Article

Some Symmetric Identities Involving the Stirling Polynomials Under the Finite Symmetric Group

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Abstract: In the paper, the authors present some symmetric identities involving the Stirling polynomials and higher order Bernoulli polynomials under all permutations in the finite symmetric group of degree $n$. These identities extend and generalize some known results.

Keywords: symmetric identity; Stirling polynomial; Stirling number of the second kind; finite symmetric group; permutation; higher order Bernoulli polynomial

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1. Introduction

The Stirling numbers arise in a variety of analytic and combinatorial problems. They were introduced in the eighteenth century by James Stirling. There are two kinds of the Stirling numbers: the Stirling numbers of the first and second kind. Some combinatorial identities for the Stirling numbers of these two kinds are studied and collected in [1–8] and closely related references.

The Stirling numbers of second kind $S(n, k)$ are the numbers of ways to partition a set of $n$ elements into $k$ nonempty subsets. It can be computed by

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

and can be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}, \quad k \geq 0.$$ 

The Stirling polynomials $S_n(x)$ can be generated [4,9–13] by

$$F(t, x) = \left(\frac{t}{1-e^{-t}}\right)^{x+1} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!} \quad (1)$$

and the first five Stirling polynomials $S_n(x)$ for $0 \leq n \leq 4$ are

$$S_0(x) = 1, \quad S_1(x) = \frac{x+1}{2}, \quad S_2(x) = \frac{(3x+2)(x+1)}{12}, \quad S_3(x) = \frac{x(x+1)^2}{8}.$$ 

The Stirling polynomials $S_n(x)$ generalize several important sequences of numbers, including the Stirling numbers of the second kind $S(n, k)$ and the Bernoulli numbers $B_n$, appearing in combinatorics, number theory, and analysis.
In the case $x = -n$ for $n \in \mathbb{N}$ in Equation (1), we can derive
\[
\sum_{k=0}^{\infty} S_k(-n) \frac{t^k}{k!} = \frac{(1 - e^{-t})^{n-1}}{t} = (-1)^{n-1} \frac{(1 - e^{-t} - 1)^{n-1}}{t^{n-1}} = \frac{(-1)^{n-1}}{t^{n-1}} (n-1)! \sum_{k=0}^{\infty} S(k, n-1) \frac{(-t)^k}{k!}
\]
\[
= \sum_{k=0}^{\infty} S(k + n - 1, n-1) \frac{(n-1)!}{(k + n - 1)!} (1 - e^{-t})^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} S(k + n - 1, n-1) \frac{(-1)^k}{k!} \frac{t^k}{k!}
\]

Equating coefficients on the very ends of the above identity arrives at
\[
S(m, n) = (-1)^{m-n} \binom{m-n+n}{k} S_{m-n}(-n-1)
\]
(2)
for $m, n \in \mathbb{N} \cup \{0\}$.

It is common knowledge \[14\] that the Bernoulli numbers $B_n$ are generated by
\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.
\]
(3)
By considering the case $x = 0$ in Equation (1) and the definition in Equation (3) for the Bernoulli numbers $B_n$, we have
\[
\sum_{n=0}^{\infty} S_n(0) \frac{t^n}{n!} = \frac{t}{e^t - 1} = \frac{-t}{e^{-t} - 1} = \sum_{n=0}^{\infty} (-1)^n B_n \frac{t^n}{n!}.
\]
Comparing the coefficients on both sides of this equation results in
\[
B_n = (-1)^n S_n(0), \quad n \geq 0.
\]
(4)
One can also find Equations (2) and (4) in \[4\], p. 154.
The higher order Bernoulli numbers $B_n^{(\alpha)}$ for $n \geq 0$ and $\alpha \in \mathbb{N}$ can be generated \[15-17\] by
\[
\left( \frac{t}{e^t - 1} \right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{t^n}{n!}, \quad |t| < 2\pi.
\]
Combining this with Equation (1) yields the relation
\[
S_k(n) = (-1)^k B_k^{(n+1)}, \quad k \in \mathbb{N} \cup \{0\}, \quad n \in \mathbb{N}.
\]
(5)
In the paper \[15\], some new symmetric identities for the $q$-Bernoulli polynomials are derived from the fermionic integral on $\mathbb{Z}_p$. In \[18,19\], the method in the paper \[15\] is extended to the $q$-Euler and $q$-Genocchi polynomials, respectively. In \[13\], some symmetric identities involving the Stirling polynomials $S_n(x)$ are investigated. The symmetric identities of some special polynomials, such as higher order Bernoulli polynomials $B_n^{(\alpha)}$, higher order $q$-Euler polynomials, degenerate generalized Bernoulli polynomials, and degenerate higher order $q$-Euler polynomials, have been studied by several mathematicians in \[10,16,20-23\] and closely related references therein.
The purpose of this paper is to investigate some interesting symmetric identities involving the Stirling polynomials $S_n(x)$ under the finite symmetric group $S_n$. By specializing these identities, we can obtain some new symmetric identities involving the Stirling polynomials $S_n(x)$.

