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On Approximation by Linear Combinations of Modified Summation Operators of Integral Type in Orlicz Spaces

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Abstract: In this paper, the authors introduce the Orlicz spaces corresponding to the Young function and, by virtue of the equivalent theorem between the modified K-functional and modulus of smoothness, establish the direct, inverse, and equivalent theorems for linear combinations of modified summation operators of integral type in the Orlicz spaces.

Keywords: approximation; linear combination; direct theorem; inverse theorem; equivalent theorem; Orlicz space; modified summation operators of integral type; K-functional; modulus

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1. Introduction and Main Results

Throughout this paper, we use C to denote an absolute constant independent of anything, which may be not necessarily the same in different cases.

There are many types of integral operators (see, for example, [1–7] and closely related references therein). In the paper [8], Ueki provided a characterization for the boundedness and compactness of the Li-Stević type integral operators

$$T_{\varphi}^{g} f(z) = \int_0^z f(\varphi(\xi)) g(\xi) \,\mathrm{d}\,\xi$$

from the weighted Bergman space $L_a^p(dA_\alpha)$ to the β-Zygmund space Z_β. Later, Li and Ma [9] investigated the boundedness and compactness of products of composition operators and integral type operators

$$(C_\varphi I_g f)(z) = \int_0^{\varphi(z)} f'(\xi) g(\xi) \,\mathrm{d}\,\xi$$

from Zygmund-type spaces to Q_k spaces. Recently, the equivalent characterizations for the boundedness and compactness of several integral type operators

$$V_\Phi^h f(z) = \int_0^z f'(\Phi(t)) h'(t) \,\mathrm{d}\,t$$

from $F(p,q,s)$ space to α-Bloch-Orlicz and β-Zygmund-Orlicz spaces were developed in [10]. Gupta and Yadav [11] estimated the approximation by complex summation integral type operator

$$M_n(f,z) = (n+1)\sum_{k=1}^{n} p_{n,k}(z)\int_0^1 f(t)p_{n,k-1}(t)\,\mathrm{d}\,t + f(0)p_{n,0}(z)$$

in compact disks, where $p_{n,k}(z) = \binom{n}{k}z^k(1-z)^{n-k}$. In [8–11], some approximating properties of integral type operators in complex spaces were established. Vural, Altin, and Yüksel [12] provided weighted approximation and obtained a rate of convergence of Schurer's generalization of the q-Hybrid summation operators of integral type

$$\begin{aligned} M_{n,p,q}^{(\alpha,\beta)}(f,x) = {} & [n+p-1]_q \sum_{k=1}^{\infty} S_{n,p,k}^{q}(x)\int_0^{\infty} P_{n,p,k-1}^{q}(t) f\left(\frac{[n+p]_q t+\alpha}{[n+p]_q+\beta}\right)\mathrm{d}_q(t) \\ & + e^{-[n+p]_q x} f\left(\frac{\alpha}{[n+p]_q+\beta}\right), \end{aligned}$$

where

$$S_{n,p,k}^{q}(x) = ([n+p]_q x)^k \frac{e^{-[n+p]_q x}}{[k]_q!} \quad \text{and} \quad P_{n,p,k}^{q} = \binom{n+p+k-1}{k}_q q^{k(k+1)} \frac{x^k}{(1+x)_q^{n+p+k}}.$$

In [13], Govil and Gupta considered the simultaneous approximation for the Stancu-type generalization of certain summation operators of integral type

$$G_n^{\alpha,\beta}(f,x) = \sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k-1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right)\mathrm{d}\,t + (1+x)^{-n} f\left(\frac{\alpha}{n+\beta}\right)$$

by hypergeometric series. Srivastava and Gupta [14] introduced and investigated a new sequence of linear positive operators

$$G_{n,c}(f,x) = n\sum_{k=1}^{\infty}\int_0^{\infty} p_{n,k}(x;c)p_{n+c,k-1}(t;c)f(t)\,\mathrm{d}\,t + \int_0^{\infty} p_{n,0}(x;c)p_{n,0}(t;c)\delta(t)f(t)\,\mathrm{d}\,t,$$

which included some well-known operators as its special cases and obtained an estimate on the rate of convergence by means of the decomposition technique for functions of bounded variation. In [15], Gupta, Mohapatra, and Finta studied the mixed summation operators of integral type

$$S_n(f,x) = \sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)f(t)\,\mathrm{d}\,t + e^{-nx}f(0)$$

and obtained the rate of point-wise convergence, where

$$s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!} \quad \text{and} \quad b_{n,v}(t) = \frac{1}{B(n,v+1)}t^v(1+t)^{-n-v-1}.$$

