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The A_α-Spectral Radii of Graphs with Given Connectivity

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Abstract: The A_α-matrix is $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ with $\alpha \in [0,1]$, given by Nikiforov in 2017, where $A(G)$ is adjacent matrix, and $D(G)$ is its diagonal matrix of the degrees of a graph G. The maximal eigenvalue of $A_\alpha(G)$ is said to be the A_α-spectral radius of G. In this work, we determine the graphs with largest $A_\alpha(G)$-spectral radius with fixed vertex or edge connectivity. In addition, related extremal graphs are characterized and equations satisfying $A_\alpha(G)$-spectral radius are proposed.

Keywords: adjacent matrix; signless Laplacian; spectral radius; connectivity

1. Introduction

We consider simple finite connected graph G with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices $|V(G)| = n$ is the order of a graph, and the number of edges $|E(G)|$ is the size of a graph. Denote the neighborhood of $v \in V(G)$ by $N(v) = \{u \in V(G), vu \in E(G)\}$, and the degree of v by $d_G(v) = |N(v)|$ (or briefly d_v). For $L \subseteq V(G)$ and $R \subseteq E(G)$, let $w(G-L)$ or $w(G-R)$ be the number of components of $G-L$ or $G-R$. L(or R) be a vertex(edge) cut set if $w(G-L \text{ (or } R)) \geq 2$ and $E(w,L) = \{wu \in E(G), u \in L\}$. For $U \subseteq V(G)$, $G[U]$ denote the induced subgraph of G, that is, $V(G[U]) = U$ and $E(G[U]) = \{uv | uv \in E(G), u, v \in U\}$.

If $A(G)$ is adjacency matrix of a graph G, and $D(G)$ is its diagonal matrix of the degrees of G, then the signless Laplacian matrix of G is $D(G) + A(G)$. With the successful studies of these matrices, Nikiforov [1] proposed the A_α-matrix

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$$

with $\alpha \in [0,1]$. Obviously, $A_0(G)$ is the adjacent matrix and $A_{\frac{1}{2}}$ is the half of signless Laplacian matrix of G, respectively. For undefined terminologies and notations, we refer to [2].

The research of (adjacency, signless Laplacian) spectral radius is an intriguing topic during past decades [3–9]. For instances, Lovász and J. Pelikán studied the spectral radius of trees [10]. The minimal Laplacian spectral radius of trees with given matching number is given by Feng et al. [7]. The properties of spectra of graphs and their line graphs are studied by Chen [11]. The signless Laplacian spectra of graphs is explored by Cvetković et al. [12]. Zhou [13] found bounds of signless Laplacian spectral radius and its hamiltonicity. Graphs having none or one signless Laplacian eigenvalue larger than three are obtained by Lin and Zhou [14]. At the same time, the maximal adjacency or signless Laplacian spectral radius have attracted many interests among the mathematical literature including algebra and graph theory. Ye et al. [6] gave the maximal adjacency or signless Laplacian spectral radius of graphs subject to fixed connectivity.

Inspired by these outcomes, we determine the graphs with largest $A_\alpha(G)$-spectral radius with given vertex or edge connectivity. In addition, the corresponding extremal graphs are provided and the equations satisfying the $A_\alpha(G)$-spectral radius are obtained.

2. Preliminary

In this section, we provide some important concepts and lemmas that will be used in the main proofs.

Denote by G a graph such that $V(G) = \{v_1, v_2, \cdots, v_n\}$ is its vertex set and $E(G)$ is its edge set. The A_α-matrix of G has the (i,j)-entry of $A_\alpha(G)$ is $1-\alpha$ if $v_iv_j \in E(G)$; $\alpha d(v_i)$ if $i = j$, and otherwise 0. For $\alpha \in [0,1]$, let $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$ be the eigenvalues of $A_\alpha(G)$. The A_α-spectral radius of G is considered as the maximal eigenvalue $\rho := \lambda_1(A_\alpha(G))$. Let $X = (x_{v_1}, x_{v_2}, \cdots, x_{v_n})^T$ be a real vector of ρ.