2. Symmetric Identities of Stirling Polynomials

Now, we start out to state and prove our main results.
Theorem 1. Let \( w_1, w_2, \ldots, w_n \in \mathbb{N} \) and \( \ell \in \mathbb{N} \cup \{0\} \). Then, the expression
\[
\sum_{I(\sigma) \neq 0} (-1)^{\ell(I)} \sum_{i(1), \ldots, i(\ell)} A_{i(\ell)} \left[ (w_1 - 1) A_{i(1)} (w_2 - 1) \cdots (w_n - 1) A_{i(n)} \right] \prod_{k=1}^n S_{i(k)} (x) \]

is invariant under any permutation \( \sigma \in S_n \), where \( \hat{w}_i = \frac{1}{n!} \prod_{k=1}^n w_k \),
\[
I(\sigma) = i(1) + \cdots + i(n) + i(1) + \cdots + i(n),
\]
and \( A_k(n) = 0^k + 1^k + \cdots + n^k \) for \( k, n \in \mathbb{N} \cup \{0\} \).

Proof. For convenience, we denote \( w = \prod_{k=1}^n w_k \). Define
\[
I = I(w_1, w_2, \ldots, w_n) = \frac{t^{n^x + (n-1) \ell}}{\prod_{k=1}^n (1 - e^{-\hat{w}_k})^{x+1}}.
\]

It is clear that we can rewrite \( I \) as
\[
\left( \frac{t}{1 - e^{-\hat{w}_1}} \right) \left( \frac{1 - e^{\hat{w}_1}}{1 - e^{-\hat{w}_1}} \right) \prod_{k=2}^n \left( \frac{t}{1 - e^{-\hat{w}_k}} \right)^{x+1} = I(1).
\]

By applying Equations (1)–(8), we can rearrange the equality in Equation (7) as
\[
I(1) = \frac{1}{\hat{w}_1^{\ell+1}} \sum_{j_1=0}^{\infty} S_{i_1}(x-1) \frac{\hat{w}_1^j}{j!} \prod_{j_1=0}^{\infty} (-1)^j \frac{\hat{w}_1^j}{j!} \prod_{k=2}^n \left( \frac{1}{\hat{w}_k^{x+1}} \sum_{i_k=0}^{\infty} S_{i_k}(x) \frac{\hat{w}_k^{i_k}}{i_k!} \right)\]
\[
= \frac{1}{\hat{w}_1^{x+1} \hat{w}_2^{x+1} \cdots \hat{w}_n^{x+1}} \sum_{i_1+i_2+\cdots+i_n=0}^{\ell} \left( i_1, i_2, \ldots, i_n \right) (-1)^{i_1} \hat{w}_1^{i_1} \hat{w}_2^{i_2} \cdots \hat{w}_n^{i_n} \sum_{i_1=0}^{\infty} S_{i_1}(x-1) A_{i}(w_1-1) \prod_{k=2}^n S_{i_k}(x) \frac{t^\ell}{\ell!}.
\]

