On Approximation by Linear Combinations of Modified Summation Operators of Integral Type in Orlicz Spaces

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1. Introduction and Main Results

Throughout this paper, we use $C$ to denote an absolute constant independent of anything, which may be not necessarily the same in different cases.

There are many types of integral operators (see, for example, [1–7] and closely related references therein). In the paper [8], Ueki provided a characterization for the boundedness and compactness of the Li-Stević type integral operators

$$T_{\phi}^g f(z) = \int_0^z f(\phi(\xi))g(\xi) \, d\xi$$

from the weighted Bergman space $L^p_\alpha(dA_\alpha)$ to the $\beta$-Zygmund space $Z_\beta$. Later, Li and Ma [9] investigated the boundedness and compactness of products of composition operators and integral type operators

$$(C_\psi I_\phi f)(z) = \int_0^{\psi(z)} f'(\xi)g(\xi) \, d\xi$$

from Zygmund-type spaces to $Q_k$ spaces. Recently, the equivalent characterizations for the boundedness and compactness of several integral type operators

$$V_{\Phi}^h f(z) = \int_0^z f'(\Phi(t))h'(t) \, dt$$
from $F(p,q,s)$ space to $\alpha$-Bloch-Orlicz and $\beta$-Zygmund-Orlicz spaces were developed in [10]. Gupta and Yadav [11] estimated the approximation by complex summation integral type operator

$$M_n(f, z) = (n + 1) \sum_{k=1}^{n} p_{n,k}(z) \int_{0}^{1} f(t)p_{n,k-1}(t) \, dt + f(0)p_{n,0}(z)$$

in compact disks, where $p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}$. In [8–11], some approximating properties of integral type operators were discussed in $\mathbb{C}$. Gupta, Mohapatra, and Finta studied the mixed summation operators of integral type

$$G_n^{(\alpha, \beta)}(f, x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_{0}^{\infty} b_{n,k-1}(t) f \left( \frac{nt + \alpha}{n + \beta} \right) \, dt + (1 + x)^{-n} f \left( \frac{\alpha}{n + \beta} \right),$$

by hypergeometric series. Srivastava and Gupta [14] introduced and investigated a new sequence of linear positive operators

$$G_n^{(\alpha, \beta)}(f, x) = n \sum_{k=1}^{\infty} p_{n,k}(x) \int_{0}^{\infty} b_{n,k-1}(t) f(t) \, dt + \int_{0}^{\infty} p_{n,0}(x; c) p_{n,0}(c; \delta(t)) f(t) \, dt,$$

which included some well-known operators as its special cases and obtained an estimate on the rate of convergence by means of the decomposition technique for functions of bounded variation. In [15], Gupta, Mohapatra, and Finta studied the mixed summation operators of integral type

$$S_n(f, x) = \sum_{v=1}^{\infty} s_{n,v}(x) \int_{0}^{\infty} b_{n,v-1}(t) f(t) \, dt + e^{-nx} f(0)$$

and obtained the rate of point-wise convergence, where

$$s_{n,v}(x) = e^{-nx} \left( \frac{n^x}{v!} \right) \quad \text{and} \quad b_{n,v}(t) = \frac{1}{B(n, v + 1)} t^{v}(1 + t)^{-n-v-1}.$$
where \( b_{n,k}(x) = \frac{(n+k)!}{(k-1)!n!} \frac{x^{k-1}}{n^{k-1}} \) for \( k, n \geq 1 \). Recently, Han and Wu [16] obtained the following direct, inverse, and equivalent theorems of modified summation operators of integral type in Orlicz spaces.