By $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, we have the quadratic formula of $X^T A_\alpha(G) X$ can be expressed that

$$X^T A_\alpha(G) X = \alpha \sum_{v_i \in V(G)} x_{v_i}^2 d_{v_i} + 2(1-\alpha) \sum_{v_iv_j \in E(G)} x_{v_i} x_{v_j}.$$

Because $A_\alpha(G)$ is a real symmetric matrix, and by Rayleigh principle, we have the formula

$$\rho(G) = max_{X \neq 0} \frac{X^T A_\alpha(G) X}{X^T X}.$$

As we know that once X is an eigenvector of $\rho(G)$ for a connected graph G, X should be unique and positive. The corresponding eigenequations for $A_\alpha(G)$ is rewritten as

$$\rho(G) x_{v_i} = \alpha d_{v_i} x_{v_i} + (1-\alpha) \sum_{v_iv_j \in E(G)} x_{v_j}. \tag{1}$$

As $A_1(G) = D(G)$, we study the A_α-matrix for $\alpha \in [0,1)$ below. Based on the definition of A_α-spectral radius, we have

Lemma 1. [4,15] *Let $A_\alpha(G)$ be the A_α-matrix of a connected graph G ($\alpha \in [0,1)$), $v, w \in V(G)$, $u \in T \subset V(G)$ such that $T \subset N(v) \setminus (N(w) \cup \{w\})$. Let G^* be a graph with vertex set $V(G)$ and edge set $E(G) \setminus \{uv, u \in T\} \cup \{uw, u \in T\}$, and X a unit eigenvector to $\rho(A_\alpha(G))$. If $x_w \geq x_v$ and $|T| \neq 0$, then $\rho(G^*) > \rho(G)$.*

If G is a connected graph, then $A_\alpha(G)$ is a nonnegative irreducible symmetric matrix. By the results of [1,16,17] and adding extra edges to a connected graph, then A_α-spectral radius will increase and the following lemma is straightforward.

Lemma 2. (i) *If G^* is any proper subgraph of connected graph G, and ρ is the A_α-spectral radius, then $\rho(G^*) < \rho(G)$.*

(ii) *If X is a positive vector and r is a positive number such that $A_\alpha(G)X < rX$, then $\rho(G) < r$.*

Recall that the vertex connectivity (respectively, edge connectivity) of a graph G is the smallest number of vertices (respectively, edges) such that if we remove them, the graph will be disconnected or be a single vertex. For convenience, let $\mathcal{F}_n$ be the set of all graphs of order n, and $\mathcal{F}_n^k$ (respectively, $\overline{\mathcal{F}}_n^k$) ($k \geq 0$) be the set of such graphs with order n and vertex (resp., edge) connectivity k. Note that $\mathcal{F}_n^0 = \overline{\mathcal{F}}_n^0$ having some disconnected graphs of order n, and $\mathcal{F}_n^{n-1} = \overline{\mathcal{F}}_n^{n-1}$ consisting of the unique graph K_n. Obviously, $\mathcal{F}_n = \cup_k \mathcal{F}_n^k = \cup_k \overline{\mathcal{F}}_n^k$.

Recall the graph $K(p,q)(p \geq q \geq 0)$ obtained from K_p by attaching a vertex together with edges connecting this vertex to q vertices of K_p. $K(p,q)$ is was found by Brualdi and Solehid in terms of stepwise adjacency matrix, but it is Peter Rowlinson who gives the purely combinatorial definition of such graph. For the property of $K(p,q)$, we refer to [18–20]. Clearly, $K(p,0)$ is K_p with an additional isolated vertex. It's not hard to see that $K(p,q)$ is of vertex (resp., edge) connectivity q. Let δ, Δ be the smallest and largest degrees of vertices in the graph G, respectively.

Lemma 3. *The graph K_n is the graph in $\mathcal{F}_n$ having the largest A_α-spectral radius, and $K_{n-1} \cup K_1 = K(n-1,0)$ is the graph in $\mathcal{F}_n^0$ or $\overline{\mathcal{F}}_n^0$ having the smallest A_α-spectral radius.*

Proof. By Lemma 2, the first statement is clear. For the second one, let G be a graph which attains the maximum A_α-spectral radius in $\mathcal{F}_n^0$, then G only has two unique connected components: K_{n-1}, K_1; if not, any component of G will be a proper subgraph of K_{n-1}. Then $\rho(G) < \rho(K_{n-1}) = \rho(K_{n-1} \cup K_1)$, a contradiction. Then this lemma is proved. □

Lemma 4. *For $k \in [1, n-2]$, $K(n-1,k)$ is the graph having the largest A_α-spectral radius in $\mathcal{F}_n^k$.*