Similarly, from Equation (7), we can also consider \( I \) as
\[
I(2) = \left( \frac{t}{1 - e^{-\hat{w}_1}} \right) \left( \frac{1 - e^{\hat{w}_1}}{1 - e^{-\hat{w}_1}} \right) \prod_{k=3}^n \left( \frac{t}{1 - e^{-\hat{w}_k}} \right)^{x+1} \]
\[
= \frac{1}{\hat{w}_1^{x+1} \hat{w}_2^{x+1} \cdots \hat{w}_n^{x+1}} \sum_{i_1+i_2+\cdots+i_n=0}^{\ell} \left( i_1, i_2, i_3, \ldots, i_n \right) (-1)^{i_1} \hat{w}_1^{i_1} \hat{w}_2^{i_2} \cdots \hat{w}_n^{i_n} \sum_{i_1=0}^{\infty} S_{i_1}(x) S_{i_2}(x) \cdots S_{i_n}(x) A_{i}(w_1-1) \prod_{k=3}^n S_{i_k}(x) \frac{t^\ell}{\ell!}.
\]

Inductively, for any \( s \in \{1, 2, \ldots, n\} \), from Equation (7), we can consider \( I \) as
\[
I(s) = \left( \frac{t}{1 - e^{-\hat{w}_1}} \right) \left( \frac{1 - e^{\hat{w}_1}}{1 - e^{-\hat{w}_1}} \right) \prod_{k=3}^n \left( \frac{t}{1 - e^{-\hat{w}_k}} \right)^{x+1} \]
\[
= \frac{1}{\hat{w}_1^{x+1} \hat{w}_2^{x+1} \cdots \hat{w}_n^{x+1}} \sum_{i_1+\cdots+i_s+i_{s+1}+\cdots+i_n=0}^{\ell} \left( i_1, \ldots, i_s, i_{s+1}, \ldots, i_n \right) \frac{t^\ell}{\ell!}.
\]
which are invariant under any permutations \( \sigma \in S_n \).

Combining Theorem 1 for \( x = 1 \) with the equalities in Equations (4) and (5) deduces Corollary 1 below.

**Corollary 1.** Let \( w_1, w_2, \ldots, w_n \in \mathbb{N} \) and \( \ell \geq 0 \). Then the quantities

\[
\sum_{I_{v(s)}=0}^\ell (-1)^{j_{v(s)}} \binom{\ell}{i_{v(1)}, \ldots, i_{v(s)}, j_{v(s)}, i_{v(s+1)}, \ldots, i_{v(n)}} \left( \sum_{k \neq s} S(I_{v(k)} + p, p) \right) \frac{B_{I_{v(s)}}^{(2)}}{(\ell_{v(s)} + p - 1)_{v(s)}} A_{I_{v(s)}}[w_{v(s)} - 1] \prod_{k=1}^n B_{I_{v(k)}}^{(p+1)}
\]

are invariant under any permutation \( \sigma \in S_n \).

Replacing \( x \) by \( -p \) for \( p \in \mathbb{N} \cup \{0\} \) in Theorem 1 and employing Equation (2) result in the following corollary.

**Corollary 2.** Let \( w_1, w_2, \ldots, w_n \in \mathbb{N} \) and \( \ell, p \geq 0 \). Then the expressions

\[
\sum_{I_{v(s)}=0}^\ell (-1)^{j_{v(s)}} \binom{\ell}{i_{v(1)}, \ldots, i_{v(s)}, j_{v(s)}, i_{v(s+1)}, \ldots, i_{v(n)}} \left( \sum_{k \neq s} S(I_{v(k)} + p, p) \right) \frac{B_{I_{v(s)}}^{(2)}}{(\ell_{v(s)} + p - 1)_{v(s)}} A_{I_{v(s)}}[w_{v(s)} - 1] \prod_{k=1}^n B_{I_{v(k)}}^{(p+1)}
\]

are invariant under any permutations \( \sigma \in S_n \).

Finally, combining Corollary 2 with Equation (4) leads to the following corollary.