**Theorem 1** (Direct theorem [16]). Let \( f \in L_{\Phi}^*([0, \infty)), \varphi^2(x) = x(1 + x), \) and \( \Psi \in \Delta_2 \). Then

\[
\| B_n(f) - f \|_{\Phi} \leq C \left[ \omega_{2,\varphi} \left( f, \frac{1}{\sqrt{n}} \right)_{\Phi} + \omega_1 \left( f, \frac{1}{n} \right)_{\Phi} + \| f \|_{\Phi} \right].
\]

**Theorem 2** (Inverse theorem [16]). Let \( f \in L_{\Phi}^*([0, \infty)), 0 \leq \alpha < 2, \) and \( \| B_n(f) - f \|_{\Phi} = O(n^{-\alpha/2}) \). Then

\[
\omega_{2,\varphi}(f, t)_{\Phi} = O(t^\alpha) \quad \text{and} \quad \omega_1(f, t)_{\Phi} = O(t^{\alpha/2}).
\]

**Theorem 3** (Equivalence theorem [16]). Let \( f \in L_{\Phi}^*([0, \infty)) \) and \( 0 \leq \alpha < 2 \). Then

\[
\| B_n(f) - f \|_{\Phi} = O(n^{-\alpha/2}) \quad \text{if and only if} \quad \omega_{2,\varphi}(f, t)_{\Phi} = O(t^\alpha) \quad \text{and} \quad \omega_1(f, t)_{\Phi} = O(t^{\alpha/2}).
\]

In recent years, since the Orlicz spaces are more general than the classical \( L_p \) spaces, which is composed of measurable functions \( f(x) \) such that \( |f(x)|^p \) are integrable, there is growing interest in problems of approximation in Orlicz spaces.

For smoothly proceeding, we recall from [17] some definitions and related results. A continuous convex function \( \Phi(t) \) on \([0, \infty)\) is called a Young function if it satisfies

\[
\lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.
\]

For a given Young function \( \Phi(t) \), its complementary Young function is denoted by \( \Psi(t) \).

It is clear that the convexity of \( \Phi(t) \) can lead to \( \Phi(at) \leq a \Phi(t) \) for \( a \in [0, 1] \). In particular, one has \( \Phi(at) < a \Phi(t) \) for \( a \in (0, 1) \).

A Young function \( \Phi(t) \) is said to satisfy the \( \Delta_2 \)-condition, denoted by \( \Phi \in \Delta_2 \), if there exist \( t_0 > 0 \) and \( C \geq 1 \) such that \( \Phi(2t) \leq C \Phi(t) \) for \( t \geq t_0 \).

Let \( \Phi(t) \) be a Young function. We define the Orlicz class \( L_{\Phi}([0, \infty)) \) as the collection of all Lebesgue measurable functions \( u(x) \) on \([0, \infty)\). Since the integral

\[
\rho(u, \Phi) = \int_0^\infty \Phi(|u(x)|) \, dx
\]

is finite, we define the Orlicz space \( L_{\Phi}^*([0, \infty)) \) as the linear hull of \( L_{\Phi}([0, \infty)) \) under the Luxemburg norm

\[
\| u \|_{\Phi} = \inf_{\lambda > 0} \left\{ \lambda : \rho \left( \frac{u}{\lambda}, \Phi \right) \leq 1 \right\}.
\]

The Orlicz norm, which is equivalent to the Luxemburg norm on \( L_{\Phi}^*([0, \infty)) \), is given by

\[
\| u \|_{\Phi} = \sup_{\rho \leq 1} \left| \int_0^\infty u(x)\nu(x) \, dx \right|
\]

and satisfies

\[
\| u \|_{\Phi} \leq \| u \|_{\Phi} \leq 2\| u \|_{\Phi}.
\]  

For \( f \in L_{\Phi}^*([0, \infty)) \), the weighted \( K \)-functional \( K_{r,\varphi}(f, t') \), the modified weighted \( K \)-functional \( K_{r,\varphi}(f, t') \), and the weighted modulus of smoothness \( \omega_{r,\varphi}(f, t)_{\Phi} \) are given, respectively, by

\[
K_{r,\varphi}(f, t')_{\Phi} = \inf_{g} \{ \| f - g \|_{\Phi} + t' \| \varphi^\prime(1) \|_{\Phi} : \| g \|_{\Phi} \leq t' \} \in AC_{loc},
\]
Theorem 6 (Direct theorem). Let \( f \in L_{\Psi}^1[0, \infty) \), \( n \in \mathbb{N} \), \( \Psi \in \Delta_2 \), and \( \psi(x) = \sqrt{x(1+x)} \). Then

\[
\|L_{n,r}(f) - f\|_{\Phi} \leq C \omega_{2r,\psi}\left(f, \frac{1}{\sqrt{n}} \right)_{\Phi}.
\]