Proof. Denote by G a graph having the largest A_α-spectral radius in $\mathcal{F}_n^k$. x is a unit (positive) Perron vector of A_α. Let U be the vertex cut of G having k vertices, and these components of $G - U$ be $G_1, G_2, \cdots, G_s$, for $s \geq 2$. We declare that $s = 2$; if not, adding all possible edges within the graph $G_1 \cup G_2 \cup \cdots \cup G_{s-1}$, we would get a graph belonging to $\mathcal{F}_n^k$ (because U is the smallest vertex cut set) and with a larger A_α-spectral radius. Similarly, induced subgraph $G[U]$, the subgraphs G_1 and G_2 are complete subgranph, and every vertex of U connects these vertices of G_1 and G_2. Next we prove that one of G_1, G_2 will be a singleton, which has a unique vertex. If not, suppose that G_1, G_2 have orders greater than one. Without loss of generality, denote by u a vertex of G_1 having a smallest value for x among vertices in $G_1 \cup G_2$. Deleting these edges of G_1 incident to u, and connecting all possible edges between $G_1 - u$ and G_2, we get a graph $\widetilde{G} = K(n-1,k)$ still in $\mathcal{F}_n^k$. By Lemma 1, $\rho(\widetilde{G}) > \rho(G)$, which yields a contradiction. So one of G_1, G_2 is a singleton, and G is the desired graph $K(n-1,k)$. □

Lemma 5. *For $k \in [1, n-2]$, $K(n-1,k)$ is the graph having maximum A_α-spectral radius in $\overline{\mathcal{F}}_n^k$.*

Proof. Denote by G a graph having the largest A_α-spectral radius in $\mathcal{F}_n^k$. x is a unit (positive) Perron vector of A_α. We know that each vertex of G has degree greater than or equal to k. Otherwise $G \notin \overline{\mathcal{F}}_n^k$. If there is a vertex u in G with degree k, then the edges adjacent to u are an edge cut such that $G - u$ is complete. The statement follows in this case. Then we will suppose that all vertices in G have degrees greater than k. Let E_c be an edge cut set of G having k edges. So $G - E_c$ consists of only two components G_1, G_2, respectively, of order n_1, n_2. Obviously G_1, G_2 are both complete. In addition, neither of G_1, G_2 is a singleton. Otherwise G would contain a vertex of degree k, which contradicted to the above assumption. So G_1, G_2 contain more than 1 vertex, i.e., $n_1 \geq 2$ and $n_2 \geq 2$.

Without loss of generality, suppose that G_1 contains a vertex w_1 having a minimal value given by x within all vertices of $G_1 \cup G_2$, and consists of vertices $w_1, w_2, \cdots, w_{n_1}$ such that $x(w_1) \leq x(w_2) \leq \cdots \leq x(w_{n_1})$. Assume that w_1 joins t vertices of G_2. Surely $t \leq min\{k, n_2\}$.

If $t = k$, there exist no edges joining $G_1 - w_1$ and G_2, and $n_2 \geq k+2$ otherwise G_2 contains a vertex of degree k. Denote by G' a new graph with vertex set $V(G)$ and edge set $E(G) \setminus E(w_1, N) \cup E(N, v')$, where $N = N(w_1) \cap V(G_1)$, and $v' \in V(G_2) - N(w_1) \cap V(G_2)$, by Lemma 1, we have $\rho(G') > \rho(G)$. Let G'' be another new graph with vertex set $V(G')$ and adding all possible edges between $G_1 - w_1$ and G_2. Note that $G'' = K(n-1,k)$, and G' is a proper subgraph of G''. By Lemma 2, we have $\rho(G'') > \rho(G')$. Thus, $\rho(G'') > \rho(G)$, a contradiction.

If $t < k$. Partition the set $V(G_1) - w_1$ as: $V_{11} = \{w_i : i = 2, 3, \cdots, n_1 - (k-t)\}$, $V_{12} = \{w_j : j = n_1 - (k-t) + 1, \cdots, n_1\}$. Thus, $|V_{11}| = n_1 - (k-t) - 1$; $|V_{12}| = k - t$.