In [12–15], some approximating properties of integral type operators were discussed in $C[0,\infty)$, which is a special case of the Orlicz space. For $f \in L_\Phi^*[0,\infty)$, the modified summation operators of integral type $B_n(f,x)$ are defined in [16] as

$$B_n(f,x) = \frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)f(t)\,\mathrm{d}\,t, \quad x \in [0,\infty), \tag{1}$$

where $b_{n,k}(x) = \frac{(n+k)!}{(k-1)!n!} \frac{x^{k-1}}{(1+x)^{n+k+1}}$ for $k, n \geq 1$. Recently, Han and Wu [16] obtained the following direct, inverse, and equivalent theorems of modified summation operators of integral type in Orlicz spaces.

Theorem 1 (Direct theorem [16])**.** *Let $f \in L^*_\Phi[0,\infty)$, $\varphi^2(x) = x(1+x)$, and $\Psi \in \Delta_2$. Then*

$$\|B_n(f) - f\|_\Phi \leq C\left[\omega_{2,\varphi}\left(f, \frac{1}{\sqrt{n}}\right)_\Phi + \omega_1\left(f, \frac{1}{n}\right)_\Phi + \frac{\|f\|_\Phi}{n}\right].$$

Theorem 2 (Inverse theorem [16])**.** *Let $f \in L^*_\Phi[0,\infty)$, $0 \leq \alpha < 2$, and $\|B_n(f) - f\|_\Phi = O\big(n^{-\alpha/2}\big)$. Then*

$$\omega_{2,\varphi}(f,t)_\Phi = O(t^\alpha) \quad \text{and} \quad \omega_1(f,t)_\Phi = O\big(t^{\alpha/2}\big).$$

Theorem 3 (Equivalence theorem [16])**.** *Let $f \in L^*_\Phi[0,\infty)$ and $0 \leq \alpha < 2$. Then*

$$\|B_n(f) - f\|_\Phi = O\big(n^{-\alpha/2}\big) \quad \text{if and only if} \quad \omega_{2,\varphi}(f,t)_\Phi = O(t^\alpha) \quad \text{and} \quad \omega_1(f,t)_\Phi = O\big(t^{\alpha/2}\big).$$

In recent years, since the Orlicz spaces are more general than the classical L_p spaces, which is composed of measurable functions $f(x)$ such that $|f(x)|^p$ are integrable, there is growing interest in problems of approximation in Orlicz spaces.

For smoothly proceeding, we recall from [17] some definitions and related results.

A continuous convex function $\Phi(t)$ on $[0,\infty)$ is called a Young function if it satisfies

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty.$$

For a given Young function $\Phi(t)$, its complementary Young function is denoted by $\Psi(t)$.

It is clear that the convexity of $\Phi(t)$ can lead to $\Phi(\alpha t) \leq \alpha\Phi(t)$ for $\alpha \in [0,1]$. In particular, one has $\Phi(\alpha t) < \alpha\Phi(t)$ for $\alpha \in (0,1)$.

A Young function $\Phi(t)$ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exist $t_0 > 0$ and $C \geq 1$ such that $\Phi(2t) \leq C\Phi(t)$ for $t \geq t_0$.

Let $\Phi(t)$ be a Young function. We define the Orlicz class $L_\Phi[0,\infty)$ as the collection of all Lebesgue measurable functions $u(x)$ on $[0,\infty)$. Since the integral

$$\rho(u,\Phi) = \int_0^\infty \Phi(|u(x)|)\,\mathrm{d}\,x$$

is finite, we define the Orlicz space $L^*_\Phi[0,\infty)$ as the linear hull of $L_\Phi[0,\infty)$ under the Luxemburg norm

$$\|u\|_{(\Phi)} = \inf_{\lambda>0}\left\{\lambda : \rho\left(\frac{u}{\lambda},\Phi\right) \leq 1\right\}.$$