Theorem 7 (Inverse theorem). Let \( f \in L_{\Psi}^1[0, \infty) \), \( n \geq 2r \), and \( \psi^2(x) = x(1+x) \). Then

\[
\omega_{2r,\psi}\left(f, \frac{1}{n^{r/2}} \right)_{\Phi} \leq \frac{C}{n^r} \sum_{k=1}^{n} k^{r-1} \|L_{n,r}(f) - f\|_{\Phi}.
\]

Theorem 8 (Equivalent theorem). Let \( f \in L_{\Psi}^1[0, \infty) \), \( n \geq 2r \), \( \psi^2(x) = x(1+x) \), and \( \Psi \in \Delta_2 \). Then

\[
\|L_{n,r}(f) - f\|_{\Phi} = O\left(\psi\left(\frac{1}{n^{1/2}}\right)\right), \quad n \to \infty \quad \text{if and only if} \quad \omega_{2r,\psi}(f,t)_{\Phi} = O(\psi(t)), \quad t \to 0^+.
\]
These main results improve some conclusions in [19] and increase the approximating speed of corresponding operators.

2. Proof of the Direct Theorem

In order to prove the direct theorem, we need several lemmas below.

Lemma 1. The modified summation operator of integral type $B_n(f, x)$ defined in Equation (1) satisfies

$$B_n(1, x) = 1 \quad \text{and} \quad B_n((t - x)^{2r}, x) \leq C \left[ \frac{\delta_n^2(x)}{n} \right]^r,$$

where $\delta_n^2(x) = \max\{\varphi^2(x), \frac{1}{n}\}$, $\varphi(x) = \sqrt{x(1+x)}$, $r \in \mathbb{N}$, and $C$ is a positive constant.

Proof. This follows from simple calculation. \qed

Lemma 2 ([19]). If $u$ locates between $x$ and $t$, then

$$\frac{(t - u)^{2r - 1}}{\varphi^{2r}(u)} \leq \frac{|t - x|^{2r - 1}}{\varphi^{2(r-1)}(x)} \left( \frac{1}{1 + x} + \frac{1}{1 + t} \right).$$

Lemma 3. Let $f \in L^*_\Phi[0, \infty)$. Then

$$\|L_{n,r}(f)\|_\Phi \leq C\|f\|_\Phi.$$

Proof. By Lemma 3.2 in [16], we have

$$\|B_n(f)\|_\Phi \leq 2\|f\|_\Phi.$$

Using Equation (5), we obtain

$$\|L_{n,r}(f)\|_\Phi = \left\| \sum_{i=0}^{2r-1} c_i(n)B_{n_i}(f) \right\|_\Phi \leq \sum_{i=0}^{2r-1} |c_i(n)|\|B_{n_i}(f)\|_\Phi \leq C\|f\|_\Phi.$$ 

The proof of Lemma 3 is complete. \qed

Lemma 4 ([18]). For $f \in L^*_\Phi[0, \infty)$ and $\Psi \in \Delta_2$, we have

$$\|\theta(f)\|_\Phi \leq C\|f\|_\Phi,$$

where

$$\theta(f, x) = \sup_{0 \leq t < \infty; t \neq x} \frac{1}{t - x} \int_x^t f(u) \, du$$

is the Hardy-Littlewood function of $f(x)$, and $C$ is a positive constant.

We are now in a position to prove Theorem 6.

