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Article

Some Symmetric Identities Involving the Stirling Polynomials Under the Finite Symmetric Group

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Abstract: In the paper, the authors present some symmetric identities involving the Stirling polynomials and higher order Bernoulli polynomials under all permutations in the finite symmetric group of degree n. These identities extend and generalize some known results.

Keywords: symmetric identity; Stirling polynomial; Stirling number of the second kind; finite symmetric group; permutation; higher order Bernoulli polynomial

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1. Introduction

The Stirling numbers arise in a variety of analytic and combinatorial problems. They were introduced in the eighteenth century by James Stirling. There are two kinds of the Stirling numbers: the Stirling numbers of the first and second kind. Some combinatorial identities for the Stirling numbers of these two kinds are studied and collected in [1–8] and closely related references.

The Stirling numbers of second kind $S(n,k)$ are the numbers of ways to partition a set of n elements into k nonempty subsets. It can be computed by

$$S(n,k)=\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

and can be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!},\quad k\geq 0.$$

The Stirling polynomials $S_n(x)$ can be generated [4,9–13] by

$$F(t,x)=\left(\frac{t}{1-e^{-t}}\right)^{x+1}=\sum_{n=0}^{\infty}S_n(x)\frac{t^n}{n!} \tag{1}$$

and the first five Stirling polynomials $S_n(x)$ for $0\leq n\leq 4$ are

$$S_0(x)=1,\qquad S_1(x)=\frac{x+1}{2},\qquad S_2(x)=\frac{(3x+2)(x+1)}{12},\qquad S_3(x)=\frac{x(x+1)^2}{8}.$$

The Stirling polynomials $S_n(x)$ generalize several important sequences of numbers, including the Stirling numbers of the second kind $S(n,k)$ and the Bernoulli numbers B_n, appearing in combinatorics, number theory, and analysis.

In the case $x = -n$ for $n \in \mathbb{N}$ in Equation (1), we can derive

$$\sum_{k=0}^{\infty} S_k(-n)\frac{t^k}{k!} = \left(\frac{1-e^{-t}}{t}\right)^{n-1} = \frac{(-1)^{n-1}}{t^{n-1}}(e^{-t}-1)^{n-1} = \frac{(-1)^{n-1}}{t^{n-1}}(n-1)! \sum_{k=n-1}^{\infty} S(k,n-1)\frac{(-t)^k}{k!}$$
$$= \sum_{k=0}^{\infty} S(k+n-1,n-1)\frac{(n-1)!k!}{(k+n-1)!}(-1)^k\frac{t^k}{k!} = \sum_{k=0}^{\infty} \frac{S(k+n-1,n-1)}{\binom{k+n-1}{k}}(-1)^k\frac{t^k}{k!}.$$

Equating coefficients on the very ends of the above identity arrives at

$$S(m,n) = (-1)^{m-n}\binom{m-n+n}{k} S_{m-n}(-n-1) \tag{2}$$

for $m, n \in \mathbb{N} \cup \{0\}$.

It is common knowledge [14] that the Bernoulli numbers B_n are generated by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi. \tag{3}$$

By considering the case $x = 0$ in Equation (1) and the definition in Equation (3) for the Bernoulli numbers B_n, we have

$$\sum_{n=0}^{\infty} S_n(0)\frac{t^n}{n!} = \frac{t}{1-e^{-t}} = \frac{-t}{e^{-t}-1} = \sum_{n=0}^{\infty}(-1)^n B_n \frac{t^n}{n!}.$$

Comparing the coefficients on both sides of this equation results in

$$B_n = (-1)^n S_n(0), \quad n \ge 0. \tag{4}$$

One can also find Equations (2) and (4) in [4], p. 154.

The higher order Bernoulli numbers $B_n^{(\alpha)}$ for $n \ge 0$ and $\alpha \in \mathbb{N}$ can be generated [15–17] by

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{t^n}{n!}, \quad |t| < 2\pi.$$

Combining this with Equation (1) yields the relation

$$S_k(n) = (-1)^k B_k^{(n+1)}, \quad k \in \mathbb{N} \cup \{0\}, \quad n \in \mathbb{N}. \tag{5}$$

In the paper [15], some new symmetric identities for the q-Bernoulli polynomials are derived from the fermionic integral on $\mathbb{Z}_p$. In [18,19], the method in the paper [15] is extended to the q-Euler and q-Genocchi polynomials, respectively. In [13], some symmetric identities involving the Stirling polynomials $S_n(x)$ are investigated. The symmetric identities of some special polynomials, such as higher order Bernoulli polynomials $B_n^{(\alpha)}$, higher order q-Euler polynomials, degenerate generalized Bernoulli polynomials, and degenerate higher order q-Euler polynomials, have been studied by several mathematicians in [10,16,20–23] and closely related references therein.