The Orlicz norm, which is equivalent to the Luxemburg norm on $L^*_\Phi[0,\infty)$, is given by

$$\|u\|_\Phi = \sup_{\rho(v,\Psi)\leq 1}\left|\int_0^\infty u(x)v(x)\,\mathrm{d}\,x\right|$$

and satisfies

$$\|u\|_{(\Phi)} \leq \|u\|_\Phi \leq 2\|u\|_{(\Phi)}. \tag{2}$$

For $f \in L^*_\Phi[0,\infty)$, the weighted K-functional $K_{r,\varphi}(f,t^r)$, the modified weighted K-functional $\overline{K}_{r,\varphi}(f,t^r)$, and the weighted modulus of smoothness $\omega_{r,\varphi}(f,t)_\Phi$ are given, respectively, by

$$K_{r,\varphi}(f,t^r)_\Phi = \inf_g\{\|f-g\|_\Phi + t^r\|\varphi^r g^{(r)}\|_\Phi : g^{(r-1)} \in AC_{loc}\},$$

$$\overline{K}_{r,\varphi}(f,t^r)_\Phi = \inf_g\{\|f-g\|_\Phi + t^r\|\varphi^r g^{(r)}\|_\Phi + t^{2r}\|g^{(r)}\|_\Phi : g^{(r-1)} \in AC_{loc}\},$$

and

$$\omega_{r,\varphi}(f,t)_\Phi = \sup_{0<h\le t}\|\Delta^r_{h\varphi}f\|_\Phi,$$

where $\varphi(x) = \sqrt{x}$, $\varphi(x) = \sqrt{x(1+x)}$, or $\varphi(x) = x$, and $g^{(r-1)} \in AC_{loc}$ means that g is $r-1$ times differentiable and $g^{(r-1)}$ is absolutely continuous in every closed finite interval $[c,d] \subseteq [0,\infty)$.

Between the weighted modulus of smoothness and the modified weighted K-functional, there exists the following equivalent theorem.

Theorem 4 ([16])**.** *Let $f \in L^*_\Phi[0,\infty)$. Then there exist some constants C and t_0 such that*

$$\frac{\omega_{r,\varphi}(f,t)_\Phi}{C} \le \overline{K}_{r,\varphi}(f,t^r)_\Phi \le C\omega_{r,\varphi}(f,t)_\Phi, \quad 0<t\le t_0. \tag{3}$$

Between the weighted modulus of smoothness and the weighted K-functional, there exists the following equivalent theorem.

Theorem 5 ([18])**.** *Let $f \in L^*_\Phi[0,\infty)$. Then there are some constants C and t_0 such that*

$$\frac{\omega_{r,\varphi}(f,t)_\Phi}{C} \le K_{r,\varphi}(f,t^r)_\Phi \le C\omega_{r,\varphi}(f,t)_\Phi, \quad 0<t\le t_0. \tag{4}$$

Currently, there are few results about linear combinations of modified summation operators of integral type $B_n(f,x)$. In this article, we investigate the approximation of linear combinations of modified summation operators of integral type $B_n(f,x)$ in Orlicz spaces $L^*_\Phi[0,\infty)$. The linear combinations of modified summation operators of integral type $B_n(f,x)$ are defined as

$$L_{n,r}(f,x) = \sum_{i=0}^{2r-1} c_i(n)B_{n_i}(f,x),$$

where

$$\begin{aligned} &n \le n_0 \le n_1 < \cdots < n_{2r-1} \le C_n, \quad \sum_{i=0}^{2r-1} c_i(n) = 1, \quad \sum_{i=0}^{2r-1}|c_i(n)| < C, \\ &\sum_{i=0}^{2r-1} c_i(n)B_{n_i}\big((t-x)^k, x\big) = 0, \quad k = 0,1,2,\ldots,2r-1. \end{aligned} \tag{5}$$

Our main results in this paper can be stated as the following three theorems.