**Corollary 3.** Let \( w_1, w_2, \ldots, w_n \in \mathbb{N} \) and \( \ell, p \geq 0 \). Then the expressions

\[
\sum_{I_{v(s)}=0}^\ell (-1)^{j_{v(s)}} \binom{\ell}{i_{v(1)}, \ldots, i_{v(s)}, j_{v(s)}, i_{v(s+1)}, \ldots, i_{v(n)}} \left( \sum_{k \neq s} S(I_{v(k)} + p, p) \right) \frac{B_{I_{v(s)}}^{(2)}}{(\ell_{v(s)} + p - 1)_{v(s)}} A_{I_{v(s)}}[w_{v(s)} - 1] \prod_{k=1}^n B_{I_{v(k)}}^{(p+1)}
\]

are invariant under any permutations \( \sigma \in S_n \).

From Theorem 1 to Corollary 3, if taking \( w_1 = w_2 = \ldots = w_n = 1 \), then we have the following corollaries.
Corollary 4. Let $w_1, w_2, w_3$ be any positive integers, $n$ be any non-negative integer. Then the expressions
\[
\sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^j w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} S_m(x-1) A_\ell(w_3-1) S_k(x) S_j(x) \\
= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^j w_1^{m+j} w_2^{m+\ell+k+1} w_3^{m+k+j+1} S_m(x) S_\ell(x-1) A_k(w_1-1) S_j(x) \\
= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^j w_1^{m+k+j+1} w_2^{m+\ell} w_3^{\ell+k+j+1} S_m(x) S_\ell(x) S_k(x-1) A_j(w_2-1)
\]
and
\[
\sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} B_m A_\ell(w_3-1) B^{(2)}_k B_j \\
= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{m+k+j+1} B_m B_\ell A_k(w_1-1) B^{(2)}_j \\
= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+k+j+1} w_2^{m+k+j+1} w_3^{\ell+k+j+1} B_m B_\ell B_k A_j(w_2-1)
\]
are invariant under any permutations of $w_1, w_2, w_3$.

Corollary 5. For $n, p \in \mathbb{N} \cup \{0\}$ and $w_1, w_2, w_3 \in \mathbb{N}$, the expressions
\[
\sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+\ell+k+1} w_2^{m+\ell+k+1} w_3^{k+j} S(m+p, p) A_\ell(w_3-1) \frac{S(k+p-1, p-1)}{m+p-1 \choose k+p-1} \frac{S(j+p-1, p-1)}{j+p-1 \choose k+p-1} \\
= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{m+k+j+1} S(m+p-1, p-1) \frac{S(\ell+p, p)}{m+p-1 \choose \ell+p} A_k(w_1-1) \frac{S(j+p-1, p-1)}{j+p-1 \choose k+p-1} \\
= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+k+j+1} w_2^{m+k+j+1} w_3^{\ell+k+j+1} S(m+p-1, p-1) \frac{S(\ell+p-1, \ell)}{m+p-1 \choose \ell+p} \frac{S(k+p, p)}{k+p \choose k} A_j(w_2-1)
\]
are invariant under any permutations of $w_1, w_2, w_3$.

Corollary 6. For $n, p \in \mathbb{N} \cup \{0\}$ and $w_1, w_2, w_3 \in \mathbb{N}$, the expressions
\[
\sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+\ell+j+1} w_2^{m+\ell+k+1} w_3^{k+j} B_m A_\ell(w_3-1) B_k A_j(w_2-1) \\
= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+j} w_2^{m+\ell+k+1} w_3^{m+k+j+1} B_m B_\ell A_k(w_1-1) B_k A_j(w_2-1) \\
= \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+k+j+1} w_2^{m+k+j+1} w_3^{\ell+k+j+1} B_m B_\ell B_k A_j(w_2-1)
\]
are invariant under any permutations of \( w_1, w_2, w_3 \).

