\, \mathcal{X}_n + \sum_{n=1}^{\infty} g_{kn}(z)\, \mathcal{Y}_n = \sum_{n=1}^{\infty} (\mathcal{X}_n h_n + \mathcal{Y}_n g_{kn})$$

□

2.3. Convex Combination

Here, we show that the class $A_{\overline{H}}(k,\alpha,\gamma)$ is closed under convex combination of its members.

Let the function $f_{k,i}(z)$ be defined, for $i = 1,2,\dots,m$ by

$$f_{k,i}(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + (-1)^k \sum_{n=1}^{\infty} |b_{n,i}| \overline{z}^n \tag{13}$$

Theorem 4. *Let the functions* $f_{k,i}(z)$, *defined by* Equation (13) *be in the class* $A_{\overline{H}}(k,\alpha,\gamma)$, *for every* $i = 1,2,\dots,m$. *Then the functions* $c_i(z)$ *defined by*

$$c_i(z) = \sum_{i=1}^{\infty} t_i \ f_{k,i}(z), 0 \le t_i \le 1$$

are also in the class $A_{\overline{H}}(k,\alpha,\gamma)$, *where* $\sum_{i=1}^{\infty} t_i = 1$.

Proof. According to the definition of $c_i(z)$, we can write

$$c_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i \left|a_{n,i}\right| \right) z^n + (-1)^k \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \ \left|b_{n,i}\right| \right) \overline{z}^n$$

Further, since $f_{k,i}(z)$ are in $A_{\overline{H}}(k,\alpha,\gamma)$ for every $i = 1,2,\dots,m$, then by Theorem (2.1.2), we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty} \emptyset(n,k,\alpha,\gamma) \left(\sum_{i=1}^{\infty} t_i \left|a_{n,i}\right| \right) + \sum_{k=1}^{\infty} \psi(n,k,\alpha,\gamma) \left(\sum_{i=1}^{\infty} t_i \ \left|b_{n,i}\right| \right) \\ &\quad = \sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \emptyset(n,k,\alpha,\gamma) \left|a_{n,i}\right| + \sum_{k=1}^{\infty} \psi(n,k,\alpha,\gamma) \left|b_{n,i}\right| \right) \le \sum_{i=1}^{\infty} t_i = 1, \end{aligned}$$

which is required coefficient condition.

2.4. Convolution (Hadamard Product) Property

Here, we show that the class $A_{\overline{H}}(k,\alpha,\gamma)$ is closed under convolution.

The convolution of two harmonic functions

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{z}^n, \tag{14}$$

and

$$Q_n(z) = z - \sum_{n=2}^{\infty} |L_n| z^n + (-1)^k \sum_{n=1}^{\infty} |M_n| \overline{z}^n \tag{15}$$

is defined as

$$\begin{aligned} (f_n * Q_n)(z) &= f_n(z) * Q_n(z) \\ &= z - \sum_{n=2}^{\infty} |a_n L_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n M_n| \overline{z}^n \end{aligned} \tag{16}$$

Using Equations (12)–(14), we prove the following theorem.

Theorem 5. *For* $0 \le \mu \le \alpha < 1, k \in \mathbb{N}_0$, *let* $f_n \in A_{\overline{H}}(k,\alpha,\gamma)$ *and* $Q_n \in A_{\overline{H}}(k,\mu,\gamma)$. *Then*

$$f_n * Q_n \in A_{\overline{H}}(k,\alpha,\gamma) \subset A_{\overline{H}}(k,\mu,\gamma).$$

Proof. Let

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{z}^n$$

be in the class $A_{\overline{H}}(k, \alpha, \gamma)$ and

$$Q_n(z) = z - \sum_{n=2}^{\infty} |L_n| z^n + (-1)^k \sum_{n=1}^{\infty} |M_n| \overline{z}^n,$$

be in $A_{\overline{H}}(k, \mu, \gamma)$.

Then the convolution $f_n * Q_n$ is given by Equation (16), we want to show that the coefficients of $f_n * Q_n$ satisfy the required condition given in Theorem 1.

For $Q_n \in A_{\overline{H}}(k, \mu, \gamma)$, we note that $|L_n| < 1$ and $|M_n| < 1$. Now consider convolution functions $f_n * Q_n$ as follows:

$$\begin{aligned} &\sum_{k=2}^{\infty} \emptyset(n, k, \mu, \gamma) |a_n| |L_n| + \sum_{k=1}^{\infty} \psi(n, k, \mu, \gamma) |b_n| |M_n| \\ &\leq \sum_{k=2}^{\infty} \emptyset(n, k, \mu, \gamma) |a_n| + \sum_{k=1}^{\infty} \psi(n, k, \mu, \gamma) |b_n| \leq 1. \end{aligned}$$

Since $0 \leq \mu \leq \alpha < 1$ and $f_n \in A_{\overline{H}}(k, \alpha, \gamma)$. Therefore $f_n * Q_n \in A_{\overline{H}}(k, \alpha, \gamma) \subset A_{\overline{H}}(k, \mu, \gamma)$. □

2.5. Integral Operator

Here, we examine the closure property of the class $A_{\overline{H}}(k, \alpha, \gamma)$ under the generalized Bernardi-Libera-Livingston integral operator (see References [10,11]) $\mathcal{L}_u(f)$ which is defined by,

$$\mathcal{L}_u(f) = \frac{u+1}{z^u} \int_0^z t^{u-1} f(t) dt, u > -1. \quad (17)$$

Theorem 6. *Let* $f_k(z) \in A_{\overline{H}}(k, \alpha, \gamma)$. *Then*

$$\mathcal{L}_u(f_k(z)) \in A_{\overline{H}}(k, \alpha, \gamma)$$

Proof. From definition of $\mathcal{L}_u(f_k(z))$ given by Equation (17), it follows that

$$\begin{aligned} \mathcal{L}_u(f_k(z)) &= \frac{u+1}{z^u} \int_0^z t^{u-1} \left(t - \sum_{n=2}^{\infty} |a_n| t^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{t}^n \right) dt \\ &= z - \sum_{n=2}^{\infty} \frac{u+1}{u+n} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} \frac{u+1}{u+n} |b_n| z^n \\ &= z - \sum_{n=2}^{\infty} G_n z^n + (-1)^{n-1} \sum_{n=1}^{\infty} L_n z^n \end{aligned}$$

where

$$G_n = \frac{u+1}{u+n} |a_n|, \text{and}$$

$$L_n = \frac{u+1}{u+n} |b_n|$$

Hence

$$\sum_{k=2}^{\infty} \emptyset(n, k, \alpha, \gamma) \frac{u+1}{u+n} |a_n| + \sum_{k=1}^{\infty} \psi(n, k, \alpha, \gamma) \frac{u+1}{u+n} |b_n|$$

$$\leq \sum_{n=2}^{\infty} \emptyset(n,k,\alpha,\gamma)|a_n| + \sum_{n=1}^{\infty} \psi(n,k,\alpha,\gamma)\,|b_n| \leq 1.$$

by Theorem 2.

Therefore, we have $\mathcal{L}_u(f_k(z)) \in A_{\overline{H}}(k,\alpha,\gamma)$.

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