The purpose of this paper is to investigate some interesting symmetric identities involving the Stirling polynomials $S_n(x)$ under the finite symmetric group S_n. By specializing these identities, we can obtain some new symmetric identities involving the Stirling polynomials $S_n(x)$.

2. Symmetric Identities of Stirling Polynomials

Now, we start out to state and prove our main results.

Theorem 1. *Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ and $\ell \in \mathbb{N} \cup \{0\}$. Then, the expression*

$$\sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} (-1)^{j_{\sigma(s)}} \binom{\ell}{i_{\sigma(1)}, \ldots, i_{\sigma(s)}, j_{\sigma(s)}, i_{\sigma(s+1)} \ldots, i_{\sigma(n)}} \widehat{w}_{\sigma(s)}^{j_{\sigma(s)}+1} S_{i_{\sigma(s)}}(x-1) A_{j_{\sigma(s)}}[w_{\sigma(s)}-1] \prod_{\substack{k=1 \\ k\neq s}}^{n} S_{i_{\sigma(k)}}(x)$$

is invariant under any permutation $\sigma \in S_n$, where $\widehat{w}_i = \frac{1}{w_i}\prod_{k=1}^{n} w_k$,

$$\mathcal{I}_{\sigma(s)} = i_{\sigma(1)} + \cdots + i_{\sigma(s)} + j_{\sigma(s)} + i_{\sigma(s+1)} + \cdots + i_{\sigma(n)}, \tag{6}$$

and $A_k(n) = 0^k + 1^k + \cdots + n^k$ for $k, n \in \mathbb{N} \cup \{0\}$.

Proof. For convenience, we denote $w = \prod_{k=1}^{n} w_k$. Define

$$I = I(w_1, w_2, \ldots, w_n) = \frac{t^{nx+(n-1)}\left(1-e^{wt}\right)}{\prod_{i=1}^{n}\left(1-e^{-\widehat{w}_i t}\right)^{x+1}}. \tag{7}$$

It is clear that we can rewrite I as

$$\left(\frac{t}{1-e^{-\widehat{w}_1 t}}\right)^{x}\left(\frac{1-e^{wt}}{1-e^{-\widehat{w}_1 t}}\right)\prod_{k=2}^{n}\left(\frac{t}{1-e^{-\widehat{w}_k t}}\right)^{x+1} \triangleq I_{(1)}. \tag{8}$$

By applying Equations (1)–(8), we can rearrange the equality in Equation (7) as

$$\begin{aligned} I_{(1)} &= \frac{1}{\widehat{w}_1^{x}}\left[\sum_{i_1=0}^{\infty} S_{i_1}(x-1)\frac{\widehat{w}_1^{i_1} t^{i_1}}{i_1!}\right]\left[\sum_{j_1=0}^{\infty}(-1)^{j_1} A_{j_1}(w_1-1)\frac{\widehat{w}_1^{j_1} t^{j_1}}{j_1!}\right]\prod_{k=2}^{n}\left[\frac{1}{\widehat{w}_k^{x+1}}\sum_{i_k=0}^{\infty} S_{i_k}(x)\frac{\widehat{w}_k^{i_k} t^{i_k}}{i_k!}\right] \\ &= \frac{1}{\widehat{w}_1^{x+1}\widehat{w}_2^{x+1}\cdots\widehat{w}_n^{x+1}}\sum_{\ell=0}^{\infty}\left[\sum_{i_1+j_1+i_2+\cdots+i_n=0}^{\ell}\binom{\ell}{i_1, j_1, i_2, \ldots, i_n}(-1)^{j_1}\right. \\ &\quad \left.\times\, \widehat{w}_1^{i_1+j_1+1}\widehat{w}_2^{i_2}\widehat{w}_3^{i_3}\cdots\widehat{w}_n^{i_n} S_{i_1}(x-1) A_{j_1}(w_1-1)\prod_{k=2}^{n} S_{i_k}(x)\right]\frac{t^{\ell}}{\ell!}. \end{aligned}$$