Theorem 6 (Direct theorem)**.** *Let $f \in L^*_\Phi[0,\infty)$, $n \in \mathbb{N}$, $\Psi \in \Delta_2$, and $\varphi(x) = \sqrt{x(1+x)}$. Then*

$$\|L_{n,r}(f) - f\|_\Phi \le C\omega_{2r,\varphi}\left(f, \frac{1}{\sqrt{n}}\right)_\Phi.$$

Theorem 7 (Inverse theorem)**.** *Let $f \in L^*_\Phi[0,\infty)$, $n \ge 2r$, and $\varphi^2(x) = x(1+x)$. Then*

$$\omega_{2r,\varphi}\left(f, \frac{1}{n^{r/2}}\right)_\Phi \le \frac{C}{n^r}\sum_{k=1}^{n} k^{r-1}\|L_{n,r}(f) - f\|_\Phi.$$

Theorem 8 (Equivalent theorem)**.** *Let $f \in L^*_\Phi[0,\infty)$, $n \ge 2r$, $\varphi^2(x) = x(1+x)$, and $\Psi \in \Delta_2$. Then*

$$\|L_{n,r}(f) - f\|_\Phi = O\left(\psi\left(\frac{1}{n^{1/2}}\right)\right), \quad n \to \infty \quad \textit{if and only if} \quad \omega_{2r,\varphi}(f,t)_\Phi = O(\psi(t)), \quad t \to 0^+.$$

These main results improve some conclusions in [19] and increase the approximating speed of corresponding operators.

2. Proof of the Direct Theorem

In order to prove the direct theorem, we need several lemmas below.

Lemma 1. *The modified summation operator of integral type $B_n(f,x)$ defined in Equation (1) satisfies*

$$B_n(1,x) = 1 \quad \text{and} \quad B_n\big((t-x)^{2r},x\big) \le C\left[\frac{\delta_n^2(x)}{n}\right]^r,$$

where $\delta_n^2(x) = \max\{\varphi^2(x), \frac{1}{n}\}$, $\varphi(x) = \sqrt{x(1+x)}$, $r \in \mathbb{N}$, and C is a positive constant.

Proof. This follows from simple calculation. □

Lemma 2 ([19])**.** *If u locates between x and t, then*

$$\frac{(t-u)^{2r-1}}{\varphi^{2r}(u)} \le \frac{|t-x|^{2r-1}}{\varphi^{2(r-1)}(x)}\frac{1}{x}\left(\frac{1}{1+x}+\frac{1}{1+t}\right).$$

Lemma 3. *Let $f \in L_\Phi^*[0,\infty)$. Then*

$$\|L_{n,r}(f)\|_\Phi \le C\|f\|_\Phi.$$

Proof. By Lemma 3.2 in [16], we have

$$\|B_n(f)\|_\Phi \le 2\|f\|_\Phi.$$

Using Equation (5), we obtain

$$\|L_{n,r}(f)\|_\Phi = \left\|\sum_{i=0}^{2r-1} c_i(n)B_{n_i}(f)\right\|_\Phi \le \sum_{i=0}^{2r-1}|c_i(n)|\|B_{n_i}(f)\|_\Phi \le C\|f\|_\Phi.$$

The proof of Lemma 3 is complete. □

Lemma 4 ([18])**.** *For $f \in L_\Phi^*[0,\infty)$ and $\Psi \in \Delta_2$, we have*

$$\|\theta(f)\|_\Phi \le C\|f\|_\Phi,$$

where

$$\theta(f,x) = \sup_{\substack{0\le t<\infty \\ t\ne x}} \frac{1}{t-x}\int_x^t f(u)\,\mathrm{d}u$$

is the Hardy-Littlewood function of $f(x)$, and C is a positive constant.

We are now in a position to prove Theorem 6.

Proof of Theorem 6. Let

$$U = W_\Phi^{2r}\{g : g^{(2r-1)} \in AC_{loc}, \varphi^{2r}g^{(2r)} \in L_\Phi^*[0,\infty)\}.$$

Taylor's formula with integral remainder of $g \in U$ reads

$$g(t) = \sum_{i=0}^{2r-1} \frac{g^{(i)}(x)}{i!}(t-x)^i + R_{2r}(g,t,x),$$

where

$$R_{2r}(g,t,x) = \frac{1}{(2r-1)!}\int_x^t (t-u)^{2r-1} g^{(2r)}(u)\,\mathrm{d}\,u, \quad x \in [0,\infty).$$