Proof of Theorem 6. Let

$$U = W^r_\Phi \{ g : g^{(2r-1)} \in AC_{loc}, \varphi^{2r}g^{(2r)} \in L^*_\Phi[0, \infty) \}.$$
Taylor’s formula with integral remainder of \( g \in \mathcal{U} \) reads
\[
g(t) = \sum_{i=0}^{2r-1} \frac{s^{(i)}(x)}{i!} (t-x)^i + R_{2r}(g, t, x),
\]
where
\[
R_{2r}(g, t, x) = \frac{1}{(2r-1)!} \int_{x}^{t} (t-u)^{2r-1} g^{(2r)}(u) \, du, \quad x \in [0, \infty).
\]
From Equation (5), it follows that \( L_{n,r}(g, x) - g(x) = L_{n,r}(R_{2r}(g, t, x), x) \) and
\[
\|L_{n,r}(g) - g\| \leq \|L_{n,r}(R_{2r}(g, t, x), x)\|. \tag{6}
\]
Now we estimate \( |R_{2r}(g, t, x)| \). As \( x \in \left[ \frac{1}{n^2}, \infty \right) \), we have \( \delta_n^2(x) = \varphi^2(x) \). Applying Lemma 2 leads to
\[
|R_{2r}(g, t, x)| \leq \frac{1}{(2r-1)!} \left| \int_{x}^{t} (t-u)^{2r-1} \frac{\delta_n^2(u) g^{(2r)}(u)}{\psi^{2r}(u)} \, du \right|
\]
\[
\leq \frac{1}{(2r-1)!} \left| \int_{x}^{t} \frac{1}{\psi^{2r-2}(x)} \frac{1}{x} \left( \frac{1}{1+x} + \frac{1}{1+t} \right)^{2r} \, du \right|
\]
\[
\leq \frac{1}{(2r-1)!} \left| \frac{1}{\psi^{2r-2}(x)} \right| \frac{x(1+x)}{x(1+t)} \left| \theta(\delta_n^2 g^{(2r)}(x)) \right|
\]
\[
\leq I_1 + I_2.
\]
From Lemma 1, we conclude that
\[
B_n(I_1, x) = \frac{1}{n+1} \sum_{k=1}^{\infty} b_n k(x) \int_{0}^{\infty} b_{n,k}(t) \frac{1}{(2r-1)!} \left( \frac{1}{\psi^{2r}(x)} \right) \left| \theta(\delta_n^2 g^{(2r)}(x)) \right| \, dt
\]
\[
= \frac{\left| \theta(\delta_n^2 g^{(2r)}(x)) \right| \varphi^{2r}(x)}{(2r-1)!} B_n((t-x)^{2r}, x) \leq \frac{C}{n^{2r}} \left| \theta(\delta_n^2 g^{(2r)}(x)) \right| \tag{7}
\]
and
\[
B_n(I_2, x) = \frac{\left| \theta(\delta_n^2 g^{(2r)}(x)) \right|}{(2r-1)!} \sum_{k=1}^{\infty} b_n k(x) \int_{0}^{\infty} b_{n,k}(t) \frac{1}{(2r-1)!} \left( \frac{1}{\psi^{2r-2}(x)} \right) \left( \frac{1}{x(1+t)} \right) \, dt
\]
\[
= \frac{\left| \theta(\delta_n^2 g^{(2r)}(x)) \right| \varphi^{2r}(x)}{(2r-1)!} \sum_{k=1}^{\infty} \frac{(n+1)}{n+k+1} \frac{1}{b_{n+1,k}(x)} \int_{0}^{\infty} b_{n+1,k}(t) (t-x)^{2r} \, dt \leq \frac{C}{n^{2r}} \left| \theta(\delta_n^2 g^{(2r)}(x)) \right| \tag{8}
\]
Hence, by Inequalities (7) and (8) and Lemma 4, it follows that