Similarly, from Equation (7), we can also consider I as

$$\begin{aligned} I_{(2)} &\triangleq \left(\frac{t}{1-e^{-\widehat{w}_1 t}}\right)^{x+1}\left(\frac{t}{1-e^{-\widehat{w}_2 t}}\right)^{x}\left(\frac{1-e^{-wt}}{1-e^{-\widehat{w}_2 t}}\right)\prod_{k=3}^{n}\left(\frac{t}{1-e^{-\widehat{w}_k t}}\right)^{x+1} \\ &= \frac{1}{\widehat{w}_1^{x+1}\widehat{w}_2^{x+1}\cdots\widehat{w}_n^{x+1}}\sum_{\ell=0}^{\infty}\left[\sum_{i_1+i_2+j_2+i_3+\cdots+i_n=0}^{\ell}\binom{\ell}{i_1, i_2, j_2, i_3, \ldots, i_n}(-1)^{j_2}\right. \\ &\quad \left.\times\, \widehat{w}_1^{i_1}\widehat{w}_2^{i_2+j_2+1}\widehat{w}_3^{i_3}\cdots\widehat{w}_n^{i_n} S_{i_1}(x) S_{i_2}(x-1) A_{j_2}(w_2-1)\prod_{k=3}^{n} S_{i_k}(x)\right]\frac{t^{\ell}}{\ell!}. \end{aligned}$$

Inductively, for any $s \in \{1, 2, \ldots, n\}$, from Equation (7), we can consider I as

$$\begin{aligned} I_{(s)} &= \left(\frac{t}{1-e^{-\widehat{w}_s t}}\right)^{x}\left(\frac{1-e^{-wt}}{1-e^{-\widehat{w}_s t}}\right)\prod_{\substack{k=1 \\ k\neq s}}^{n}\left(\frac{t}{1-e^{-\widehat{w}_k t}}\right)^{x+1} \\ &= \frac{1}{\widehat{w}_1^{x+1}\widehat{w}_2^{x+1}\cdots\widehat{w}_n^{x+1}}\sum_{\ell=0}^{\infty}\left[\sum_{i_1+\cdots+i_s+j_s+i_{s+1}+\cdots+i_n=0}^{\ell}\binom{\ell}{i_1, \cdots, i_s, j_s, i_{s+1}, \ldots, i_n}\right. \end{aligned}$$

$$\times (-1)^{j_s} \widehat{w}_1^{i_1} \widehat{w}_2^{i_2} \cdots \widehat{w}_s^{i_s+j_s+1} \cdots \widehat{w}_n^{i_n} S_{i_s}(x-1) A_{j_s}(w_s - 1) \prod_{\substack{k=1 \\ k\neq s}}^{n} S_{i_k}(x) \Bigg] \frac{t^\ell}{\ell!}.$$

Combining the above three equalities leads to the expression

$$\begin{aligned} I_{(\sigma(s))} &= \left[\frac{t}{1-e^{-\widehat{w}_{\sigma(s)}t}}\right]^x \left[\frac{1-e^{-wt}}{1-e^{-\widehat{w}_{\sigma(s)}t}}\right] \prod_{\substack{k=1 \\ k\neq s}}^{n} \left[\frac{t}{1-e^{-\widehat{w}_{\sigma(k)}t}}\right]^{x+1} \\ &= \frac{1}{\widehat{w}_{\sigma(1)}^{x+1} \widehat{w}_{\sigma(2)}^{x+1} \cdots \widehat{w}_{\sigma(n)}^{x+1}} \sum_{\ell=0}^{\infty} \Bigg[\sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} \binom{\ell}{i_{\sigma(1)}, \ldots, i_{\sigma(s)}, j_{\sigma(s)}, i_{\sigma(s+1)} \ldots, i_{\sigma(n)}} (-1)^{j_{\sigma(s)}} \\ &\quad \times \widehat{w}_{\sigma(1)}^{i_{\sigma(1)}} \widehat{w}_{\sigma(2)}^{i_{\sigma(2)}} \cdots \widehat{w}_{\sigma(s)}^{i_{\sigma(s)}+j_{\sigma(s)}+1} \cdots \widehat{w}_{\sigma(n)}^{i_{\sigma(n)}} S_{i_{\sigma(s)}}(x-1) A_{j_{\sigma(s)}}[w_{\sigma(s)} - 1] \prod_{\substack{k=1 \\ k\neq s}}^{n} S_{i_{\sigma(k)}}(x) \Bigg] \frac{t^\ell}{\ell!} \end{aligned}$$

which are invariant under any permutations $\sigma \in S_n$. $\square$

Combining Theorem 1 for $x = 1$ with the equalities in Equations (4) and (5) deduces Corollary 1 below.