From Equation (5), it follows that $L_{n,r}(g,x) - g(x) = L_{n,r}(R_{2r}(g,t,x),x)$ and

$$\|L_{n,r}(g) - g\|_\Phi = \|L_{n,r}(R_{2r}(g,\cdot,x),x)\|_\Phi. \tag{6}$$

Now we estimate $|R_{2r}(g,t,x)|$. As $x \in [\frac{1}{n},\infty)$, we have $\delta_n^2(x) = \varphi^2(x)$. Applying Lemma 2 leads to

$$\begin{aligned}
|R_{2r}(g,t,x)| &\le \frac{1}{(2r-1)!}\left|\int_x^t \frac{(t-u)^{2r-1}}{\varphi^{2r}(u)}\delta_n^{2r}(u) g^{(2r)}(u)\,\mathrm{d}\,u\right| \\
&\le \frac{1}{(2r-1)!}\frac{|t-x|^{2r-1}}{\varphi^{2r-2}(x)}\frac{1}{x}\left(\frac{1}{1+x}+\frac{1}{1+t}\right)\left|\int_x^t \delta_n^{2r}(u) g^{(2r)}(u)\,\mathrm{d}\,u\right| \\
&\le \frac{1}{(2r-1)!}\frac{(t-x)^{2r}}{\varphi^{2r-2}(x)}\left[\frac{1}{x(1+x)}+\frac{1}{x(1+t)}\right]|\theta(\delta_n^{2r} g^{(2r)},x)| \\
&\triangleq I_1 + I_2.
\end{aligned}$$

From Lemma 1, we conclude that

$$\begin{aligned}
B_n(I_1,x) &= \frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^\infty b_{n,k}(t)\frac{1}{(2r-1)!}\frac{(t-x)^{2r}}{\varphi^{2r}(x)}|\theta(\delta_n^{2r} g^{(2r)},x)|\,\mathrm{d}\,t \\
&= \frac{|\theta(\delta_n^{2r} g^{(2r)},x)|\varphi^{-2r}(x)}{(2r-1)!}B_n((t-x)^{2r},x) \le \frac{C}{n^r}|\theta(\delta_n^{2r} g^{(2r)},x)|
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
B_n(I_2,x) &= \frac{|\theta(\delta_n^{2r} g^{(2r)},x)|}{n+1}\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^\infty b_{n,k}(t)\frac{1}{(2r-1)!}\frac{(t-x)^{2r}}{\varphi^{2r-2}(x)}\frac{1}{x(1+t)}\,\mathrm{d}\,t \\
&= \frac{|\theta(\delta_n^{2r} g^{(2r)},x)|\varphi^{-2r}(x)}{(2r-1)!(n+1)}\sum_{k=1}^{\infty}\left(\frac{n+1}{n+k+1}\right)^2 b_{n+1,k}(x)\int_0^\infty b_{n+1,k}(t)(t-x)^{2r}\,\mathrm{d}\,t \\
&\le \frac{C}{n^r}|\theta(\delta_n^{2r} g^{(2r)},x)|.
\end{aligned} \tag{8}$$

Hence, by Inequalities (7) and (8) and Lemma 4, it follows that

$$\|B_n(R_{2r}(g,\cdot,x),x)\|_{\Phi[\frac{1}{n},\infty)} \le \frac{C}{n^r}\|\theta(\delta_n^{2r} g^{(2r)},x)\|_{\Phi[\frac{1}{n},\infty)} \le \frac{C}{n^r}\|\delta_n^{2r} g^{(2r)}\|_{\Phi[\frac{1}{n},\infty)}. \tag{9}$$

For $x \in [0,\frac{1}{n})$ and $\delta_n^2(x) = \frac{1}{n}$, we have

$$|R_{2r}(g,t,x)| = \frac{1}{(2r-1)!}\left|\int_x^t (t-u)^{2r-1} g^{(2r)}(u)\,\mathrm{d}\,u\right|$$

$$= \frac{1}{(2r-1)!}\left|\int_x^t \frac{(t-u)^{2r-1}}{1/n^r}\delta_n^{2r}(u) g^{(2r)}(u)\,\mathrm{d}\,u\right| \le \frac{1}{(2r-1)!}(x-t)^{2r} n^r |\theta(\delta_n^{2r} g^{(2r)},x)|.$$

Using Lemmas 1 and 4 arrives at

$$B_n(|R_{2r}(g,t,x)|,x) \leq \frac{n^r}{(2r-1)!}\frac{C}{n^r}\delta_n^{2r}(x)\left|\theta(\delta_n^{2r}g^{(2r)},x)\right| \leq \frac{C}{n^r}\left|\theta(\delta_n^{2r}g^{(2r)},x)\right|$$

and

$$\|B_n(R_{2r}(g,\cdot,x),x)\|_{\Phi[0,\frac{1}{n})} \leq \frac{C}{n^r}\|\theta(\delta_n^{2r}g^{(2r)})\|_{\Phi[0,\frac{1}{n})} \leq \frac{C}{n^r}\|\delta_n^{2r}g^{(2r)}\|_{\Phi[0,\frac{1}{n})}.$$