\[
\|B_n(R_{2r}(g, t, x))\| \Phi_{\left[ \frac{1}{n^2}, \infty \right]} \leq \frac{C}{n^{2r}} \left| \theta(\delta_n^2 g^{(2r)}(x)) \right| \|\Phi_{\left[ \frac{1}{n^2}, \infty \right]} \leq \frac{C}{n^{2r}} \|\delta_n^2 g^{(2r)}(x)\| \|\Phi_{\left[ \frac{1}{n^2}, \infty \right]} \|. \tag{9}
\]
For \( x \in \left[ \frac{1}{n^2}, \frac{1}{n} \right) \) and \( \delta_n^2(x) = \frac{1}{n^{2r}} \), we have
\[
|R_{2r}(g, t, x)| = \left| \frac{1}{(2r-1)!} \left| \int_{x}^{t} (t-u)^{2r-1} \frac{\delta_n^2(u) g^{(2r)}(u)}{1/n^r} \, du \right| \right|
\]
\[
= \frac{1}{(2r-1)!} \left| \int_{x}^{t} (t-u)^{2r-1} \frac{\delta_n^2(u) g^{(2r)}(u)}{1/n^r} \, du \right| \leq \frac{1}{(2r-1)!} (x-t)^{2r} n^r \left| \theta(\delta_n^2 g^{(2r)}(x)) \right|.
\]
Using Lemmas 1 and 4 arrives at
\[ B_n((\mathcal{R}_2, g, t, x), x) \leq \frac{n!}{(2r - 1)!} C n^{2r} \mathcal{L}(2r, \mathcal{L}(2r, x)) \leq \frac{C}{n^r} |g(2r, \mathcal{L}(2r, x))| \]
and
\[ \|B_n((\mathcal{R}_2, g, \cdot, x), x)\|_{\mathcal{L}(2r, \mathcal{L}(2r, x))} \leq \frac{C}{n^r} \|\mathcal{L}(2r, \mathcal{L}(2r, x))\|_{\mathcal{L}(2r, \mathcal{L}(2r, x))} \leq \frac{C}{n^r} \|\mathcal{L}(2r, \mathcal{L}(2r, x))\|_{\mathcal{L}(2r, \mathcal{L}(2r, x))}. \]
Combining this with Equation (9) leads to
\[ \|B_n((\mathcal{R}_2, g, \cdot, x), x)\|_{\Phi(0, \infty)} \leq \frac{C}{n^r} \|\mathcal{L}(2r, \mathcal{L}(2r, x))\|_{\Phi(0, \infty)} \]
and, consequently,
\[ \|L_{2r}(\mathcal{R}_2, g, \cdot, x), x)\|_{\Phi(0, \infty)} \leq \frac{2r - 1}{\sum_{i=0}^{2r - 1} |c_i| \|B_n((\mathcal{R}_2, g, \cdot, x), x)\|_{\Phi}} \leq \frac{2r - 1}{\sum_{i=0}^{2r - 1} |c_i| \frac{C}{n^r} \|\mathcal{L}(2r, \mathcal{L}(2r, x))\|_{\Phi}} \leq \frac{C}{n^r} \|\mathcal{L}(2r, \mathcal{L}(2r, x))\|_{\Phi}. \]
Then, applying the above inequality, Inequalities (3) and (6), and Lemma 3, we obtain
\[ \|L_{2r}(f) - f\|_{\Phi} \leq \|L_{2r}(f - g) - (f - g)\|_{\Phi} + \|L_{2r}(g) - g\|_{\Phi} \leq C \|f - g\|_{\Phi} \frac{C}{n^r} \|\mathcal{L}(2r, \mathcal{L}(2r, x))\|_{\Phi} \leq C\omega_{2r, \Phi}(f, \frac{1}{\sqrt{n}}) \Phi. \]
The proof of the direct theorem is complete. \( \square \)