Corollary 1. *Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ and $\ell \geq 0$. Then the quantities*

$$\sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} (-1)^{j_{\sigma(s)}} \binom{\ell}{i_{\sigma(1)}, \ldots, i_{\sigma(s)}, j_{\sigma(s)}, i_{\sigma(s+1)}, \ldots, i_{\sigma(n)}} \widehat{w}_{\sigma(s)}^{j_{\sigma(s)}+1} B_{i_{\sigma(s)}} A_{j_{\sigma(s)}}[w_{\sigma(s)} - 1] \prod_{\substack{k=1 \\ k\neq s}}^{n} B_{i_{\sigma(k)}}^{(2)}$$

are invariant under any permutation $\sigma \in S_n$.

Replacing x by $-p$ for $p \in \mathbb{N} \cup \{0\}$ in Theorem 1 and employing Equation (2) result in the following corollary.

Corollary 2. *Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ and $\ell, p \geq 0$. Then the expressions*

$$\begin{aligned} &\sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} (-1)^{j_{\sigma(s)}} \binom{\ell}{i_{\sigma(1)}, \ldots, i_{\sigma(s)}, j_{\sigma(s)}, i_{\sigma(s+1)} \ldots, i_{\sigma(n)}} \widehat{w}_{\sigma(s)}^{j_{\sigma(s)}+1} \\ &\qquad \times \frac{S(i_{\sigma(s)}+p, p)}{\binom{i_{\sigma(s)}+p}{i_{\sigma(s)}}} A_{j_{\sigma(s)}}[w_{\sigma(s)} - 1] \prod_{\substack{k=1 \\ k\neq s}}^{n} \frac{S(i_{\sigma(k)}+p-1, p-1)}{\binom{i_{\sigma(k)}+p-1}{i_{\sigma(k)}}} \end{aligned}$$

are invariant under any permutations $\sigma \in S_n$.

Finally, combining Corollary 2 with Equation (4) leads to the following corollary.

Corollary 3. *Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ and $\ell, p \geq 0$. Then the expressions*

$$\sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} (-1)^{j_{\sigma(s)}} \binom{\ell}{i_{\sigma(1)}, \ldots, i_{\sigma(s)}, j_{\sigma(s)}, i_{\sigma(s+1)} \ldots, i_{\sigma(n)}} \widehat{w}_{\sigma(s)}^{j_{\sigma(s)}+1} B_{i_{\sigma(s)}}^{(p)} A_{j_{\sigma(s)}}[w_{\sigma(s)} - 1] \prod_{\substack{k=1 \\ k\neq s}}^{n} B_{i_{\sigma(k)}}^{(p+1)}$$

are invariant under any permutations $\sigma \in S_n$.

From Theorem 1 to Corollary 3, if taking $w_4 = w_5 = \cdots = w_n = 1$, then we have the following corollaries.

Corollary 4. *Let w_1, w_2, w_3 be any positive integers, n be any non-negative integer. Then the expressions*

$$\begin{aligned}
&\sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{\ell} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} S_m(x-1) A_\ell(w_3-1) S_k(x) S_j(x) \\
&= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{k} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{m+k+j+1} S_m(x) S_\ell(x-1) A_k(w_1-1) S_j(x) \\
&= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{j} w_1^{m+k+j+1} w_2^{m+\ell} w_3^{\ell+k+j+1} S_m(x) S_\ell(x) S_k(x-1) A_j(w_2-1)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} B_m A_\ell(w_3-1) B_k^{(2)} B_j^{(2)} \\
&= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{m+k+j+1} B_m^{(2)} B_\ell A_k(w_1-1) B_j^{(2)} \\
&= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+k+j+1} w_2^{m+\ell} w_3^{\ell+k+j+1} B_m^{(2)} B_\ell^{(2)} B_k A_j(w_2-1)
\end{aligned}$$

are invariant under any permutations of w_1, w_2, w_3.

Corollary 5. *For $n, p \in \mathbb{N} \cup \{0\}$ and $w_1, w_2, w_3 \in \mathbb{N}$, the expressions*

$$\begin{aligned}
&\sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} \\
&\quad \times \frac{S(m+p,p)}{\binom{m+p}{m}} A_\ell(w_3-1) \frac{S(k+p-1,p-1)}{\binom{k+p-1}{k}} \frac{S(j+p-1,p-1)}{\binom{j+p-1}{j}} \\
&= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{m+k+j+1} \\
&\quad \times \frac{S(m+p-1,p-1)}{\binom{m+p-1}{m}} \frac{S(\ell+p,p)}{\binom{\ell+p}{\ell}} A_k(w_1-1) \frac{S(j+p-1,p-1)}{\binom{j+p-1}{j}} \\
&= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+\ell+j+1} w_2^{m+\ell} w_3^{\ell+k+j+1} \\
&\quad \times \frac{S(m+p-1,p-1)}{\binom{m+p-1}{m}} \frac{S(\ell+p-1,\ell)}{\binom{\ell+p-1}{\ell}} \frac{S(k+p,p)}{\binom{k+p}{k}} A_j(w_2-1)
\end{aligned}$$

are invariant under any permutations of w_1, w_2, w_3.