Combining this with Equation (9) leads to

$$\|B_n(R_{2r}(g,\cdot,x),x)\|_{\Phi[0,\infty)} \leq \frac{C}{n^r}\|\delta_n^{2r}g^{(2r)}\|_{\Phi[0,\infty)}$$

and, consequently,

$$\begin{aligned}\|L_{n,r}(R_{2r}(g,\cdot,x),x)\|_\Phi &\leq \sum_{i=0}^{2r-1}|c_i|\|B_{n_i}(R_{2r}(g,\cdot,x),x)\|_\Phi \\ &\leq \sum_{i=0}^{2r-1}|c_i|\frac{C}{n_i^r}\|\delta_{n_i}^{2r}g^{(2r)}\|_\Phi \leq \sum_{i=0}^{2r-1}|c_i|\frac{C}{n^r}\|\delta_n^{2r}g^{(2r)}\|_\Phi \leq \frac{C}{n^r}\|\delta_n^{2r}g^{(2r)}\|_\Phi.\end{aligned}$$

Then, applying the above inequality, Inequalities (3) and (6), and Lemma 3, we obtain

$$\begin{aligned}\|L_{n,r}(f)-f\|_\Phi &\leq \|L_{n,r}(f-g)-(f-g)\|_\Phi + \|L_{n,r}(g)-g\|_\Phi \\ &\leq C\|f-g\|_\Phi + \frac{C}{n^r}\|\delta_n^{2r}g^{(2r)}\|_\Phi \leq C\omega_{2r,\varphi}\left(f,\frac{1}{\sqrt{n}}\right)_\Phi.\end{aligned}$$

The proof of the direct theorem is complete. □

3. Proofs of the Inverse and Equivalent Theorems

For proving Theorems 7 and 8, we need the following lemmas.

Lemma 5. *If $f \in L_\Phi^*[0,\infty)$ and $n \geq 2r$, then*

$$\|\varphi^{2r}L_{n,r}^{(2r)}(f)\|_\Phi \leq Cn^r\|f\|_\Phi.$$

Proof. Since

$$\begin{aligned} B_n^{(2r)}(f,x) &= \frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,k}^{(2r)}(x)\int_0^\infty b_{n,k}(t)f(t)\,\mathrm{d}t \\ &= \frac{1}{n+1}\sum_{k=1}^{\infty}\frac{(n+k)!}{(k-1)!n!}\sum_{i=0}^{2r}\binom{2r}{i}(-1)^i D^{2r-i}(x^{k-1})D^i\left(\frac{1}{(1+x)^{n+k}}\right)\int_0^\infty b_{n,k}(t)f(t)\,\mathrm{d}t \\ &= \prod_{j=2}^{2r}(n+j)\sum_{k=1}^{\infty} b_{n+2r,k}(x)\int_0^\infty \sum_{i=0}^{2r}\binom{2r}{i}(-1)^i b_{n,k+2r-i}(t)f(t)\,\mathrm{d}t \qquad (10) \\ &\leq \prod_{j=2}^{2r}(n+j)\sum_{k=1}^{\infty} b_{n+2r,k}(x)\int_0^\infty \sum_{i=0}^{2r}\binom{2r}{i} b_{n,k+2r-i}(t)f(t)\,\mathrm{d}t, \end{aligned}$$

we have

$$\varphi^{2r}(x)B_n^{(2r)}(f,x) \leq \prod_{j=2}^{2r}(n+j)\sum_{k=1}^{\infty} b_{n+2r,k}\varphi^{2r}(x)\int_0^\infty \sum_{i=0}^{2r}\binom{2r}{i} b_{n,k+2r-i}(t)f(t)\,\mathrm{d}t$$

$$\leq \prod_{j=2}^{2r}(n+j)\sum_{k=1}^{\infty} b_{n,k+r}(x)\frac{(k+r-1)!n!}{(n+r+k)!}\frac{(n+2r+k)!}{(k-1)!(n+2r)!}\int_0^\infty \sum_{i=0}^{2r}\binom{2r}{i} b_{n,k+2r-i}(t)f(t)\,\mathrm{d}\,t$$
$$\leq \frac{Cn^r}{n+1}\sum_{i=0}^{2r}\binom{2r}{i}\sum_{k=1}^{\infty} b_{n,k+r}(x)\int_0^\infty b_{n,k+2r-i}(t)f(t)\,\mathrm{d}\,t.$$