3. Proofs of the Inverse and Equivalent Theorems

For proving Theorems 7 and 8, we need the following lemmas.

**Lemma 5.** If \( f \in L_{2r}^\infty(0, \infty) \) and \( n \geq 2r \), then
\[ \|\varphi^{2r}L_{2r}(f)\|_{\Phi} \leq Cn^r \|f\|_{\Phi}. \]

**Proof.** Since
\[ B_n^{(2r)}(f, x) = \frac{1}{n + 1} \sum_{k=1}^{\infty} b_{n, k}^{(2r)}(x) \int_0^x b_{n, k}(t) f(t) \, dt \]
\[ = \frac{1}{n + 1} \sum_{k=1}^{\infty} \frac{(n + k)!}{(k - 1)! n^r} \sum_{i=0}^{2r} \binom{2r}{i} (-1)^i D^{2r - i}(x^{k - 1} D^{i} \left( \frac{1}{(1 + x)^{n + r}} \right)) \int_0^x b_{n, k}(t) f(t) \, dt \]
\[ = \prod_{j=2}^{2r} (n + j) \sum_{k=1}^{\infty} b_{n, 2r, k}(x) \int_0^x \sum_{i=0}^{2r} \binom{2r}{i} (-1)^i b_{n, 2r - i}(t) f(t) \, dt \]
\[ \leq \prod_{j=2}^{2r} (n + j) \sum_{k=1}^{\infty} b_{n, 2r, k}(x) \int_0^x \sum_{i=0}^{2r} \binom{2r}{i} b_{n, 2r - i}(t) f(t) \, dt, \]
we have
\[ \varphi^{2r}(x) B_n^{(2r)}(f, x) \leq \prod_{j=2}^{2r} (n + j) \sum_{k=1}^{\infty} b_{n, 2r, k} \varphi^{2r}(x) \int_0^x \sum_{i=0}^{2r} \binom{2r}{i} b_{n, 2r - i}(t) f(t) \, dt \]
where $b \equiv \frac{Cn^r}{n + 1} \sum_{i=0}^{2r} \frac{(2r)}{i} \sum_{k=1}^{\infty} b_{n,k+r}(x) \int_0^\infty b_{n,k+2r-i}(t)f(t) \, dt$.

Therefore, by Jensen’s inequality [20] and the inequality (2), we obtain

$$\|q^{2r}(x)B_n^{(2r)}(f)\|_\Phi \leq \frac{2r-1}{2r} \sum_{i=0}^{2r-1} c_i(n) \int_0^\infty b_{n,k+r}(x) \int_0^\infty b_{n,k+2r-i}(t) \Phi \left( \frac{Cn^r |f(t)|}{\lambda} \right) \, dt \, dx \leq \frac{2r-1}{2r} \sum_{i=0}^{2r-1} c_i(n) \|B_n^{(2r)}(f)\|_\Phi \leq \frac{2r-1}{2r} \sum_{i=0}^{2r-1} c_i(n) \|B_n^{(2r)}(f)\|_\Phi \leq Cn^r \|f\|_\Phi.$$

where $b_{n,k+r}(x) = 0$ for $n + r - i \leq 0$. Combining this with Equation (5) leads to

$$\|q^{2r}(x)L_{n,r}^{(2r)}(f)\|_\Phi \leq \sum_{i=0}^{2r-1} c_i(n) \|q^{2r}(x)B_n^{(2r)}(f)\|_\Phi \leq \sum_{i=0}^{2r-1} c_i(n) \|B_n^{(2r)}(f)\|_\Phi \leq Cn^r \|f\|_\Phi.$$

Lemma 5 is thus proved. \(\square\)

**Lemma 6.** Let $f \in L^r_\Phi[0, \infty)$ and $n \geq 2r$. Then

$$\|q^{2r}L_{n,r}^{(2r)}(f)\|_\Phi \leq C \|q^{2r}f^{(2r)}\|_\Phi.$$

**Proof.** Integrating by parts $2r$ times in Equation (10) gives

$$B_n^{(2r)}(f, x) = \frac{1}{n+1} \sum_{k=1}^{2r} b_{n+2r,k}(x) \int_0^\infty b_{n,k+2r-i}(t) f(t) \, dt \leq \frac{2r}{n+1} \sum_{k=1}^{\infty} b_{n,k+2r}(x) \int_0^\infty b_{n,k+2r-i}(t) f(t) \, dt \leq \frac{2r}{n+1} \sum_{k=1}^{\infty} b_{n,k+2r-i}(t) f(t) \, dt.$$