3. Symmetric Identities via Higher Order Bernoulli Polynomials

Recall from [16] that higher order Bernoulli polynomials \( B_n^{(a)}(x) \) can be generated by

\[
\left( \frac{t}{e^t - 1} \right)^{a} e^{t} = \sum_{n=0}^{\infty} B_n^{(a)}(x) \frac{t^n}{n!}, \quad a \in \mathbb{N}. \tag{9}
\]

Now, we start out to investigate symmetric identities for the Stirling polynomials \( S_n(x) \) under the finite symmetric group \( S_n \) via higher order Bernoulli polynomials \( B_n^{(a)}(x) \).

**Theorem 2.** Let \( w_1, w_2, \ldots, w_n \in \mathbb{N} \) and \( \ell \geq 0 \). Then the quantities

\[
\sum_{i=0}^{w_{\nu(s)}-1} \sum_{i_{\nu(s)}=0}^{\ell} (-1)^{i_{\nu(s)}} \left( \prod_{\nu(1), \ldots, \nu(s), \nu(s+1), \ldots, \nu(n)} \right) S_{\nu(s)}(x + r - 1) B_{\nu(s)}^{(r)}(i) \prod_{k=1}^{n} S_{\nu(k)}(x + r)
\]

are invariant under any permutations \( \sigma \in S_n \), where \( I_{\nu(s)} \) is defined by Equation (6).

**Proof.** Define \( I^{(r)} = I^{(r)}(w_1, w_2, \ldots, w_n) \) as

\[
I^{(r)} = \frac{t^{n+nr+(n-1)}}{\prod_{i=1}^{n} (1 - e^{-w_{\ell}})^{x+r+1}}.
\]

Then we can rewrite \( I^{(r)} \) as

\[
I^{(r)}_{(1)} = \left( \frac{t}{1 - e^{-w_{\ell}}} \right)^{x} \left( \frac{1 - e^{w_{\ell}}}{1 - e^{-w_{\ell}}} \right) \left( \frac{t}{1 - e^{-w_{\ell}}} \right) \prod_{k=2}^{n} \left( \frac{t}{1 - e^{-w_{\ell}}} \right)^{x+r+1}.
\]

Applying Equations (1) and (9) to the equality in Equation (10) gives

\[
I^{(r)}_{(1)} = \frac{1}{\partial_{1}^{x+t+1} \partial_{2}^{x+r+1} \cdots \partial_{n}^{x+1}} \sum_{i_{1}=0}^{\infty} S_{i_{1}}(x + r - 1) \frac{1}{\partial_{1}^{x}} \sum_{j_{1}=0}^{\infty} \sum_{i_{2}=0}^{\infty} \sum_{j_{2}=0}^{\infty} \cdots \sum_{i_{n}=0}^{\infty} \sum_{j_{n}=0}^{\infty} S_{i_{1}}(x + r) \frac{1}{\partial_{1}^{x}} \frac{t}{1 - e^{-w_{\ell}}} \prod_{k=2}^{n} S_{i_{k}}(x + r) \left( \frac{t}{1 - e^{-w_{\ell}}} \right)^{x+r+1}
\]

Similarly, we can rewrite \( I^{(r)} \) as

\[
I^{(r)}_{(2)} = \left( \frac{t}{1 - e^{-w_{\ell}}} \right)^{x+r+1} \left( \frac{t}{1 - e^{-w_{\ell}}} \right)^{x} \left( \frac{1 - e^{w_{\ell}}}{1 - e^{-w_{\ell}}} \right) \left( \frac{1 - e^{w_{\ell}}}{1 - e^{-w_{\ell}}} \right) \prod_{k=3}^{n} \left( \frac{t}{1 - e^{-w_{\ell}}} \right)^{x+r+1}
\]

\[
I^{(r)}_{(2)} = \frac{1}{\partial_{1}^{x+t+1} \partial_{2}^{x+r+1} \cdots \partial_{n}^{x+1}} \sum_{i_{1}=0}^{\infty} \sum_{i_{2}=0}^{\infty} \cdots \sum_{i_{n}=0}^{\infty} \sum_{i_{1}+i_{2}+i_{3}+\cdots+i_{n}=0}^{\infty} (-1)^{j_{1}} \left( \frac{t}{1 - e^{-w_{\ell}}} \right)^{x+r+1}
\]