Corollary 6. *For $n, p \in \mathbb{N} \cup \{0\}$ and $w_1, w_2, w_3 \in \mathbb{N}$, the expressions*

$$\begin{aligned}
&\sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} B_m^{(p)} A_\ell(w_3-1) B_k^{(p+1)} B_j^{(p+1)} \\
&= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{m+k+j+1} B_m^{(p+1)} B_\ell^{(p)} A_k(w_1-1) B_j^{(p+1)} \\
&= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+k+j+1} w_2^{m+\ell} w_3^{\ell+k+j+1} B_m^{(p+1)} B_\ell^{(p+1)} B_k^{(p)} A_j(w_2-1)
\end{aligned}$$

are invariant under any permutations of w_1, w_2, w_3.

3. Symmetric Identities via Higher Order Bernoulli Polynomials

Recall from [16] that higher order Bernoulli polynomials $B_n^{(\alpha)}(x)$ can be generated by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad \alpha \in \mathbb{N}. \tag{9}$$

Now, we start out to investigate symmetric identities for the Stirling polynomials $S_n(x)$ under the finite symmetric group S_n via higher order Bernoulli polynomials $B_n^{(\alpha)}(x)$.

Theorem 2. *Let* $w_1, w_2, \ldots, w_n \in \mathbb{N}$ *and* $\ell \geq 0$. *Then the quantities*

$$\sum_{i=0}^{w_{\sigma(s)}-1} \sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} (-1)^{j_{\sigma(s)}} \binom{\ell}{i_{\sigma(1)}, \ldots, i_{\sigma(s)}, j_{\sigma(s)}, i_{\sigma(s+1)} \ldots, i_{\sigma(n)}} \\ \times \widehat{w}_{\sigma(s)}^{j_{\sigma(s)}+1} S_{i_{\sigma(s)}}(x+r-1) B_{j_{\sigma(s)}}^{(r)}(i) \prod_{\substack{k=1 \\ k\neq s}}^{n} S_{i_{\sigma(k)}}(x+r)$$

are invariant under any permutations $\sigma \in S_n$, *where* $\mathcal{I}_{\sigma(s)}$ *is defined by Equation* (6).

Proof. Define $I^{(r)} = I^{(r)}(w_1, w_2, \ldots, w_n)$ as

$$I^{(r)} = \frac{t^{nx+nr+(n-1)}\left(1-e^{wt}\right)}{\prod_{i=1}^{n}\left(1-e^{-\widehat{w}_i t}\right)^{x+r+1}}.$$

Then we can rewrite $I^{(r)}$ as

$$I_{(1)}^{(r)} = \left(\frac{t}{1-e^{-\widehat{w}_1 t}}\right)^{x} \left(\frac{1-e^{wt}}{1-e^{-\widehat{w}_1 t}}\right) \left(\frac{t}{1-e^{-\widehat{w}_1 t}}\right)^{r} \prod_{k=2}^{n} \left(\frac{t}{1-e^{-\widehat{w}_k t}}\right)^{x+r+1}. \tag{10}$$

Applying Equations (1) and (9) to the equality in Equation (10) gives

$$\begin{aligned} I_{(1)}^{(r)} &= \frac{1}{\widehat{w}_1^{x+r}} \sum_{i_1=0}^{\infty} S_{i_1}(x+r-1)\frac{\widehat{w}_1^{i_1} t^{i_1}}{i_1!} \sum_{i=0}^{w_1-1} \sum_{j_1=0}^{\infty} B_{j_1}^{(r)}(w_1-1)\frac{\widehat{w}_1^{j_1} t^{j_1}}{j_1!} \prod_{k=2}^{n} \frac{1}{\widehat{w}_k^{x+r+1}} \sum_{i_k=0}^{\infty} S_{i_k}(x+r)\frac{\widehat{w}_k^{i_k} t^{i_k}}{i_k!} \\ &= \frac{1}{\widehat{w}_1^{x+r+1}\widehat{w}_2^{x+r+1}\cdots\widehat{w}_n^{x+r+1}} \sum_{\ell=0}^{\infty} \sum_{i=0}^{w_1-1} \left[\sum_{i_1+j_1+i_2+\cdots+i_n=0}^{\ell} (-1)^{j_1} \binom{\ell}{i_1, j_1, i_2, \ldots, i_n} \right. \\ &\quad \left. \times \widehat{w}_1^{i_1+j_1+1}\widehat{w}_2^{i_2}\widehat{w}_3^{i_3}\cdots\widehat{w}_n^{i_n} S_{i_1}(x+r-1) B_{j_1}^{(r)}(i) \prod_{k=2}^{n} S_{i_k}(x+r) \right] \frac{t^\ell}{\ell!}. \end{aligned} \tag{11}$$