Accordingly,

$$q^{2r}(x)B_n^{(2r)}(f, x) \leq \frac{2r}{n+1} \sum_{k=1}^{\infty} b_{n,k+2r-i}(t) f(t) \, dt \leq \frac{C}{n+1} \sum_{k=1}^{\infty} b_{n,k+2r-i}(t) f(t) \, dt.$$
Employing Inequality (2) and Jensen’s inequality [20] reveals
\[
\|\varphi^{2r}(x)B_n^{(2r)}(f)\|_\Phi \\
\leq 2 \inf_{\lambda > 0} \left\{ \lambda : \int_0^\infty \Phi \left( \frac{\sum_{k=1}^{n+1} b_n^{-2r+2k}(x) \int_0^\infty b_n(k)(t) \frac{\varphi^{2r}(t)}{\lambda} \|f^{(2r)}(t)\|_\Phi}{\lambda} \right) dt \right. \right. d x \leq 1 \\
\leq 2 \inf_{\lambda > 0} \left\{ \lambda : \int_0^\infty \Phi \left( \frac{\sum_{k=1}^{n+1} b_n^{-2r+2k}(x) \int_0^\infty b_n(k)(t) \frac{\varphi^{2r}(t)}{\lambda} \|f^{(2r)}(t)\|_\Phi}{\lambda} \right) dt \right. \right. d x \leq 1 \\
\leq 2 \inf_{\lambda > 0} \left\{ \lambda : \int_0^\infty \sum_{k=1}^{n+1} b_n^{-2r+2k}(x) \int_0^\infty b_n(k)(t) \left( \frac{\sum_{k=1}^{n+1} b_n^{-2r+2k}(x) \int_0^\infty b_n(k)(t) \frac{\varphi^{2r}(t)}{\lambda} \|f^{(2r)}(t)\|_\Phi}{\lambda} \right) dt \right. \right. d x \leq 1 \\
\leq 2 \inf_{\lambda > 0} \left\{ \lambda : \frac{n+1}{n-2r+3} \int_0^\infty \Phi \left( \frac{\sum_{k=1}^{n+1} b_n^{-2r+2k}(x) \int_0^\infty b_n(k)(t) \frac{\varphi^{2r}(t)}{\lambda} \|f^{(2r)}(t)\|_\Phi}{\lambda} \right) dt \right. \right. d x \leq 1 \\
\leq C \|\varphi^{2r}(f)\|_\Phi.
\]

Applying the above inequality and Inequality (5) results in
\[
\|\varphi^{2r}L_{n,r}^{(2r)}(f)\|_\Phi = \left\| \sum_{i=0}^{2r-1} c_i(n)\varphi^{2r}B_{n_i}^{(2r)}(f) \right\|_\Phi \leq \sum_{i=0}^{2r-1} |c_i(n)| \|\varphi^{2r}B_{n_i}^{(2r)}(f)\|_\Phi \leq C \sum_{i=0}^{2r-1} |c_i(n)| \|\varphi^{2r}f^{(2r)}\|_\Phi \leq C \|\varphi^{2r}f^{(2r)}\|_\Phi.
\]

The lemma is proved. □

**Proof of Theorem 7.** From Lemmas 5 and 6 and [21] (Theorem 2.2), we obtain
\[
K_{2r,\varphi} \left( f, \frac{1}{m^{1/2}} \right) \leq C \frac{n}{m^r} \sum_{k=1}^{n} k^{r-1} \|L_{n,r}(f) - f\|_\Phi.
\]

Utilizing Inequality (4) concludes the inverse theorem. □

**Proof of Theorem 8.** Using the so-called order function \( \rho(t) = t^\alpha |\ln t|^\beta |\ln t|^{\gamma} \) for \( 0 < \alpha < 1, \beta, \gamma \in \mathbb{R} \), and \( \gamma < 1 \) and combining Theorems 7 and 8 conclude the equivalent theorem. □

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