\[
I^{(r)}_{(2)} = \frac{1}{\partial_{1}^{x+t+1} \partial_{2}^{x+r+1} \cdots \partial_{n}^{x+1}} \sum_{i_{1}=0}^{\infty} \sum_{i_{2}=0}^{\infty} \cdots \sum_{i_{n}=0}^{\infty} \sum_{i_{1}+i_{2}+i_{3}+\cdots+i_{n}=0}^{\infty} (-1)^{j_{1}} \left( \frac{t}{1 - e^{-w_{\ell}}} \right)^{x+r+1}
\]
Inductively, for any \( s \in \{1, 2, \ldots, n\} \), we can rearrange \( I^{(r)} \) as

\[
I^{(r)}_{(s)} = \left( \frac{t}{1 - e^{-\alpha_0 t}} \right)^x \left( \frac{1}{1 - e^{-\alpha_0 t}} \right) \left( \frac{t}{1 - e^{-\alpha_0 t}} \right)^r \prod_{k=1}^{n} \left( \frac{t}{1 - e^{-\alpha_0 t}} \right)^{x+r+1}
\]

\[
= \frac{1}{\hat{w}_1^{x+r+1} \hat{w}_2^{x+r+1} \cdots \hat{w}_n^{x+r+1}} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=1}^\infty (-1)^{\ell} \left( \frac{1}{1 - e^{-\alpha_0 t}} \right)^\ell \left( \frac{t}{1 - e^{-\alpha_0 t}} \right)^{x+r+1}
\]

Combining the equalities in Equations (11)–(13), we can see that the expressions

\[
I^{(r)}_{(s)} = \left[ \frac{t}{1 - e^{-\alpha_0 t}} \right]^x \left[ \frac{1}{1 - e^{-\alpha_0 t}} \right] \left[ \frac{t}{1 - e^{-\alpha_0 t}} \right]^r \prod_{k=1}^{n} \left[ \frac{t}{1 - e^{-\alpha_0 t}} \right]^{x+r+1}
\]

\[
= \frac{1}{\hat{w}_1^{x+r+1} \hat{w}_2^{x+r+1} \cdots \hat{w}_n^{x+r+1}} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=1}^\infty (-1)^{\ell} \left( \frac{1}{1 - e^{-\alpha_0 t}} \right)^\ell \left( \frac{t}{1 - e^{-\alpha_0 t}} \right)^{x+r+1}
\]

are invariant under all permutations \( \sigma \in S_n \). \( \square \)

From Theorem 2, we can derive the following interesting results in a simple way.

Combining Theorem 2 for \( x = 1 \) with Equations (4) and (5) yields the following corollary.

**Corollary 7.** Let \( w_1, w_2, \ldots, w_n \in \mathbb{N} \) and \( \ell \geq 0 \). Then the expressions

\[
\sum_{i=0}^\infty \left[ \sum_{I_{i(v)}(s)} (-1)^{\ell} \left( i_{v(1)}, \ldots, i_{v(u)}, i_{v(s)} \right) \hat{w}_1^{\ell} \hat{w}_2^{\ell} \cdots \hat{w}_n^{\ell} B^{(r+1)}_{i(v)}(i) \prod_{k=1}^{n} B^{(r+2)}_{i(k)} \right]
\]

are invariant under all permutations \( \sigma \in S_n \).

Replacing \( x \) by \( -p \) for \( p \in \mathbb{N} \cup \{0\} \) in Theorem 2 and using the equality in Equation (2) arrive at the following corollary.