Similarly, we can rewrite $I^{(r)}$ as

$$\begin{aligned} I_{(2)}^{(r)} &= \left(\frac{t}{1-e^{-\widehat{w}_1 t}}\right)^{x+r+1} \left(\frac{t}{1-e^{-\widehat{w}_2 t}}\right)^{x} \left(\frac{1-e^{-wt}}{1-e^{-\widehat{w}_2 t}}\right) \left(\frac{t}{1-e^{-\widehat{w}_2 t}}\right)^{r} \prod_{k=3}^{n} \left(\frac{t}{1-e^{-\widehat{w}_k t}}\right)^{x+r+1} \\ &= \frac{1}{\widehat{w}_1^{x+r+1}\widehat{w}_2^{x+r+1}\cdots\widehat{w}_n^{x+r+1}} \sum_{\ell=0}^{\infty} \sum_{i=0}^{w_2-1} \left[\sum_{i_1+i_2+j_2+i_3+\cdots+i_n=0}^{\ell} (-1)^{j_2} \binom{\ell}{i_1, i_2, j_2, i_3, \ldots, i_n} \right. \\ &\quad \left. \times \widehat{w}_1^{i_1}\widehat{w}_2^{i_2+j_2+1}\widehat{w}_3^{i_3}\cdots\widehat{w}_n^{i_n} S_{i_1}(x+r) S_{i_2}(x+r-1) B_{j_2}^{(r)}(i) \prod_{k=3}^{n} S_{i_k}(x+r) \right] \frac{t^\ell}{\ell!}. \end{aligned} \tag{12}$$

Inductively, for any $s \in \{1, 2, \ldots, n\}$, we can rearrange $I^{(r)}$ as

$$I_{(s)}^{(r)} = \left(\frac{t}{1-e^{-\widehat{w}_s t}}\right)^x \left(\frac{1-e^{-wt}}{1-e^{-\widehat{w}_s t}}\right) \left(\frac{t}{1-e^{-\widehat{w}_s t}}\right)^r \prod_{\substack{k=1\\k\neq s}}^{n} \left(\frac{t}{1-e^{-\widehat{w}_k t}}\right)^{x+r+1}$$
$$= \frac{1}{\widehat{w}_1^{x+r+1}\widehat{w}_2^{x+r+1}\cdots\widehat{w}_n^{x+r+1}} \sum_{\ell=0}^{\infty} \sum_{i=0}^{w_s-1} \left[\sum_{i_1+\cdots i_s+j_s+i_{s+1}+\cdots+i_n=0}^{\ell} (-1)^{j_s} \binom{\ell}{i_1,\cdots,i_s,j_s,i_{s+1}\ldots,i_n} \right.$$
$$\left. \times\, \widehat{w}_1^{i_1}\widehat{w}_2^{i_2}\cdots\widehat{w}_s^{i_s+j_s+1}\cdots\widehat{w}_n^{i_n} S_{i_s}(x+r-1) B_{j_s}^{(r)}(i) \prod_{\substack{k=1\\k\neq s}}^{n} S_{i_k}(x+r) \right] \frac{t^\ell}{\ell!}. \tag{13}$$