Therefore, by Jensen's inequality [20] and the inequality (2), we obtain

$$\begin{aligned}
&\|\varphi^{2r}(x)B_n^{(2r)}(f)\|_\Phi \\
&\leq 2\inf_{\lambda>0}\left\{\lambda:\int_0^\infty \Phi\left(\frac{1}{n+1}\sum_{i=0}^{2r}\binom{2r}{i}\sum_{k=1}^{\infty} b_{n,k+r}(x)\int_0^\infty b_{n,k+2r-i}(t)\frac{Cn^r|f(t)|}{\lambda}\,\mathrm{d}\,t\right)\mathrm{d}\,x\leq 1\right\} \\
&\leq 2\inf_{\lambda>0}\left\{\lambda:\int_0^\infty \frac{1}{2^{2r}}\sum_{i=0}^{2r}\binom{2r}{i}\Phi\left(\frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,k+r}(x)\int_0^\infty b_{n,k+2r-i}(t)\frac{Cn^r|f(t)|}{\lambda}\,\mathrm{d}\,t\right)\mathrm{d}\,x\leq 1\right\} \\
&\leq 2\inf_{\lambda>0}\left\{\lambda:\int_0^\infty \frac{1}{2^{2r}}\sum_{i=0}^{2r}\binom{2r}{i}\Phi\left(\frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^\infty b_{n,k+r-i}(t)\frac{Cn^r|f(t)|}{\lambda}\,\mathrm{d}\,t\right)\mathrm{d}\,x\leq 1\right\} \\
&\leq 2\inf_{\lambda>0}\left\{\lambda:\int_0^\infty \frac{1}{2^{2r}}\sum_{i=0}^{2r}\binom{2r}{i}\frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^\infty b_{n,k+r-i}(t)\Phi\left(\frac{Cn^r|f(t)|}{\lambda}\right)\mathrm{d}\,t\,\mathrm{d}\,x\leq 1\right\} \\
&\leq Cn^r\|f\|_\Phi,
\end{aligned}$$

where $b_{n,k+r-i}(t)=0$ for $n+r-i\leq 0$. Combining this with Equation (5) leads to

$$\|\varphi^{2r}(x)L_{n,r}^{(2r)}(f)\|_\Phi \leq \sum_{i=0}^{2r-1}|c_i(n)|\|\varphi^{2r}(x)B_{n_i}^{(2r)}(f)\|_\Phi \leq \sum_{i=0}^{2r-1}|c_i(n)|Cn_i^r\|f\|_\Phi \leq Cn^r\|f\|_\Phi.$$

Lemma 5 is thus proved. □

Lemma 6. *Let $f\in L_\Phi^*[0,\infty)$ and $n\geq 2r$. Then*

$$\|\varphi^{2r}L_{n,r}^{(2r)}(f)\|_\Phi \leq C\|\varphi^{2r}f^{(2r)}\|_\Phi.$$

Proof. Integrating by parts $2r$ times in Equation (10) gives

$$\begin{aligned}
B_n^{(2r)}(f,x) &= \frac{1}{n+1}\prod_{j=1}^{2r}(n+j)\sum_{k=1}^{\infty} b_{n+2r,k}(x)\int_0^\infty \sum_{i=0}^{2r}\binom{2r}{i}(-1)^i b_{n,k+2r-i}(t)f(t)\,\mathrm{d}\,t \\
&= \frac{(n+1)(n+2)\cdots(n+2r)}{n(n-1)\cdots(n-2r+1)}\frac{1}{n+1}\sum_{k=1}^{\infty} b_{n+2r,k}(x)\int_0^\infty b_{n-2r,k+2r}^{(2r)}(t)f(t)\,\mathrm{d}\,t \\
&= \frac{(n+2r)!(n-2r)!}{(n!)^2}\frac{1}{n+1}\sum_{k=1}^{\infty} b_{n+2r,k}(x)\int_0^\infty b_{n-2r,k+2r}(t)f^{(2r)}(t)\,\mathrm{d}\,t.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
&\varphi^{2r}(x)B_n^{(2r)}(f,x) \\
&\quad= \frac{(n+2r)!(n-2r)!}{(n!)^2}\frac{1}{n+1}\sum_{k=1}^{\infty} b_{n+2r,k}(x)\varphi^{2r}(x)\int_0^\infty b_{n-2r,k+2r}(t)\varphi^{-2r}(t)\varphi^{2r}(t)f^{(2r)}(t)\,\mathrm{d}\,t \\
&\quad\leq \frac{C}{n+1}\sum_{k=1}^{\infty} b_{n-2r+2,k+r}(x)\int_0^\infty b_{n,k+r}(t)\varphi^{2r}(t)f^{(2r)}(t)\,\mathrm{d}\,t.
\end{aligned}$$