**Corollary 8.** Let \( w_1, w_2, \ldots, w_n \in \mathbb{N} \) and \( \ell, p \geq 0 \). Then the expressions

\[
\sum_{i=0}^\infty \left[ \sum_{I_{i(v)}(s)} (-1)^{\ell} \left( i_{v(1)}, \ldots, i_{v(u)}, i_{v(s)} \right) \hat{w}_1^{\ell} \hat{w}_2^{\ell} \cdots \hat{w}_n^{\ell} S(i_{v(s)} + p, p) \prod_{k=1}^{n} B^{(r)}_{i(v)}(i) \prod_{k=1}^{n} B^{(r)}_{i(k)} \right]
\]

are invariant under all permutations \( \sigma \in S_n \).

Finally, combining Corollary 8 with Equation (4) leads to the following corollary.
Corollary 9. For \( w_1, w_2, \ldots, w_n \in \mathbb{N} \) and \( \ell, p, r \in \mathbb{N} \cup \{0\} \) such that \( p \geq r \), the quantities
\[
\sum_{i=0}^{w_{r+1}^{(0)}} \sum_{i=0}^{\ell} (-1)^{i+1} \left( i_{e(1)}, \ldots, i_{e(s)}, j_{e(s)}, i_{e(s+1)}, \ldots, i_{e(n)} \right) \sum_{k=1, k \neq s}^{n} B_{k}^{(p+r+1)} B_{r+1}^{(r)} \prod_{k=1, k \neq s}^{n} B_{k}^{(p+r+1)}
\]
are invariant under all permutations \( \sigma \in S_n \).

Corollary 10. For \( n, r \in \mathbb{N} \cup \{0\} \) and \( w_1, w_2, w_3 \in \mathbb{N} \), the quantities
\[
\sum_{i=0}^{w_{r+1}^{(0)}} \sum_{m+\ell+k+j=0}^{n} \left( m, \ell, k, j \right) (-1)^{m+\ell+k+j} \frac{\ell+1}{m} B_{m+\ell+k+j+1} S_m(x+r) S_{\ell}(x+r) B_{k}^{(r+1)} (i) S_j(x+r)
\]
are invariant under all permutations \( \sigma \in S_n \).

Corollary 11. For \( n, r \in \mathbb{N} \cup \{0\} \) and \( w_1, w_2, w_3 \in \mathbb{N} \), the quantities
\[
\sum_{i=0}^{w_{r+1}^{(0)}} \sum_{m+\ell+k+j=0}^{n} \left( m, \ell, k, j \right) (-1)^{m+\ell+k+j} \frac{\ell+1}{m} B_{m+\ell+k+j+1} B_{m}^{(r+2)} B_{k}^{(r+1)} (i) B_{j}^{(r+2)}
\]
are invariant under all permutations \( \sigma \in S_n \).

Corollary 12. For \( n, p, r \in \mathbb{N} \cup \{0\} \) such that \( p \geq r \) and \( w_1, w_2, w_3 \in \mathbb{N} \), the quantities
\[
\sum_{i=0}^{w_{r+1}^{(0)}} \sum_{m+\ell+k+j=0}^{n} \left( m, \ell, k, j \right) (-1)^{m+\ell+k+j} \frac{\ell+1}{m} B_{m+\ell+k+j+1} B_{m}^{(r+2)} B_{k}^{(r+1)} (i) S(j+p-r-1, j-r-1)
\]
are invariant under all permutations \( \sigma \in S_n \).
Corollary 13. For \( n, p, r \in \mathbb{N} \cup \{0\} \) such that \( p \geq r \) and \( w_1, w_2, w_3 \in \mathbb{N} \), the quantities

\[
\sum_{i=0}^{w_1-1} \sum_{m+\ell+k+j=0}^{n} \binom{n}{m, \ell, k, j} (-1)^{m+\ell+k+j} w_1^{m+\ell+k+j+1} w_2^{\ell+k+j+1} w_3^k B_m^{(p+x+1)} B_j^{(p+r)} B_l^{(r)} (i) B_j^{(p+x+1)}
\]

are invariant under all permutations \( \sigma \in S_n \).

Remark 1. In view of Corollaries 10–13, by specializing \( w_3 = 1 \) or \( w_2 = w_3 = 1 \), we can obtain many interesting symmetric identities for Stirling polynomials \( S_n(x) \).

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