Combining the equalities in Equations (11)–(13), we can see that the expressions

$$I_{(\sigma(s))}^{(r)} = \left[\frac{t}{1-e^{-\widehat{w}_{\sigma(s)} t}}\right]^x \left[\frac{1-e^{-wt}}{1-e^{-\widehat{w}_{\sigma(s)} t}}\right] \left[\frac{t}{1-e^{-\widehat{w}_{\sigma(s)} t}}\right]^r \prod_{\substack{k=1\\k\neq s}}^{n} \left[\frac{t}{1-e^{-\widehat{w}_{\sigma(k)} t}}\right]^{x+r+1}$$
$$= \frac{1}{\widehat{w}_{\sigma(1)}^{x+r+1}\widehat{w}_{\sigma(2)}^{x+r+1}\cdots\widehat{w}_{\sigma(n)}^{x+r+1}} \sum_{\ell=0}^{\infty} \sum_{i=0}^{w_{\sigma(s)}-1} \left[\sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} (-1)^{j_{\sigma(s)}} \binom{\ell}{i_{\sigma(1)},\ldots,i_{\sigma(s)},j_{\sigma(s)},i_{\sigma(s+1)}\ldots,i_{\sigma(n)}} \right.$$
$$\left. \times\, \widehat{w}_{\sigma(1)}^{i_{\sigma(1)}}\widehat{w}_{\sigma(2)}^{i_{\sigma(2)}}\cdots\widehat{w}_{\sigma(s)}^{i_{\sigma(s)}+j_{\sigma(s)}+1}\cdots\widehat{w}_{\sigma(n)}^{i_{\sigma(n)}} S_{i_{\sigma(s)}}(x+r-1) B_{j_{\sigma(s)}}^{(r)}(i) \prod_{\substack{k=1\\k\neq s}}^{n} S_{i_{\sigma(k)}}(x+r) \right] \frac{t^\ell}{\ell!}$$

are invariant under any permutations $\sigma \in S_n$. □

From Theorem 2, we can derive the following interesting results in a simple way.
Combining Theorem 2 for $x = 1$ with Equations (4) and (5) yields the following corollary.

Corollary 7. *Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ and $\ell \geq 0$. Then the expressions*

$$\sum_{i=0}^{w_{\sigma(s)}-1} \left[\sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} (-1)^{j_{\sigma(s)}} \binom{\ell}{i_{\sigma(1)},\ldots,i_{\sigma(s)},j_{\sigma(s)},i_{\sigma(s+1)}\ldots,i_{\sigma(n)}} \widehat{w}_{\sigma(s)}^{j_{\sigma(s)}+1} B_{i_{\sigma(s)}}^{(r+1)} B_{j_{\sigma(s)}}^{(r)}(i) \prod_{\substack{k=1\\k\neq s}}^{n} B_k^{(r+2)} \right]$$

are invariant under all permutations $\sigma \in S_n$.

Replacing x by $-p$ for $p \in \mathbb{N} \cup \{0\}$ in Theorem 2 and using the equality in Equation (2) arrive at the following corollary.

Corollary 8. *Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ and $\ell, p \geq 0$. Then the expressions*

$$\sum_{i=0}^{w_{\sigma(s)}-1} \left[\sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} (-1)^{j_{\sigma(s)}} \binom{\ell}{i_{\sigma(1)},\ldots,i_{\sigma(s)},j_{\sigma(s)},i_{\sigma(s+1)}\ldots,i_{\sigma(n)}} \right.$$
$$\left. \times\, \widehat{w}_{\sigma(s)}^{j_{\sigma(s)}+1} \frac{S(i_{\sigma(s)}+p,p)}{\binom{i_{\sigma(s)}+p}{i_{\sigma(s)}}} B_{j_{\sigma(s)}}^{(r)}(i) \prod_{\substack{k=1\\k\neq s}}^{n} \frac{S(i_{\sigma(k)}+p-1,p-1)}{\binom{i_{\sigma(k)}+p-1}{i_{\sigma(k)}}} \right]$$

are invariant under all permutations $\sigma \in S_n$.

Finally, combining Corollary 8 with Equation (4) leads to the following corollary.

Corollary 9. *For $w_1, w_2, \dots, w_n \in \mathbb{N}$ and $\ell, p, r \in \mathbb{N} \cup \{0\}$ such that $p \geq r$, the quantities*

$$\sum_{i=0}^{w_{\sigma(s)}-1} \left[\sum_{\mathcal{I}_{\sigma(s)}=0}^{\ell} (-1)^{j_{\sigma(s)}} \binom{\ell}{i_{\sigma(1)}, \dots, i_{\sigma(s)}, j_{\sigma(s)}, i_{\sigma(s+1)} \dots, i_{\sigma(n)}} \widehat{w}_{\sigma(s)}^{j_{\sigma(s)}+1} B_{i_{\sigma(s)}}^{(p+r)} B_{j_{\sigma(s)}}^{(r)} \prod_{k=1, k \neq s}^{n} B_{i_{\sigma(k)}}^{(p+x+1)} \right]$$

are invariant under all permutations $\sigma \in S_n$.