Employing Inequality (2) and Jensen's inequality [20] reveals

$$
\begin{aligned}
&\|\varphi^{2r}(x)B_n^{(2r)}(f)\|_\Phi \\
&\leq 2\inf_{\lambda>0}\left\{\lambda : \int_0^\infty \Phi\left(\frac{C}{n+1}\sum_{k=1}^{\infty} b_{n-2r+2,k+r}(x)\int_0^\infty b_{n,k+r}(t)\frac{\varphi^{2r}(t)|f^{(2r)}(t)|}{\lambda}\,\mathrm{d}t\right)\mathrm{d}x \leq 1\right\} \\
&\leq 2\inf_{\lambda>0}\left\{\lambda : \int_0^\infty \Phi\left(\frac{C}{n+1}\sum_{k=1}^{\infty} b_{n-2r+2,k}(x)\int_0^\infty b_{n,k}(t)\frac{\varphi^{2r}(t)|f^{(2r)}(t)|}{\lambda}\,\mathrm{d}t\right)\mathrm{d}x \leq 1\right\} \\
&\leq 2\inf_{\lambda>0}\left\{\lambda : \int_0^\infty \frac{\sum_{k=1}^{\infty} b_{n-2r+2,k}(x)}{n-2r+3}\int_0^\infty b_{n,k}(t)\Phi\left(\frac{\frac{C(n-2r+3)}{n+1}\varphi^{2r}(t)|f^{(2r)}(t)|}{\lambda}\right)\mathrm{d}t\,\mathrm{d}x \leq 1\right\} \\
&\leq 2\inf_{\lambda>0}\left\{\lambda : \frac{n+1}{n-2r+3}\int_0^\infty \Phi\left(\frac{\frac{C(n-2r+3)}{n+1}\varphi^{2r}(t)|f^{(2r)}(t)|}{\lambda}\right)\mathrm{d}t \leq 1\right\} \\
&\leq C\|\varphi^{2r}f^{(2r)}\|_\Phi.
\end{aligned}
$$

Applying the above inequality and Inequality (5) results in

$$
\begin{aligned}
\|\varphi^{2r}L_{n,r}^{(2r)}(f)\|_\Phi &= \left\|\sum_{i=0}^{2r-1} c_i(n)\varphi^{2r}B_{n_i}^{(2r)}(f)\right\|_\Phi \leq \sum_{i=0}^{2r-1}|c_i(n)|\|\varphi^{2r}B_{n_i}^{(2r)}(f)\|_\Phi \\
&\leq C\sum_{i=0}^{2r-1}|c_i(n)|\|\varphi^{2r}f^{(2r)}\|_\Phi \leq C\|\varphi^{2r}f^{(2r)}\|_\Phi.
\end{aligned}
$$

The lemma is proved. □

Proof of Theorem 7. From Lemmas 5 and 6 and [21] (Theorem 2.2), we obtain

$$
K_{2r,\varphi}\left(f,\frac{1}{n^{r/2}}\right)_\Phi \leq \frac{C}{n^r}\sum_{k=1}^{n} k^{r-1}\|L_{n,r}(f)-f\|_\Phi.
$$

Utilizing Inequality (4) concludes the inverse theorem. □

Proof of Theorem 8. Using the so-called order function $\psi(t) = t^\alpha|\ln t|^\beta e^{|\ln t|^\gamma}$ for $0<\alpha<1$, $\beta\in\mathbb{R}$, and $\gamma<1$ and combining Theorems 7 and 8 conclude the equivalent theorem. □

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