Corollary 10. *For $n, r \in \mathbb{N} \cup \{0\}$ and $w_1, w_2, w_3 \in \mathbb{N}$, the quantities*

$$\begin{aligned}
&\sum_{i=0}^{w_1-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{\ell} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{\ell+k+j+1} S_m(x+r) S_\ell(x+r-1) B_k^{(r)}(i) S_j(x+r) \\
= &\sum_{i=0}^{w_2-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{j} w_1^{m+k+j+1} w_2^{m+\ell} w_3^{m+k+j+1} S_m(x+r) S_\ell(x+r) S_k(x+r-1) B_j^{(r)}(i) \\
= &\sum_{i=0}^{w_3-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{\ell} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} S_m(x+r-1) B_\ell^{(r)}(i) S_k(x+r) S_j(x+r)
\end{aligned}$$

are invariant under all permutations $\sigma \in S_n$.

Corollary 11. *For $n, r \in \mathbb{N} \cup \{0\}$ and $w_1, w_2, w_3 \in \mathbb{N}$, the quantities*

$$\begin{aligned}
&\sum_{i=0}^{w_1-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{\ell+k+j+1} B_m^{(r+2)} B_l^{(r+1)} B_k^{(r)}(i) B_j^{(r+2)} \\
= &\sum_{i=0}^{w_2-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+k+j+1} w_2^{m+\ell} w_3^{\ell+k+j+1} B_m^{(r+2)} B_l^{(r+2)} B_k^{(r+1)} B_j^{(r)}(i) \\
= &\sum_{i=0}^{w_3-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} B_m^{(r+1)} B_\ell^{(r)}(i) B_k^{(r+2)} B_j^{(r+2)}
\end{aligned}$$

are invariant under all permutations $\sigma \in S_n$.

Corollary 12. *For $n, p, r \in \mathbb{N} \cup \{0\}$ such that $p \geq r$ and $w_1, w_2, w_3 \in \mathbb{N}$, the quantities*

$$\begin{aligned}
&\sum_{i=0}^{w_1-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{\ell+k+j+1} \binom{m+p-r-1}{m}^{-1} \binom{\ell+p-r}{\ell}^{-1} \\
&\times \binom{j+p-r-1}{j}^{-1} S(m+p-r-1, p-r-1) S(\ell+p-r, p-r) B_k^{(r)}(i) S(j+p-r-1, j-r-1) \\
&= \sum_{i=0}^{w_2-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+k+j+1} w_2^{m+\ell} w_3^{\ell+k+j+1} \binom{m+p-r-1}{m}^{-1} \\
&\times \binom{\ell+p-r-1}{\ell}^{-1} \binom{\ell+p-r}{j}^{-1} S(m+p-r-1, p-r-1) \\
&\times S(\ell+p-r-1, p-r-1) S(j+p-r, p-r) B_j^{(r)}(i) \\
&= \sum_{i=0}^{w_3-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} \\
&\times \binom{m+p-r}{m}^{-1} \binom{k+p-r-1}{k}^{-1} \binom{j+p-r-1}{j}^{-1} \\
&\times S(m+p-r, p-r) B_\ell^{(r)}(i) S(k+p-r-1, p-r-1) S(j+p-r-1, p-r-1)
\end{aligned}$$

are invariant under all permutations $\sigma \in S_n$.

Corollary 13. *For $n, p, r \in \mathbb{N} \cup \{0\}$ such that $p \geq r$ and $w_1, w_2, w_3 \in \mathbb{N}$, the quantities*

$$\sum_{i=0}^{w_1-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{\ell+k+j+1} B_m^{(p+x+1)} B_l^{(p+r)} B_k^{(r)}(i) B_j^{(p+x+1)}$$

$$= \sum_{i=0}^{w_2-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+k+j+1} w_2^{m+\ell} w_3^{\ell+k+j+1} B_m^{(p+x+1)} B_l^{(p+x+1)} B_k^{(p+r)} B_j^{(r)}(i)$$

$$= \sum_{i=0}^{w_3-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m,\ell,k,j} (-1)^{m+\ell+k+j} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} B_m^{(p+r)} B_\ell^{(r)}(i) B_k^{(p+x+1)} B_j^{(p+x+1)}$$

are invariant under all permutations $\sigma \in S_n$.

Remark 1. *In view of Corollaries 10–13, by specializing $w_3 = 1$ or $w_2 = w_3 = 1$, we can obtain many interesting symmetric identities for Stirling polynomials $S_n(x)$.*

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