Let $N = N(w_1) \cap V_{11}$, then $N \neq \emptyset$ since $d(w_1) > k$. Note there is vertex $v' \in V(G_2) - N(w_1) \cap V(G_2)$ since $n_2 \geq k+2$. Let G' be a new graph having vertex set $V(G)$ and edge set $E(G) \setminus E(w_1, N) \cup E(N, v')$, where $N = N(w_1) \cap V_{11}$, and $v' \in V(G_2) - N(w_1) \cap V(G_2)$, by Lemma 1, we have $\rho(G') > \rho(G)$. Let G'' be another new graph having vertex set $V(G')$ and adding all possible edges between $G_1 - w_1$ and G_2, adding all edges between w_1 and V_{12}. Note that $G'' = K(n-1,k)$, and G' is a proper subgraph of G''. Lemma 2 implies that $\rho(G'') > \rho(G')$. Thus, $\rho(G'') > \rho(G)$, a contradiction. The result follows. □

3. Main Results

In this section, we will determine maximizing A_α-spectral radius of of graphs with given connectivity. By Lemmas 4 and 5, we obtain the following Theorem:

Theorem 1. *The graph K_n is the graph in $\mathcal{F}_n$ with A_α-spectral radius , and $K_{n-1} \cup K_1 = K(n-1,0)$ is the unique one in $\mathcal{F}_n^0$ or $\overline{\mathcal{F}}_n^0$ with A_α-spectral radius. For $k \in [1, n-2]$, $K(n-1,k)$ is the graph with maximum A_α-spectral radius in $\mathcal{F}_n^k$ or $\overline{\mathcal{F}}_n^k$.*

Proof. By the Lemmas 3–5, we obtain the results. □

Lemma 6. *[20] Given a partition $\{1, 2, \cdots, n\} = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$ with $|\Delta_i| = n_i > 0$, A be any matrix partitioned into blocks A_{ij}, where A_{ij} is an $n_i \times n_j$ block. Suppose that the block A_{ij} has constant row sums b_{ij} , and let $B = (b_{ij})$. Then the spectrum of B is contained in the spectrum of A (taking into account the multiplicities of the eigenvalues).*

Since $K(n-1,k)$ contains K_{n-1}, we can partition $K(n-1,k)$ into three different subsets: $\{u\}, T, S$, in which u is the vertex connecting a complete subgraph K_{n-1} with k edges, a subset S is in K_{n-1} connecting u, and $T = V(K_{n-1} \setminus S)$. Let x be a Perron vector of $K(n-1,k)$. $S = \{u_1, u_2, \cdots, u_k\}$ and $T = \{v_1, v_2, \cdots, v_t\}$. Note that $k + t + 1 = n$.

Theorem 2. *Label the vertices of $K(n-1,k)$ as $u, u_1, u_2, \cdots, u_k, v_1, v_2 \cdots, v_t$ with $k, t \geq 0$. The maximum eigenvalues of $A_\alpha(K(n-1,k))$ satisfy the equation: $f(\rho) = (\rho - k\alpha)(\rho - k\alpha - n + k + 2)(\rho - n\alpha + 1) - k(1-\alpha)(\rho - k\alpha - \alpha + 1)(\rho - n\alpha + \alpha + 1) + k(1-\alpha)^3(n-k-1) = 0$.*

Proof. Since the matrix $A_\alpha = \alpha D + (1-\alpha)A$, where D has on the diagonal the vector $(k, n-1, n-2)$ and A consists of the following three row-vectors, in the order: $(0, k, 0)$; $(1, k-1, n-k-1)$; $(0, k, n-k-2)$. Thus, by the Lemma 6, x is a constant value β_2 on the vertex set S, and constant value β_3 on the vertex set T. Defining $x(u) =: \beta_1$, $\rho(K(n-1,k)) =: \rho$, also by (1), we get

$$(\rho - \alpha k)\beta_1 = k(1-\alpha)\beta_2$$

$$(\rho - \alpha(n-1))\beta_2 = (1-\alpha)(\beta_1 + (k-1)\beta_2 + t\beta_3), \text{ and}$$

$$(\rho - \alpha(n-2))\beta_3 = (1-\alpha)(k\beta_2 + (t-1)\beta_3).$$

Then we get

$$(\rho - \alpha(n-1)) = \frac{k(1-\alpha)^2}{\rho - k\alpha} + \frac{kt(1-\alpha)^2}{\rho - k\alpha - t + 1} + (k-1)(1-\alpha).$$

Note that for $n = t + k + 1$, that is, $n - 1 = k + t$. Then we have:

$$(\rho - k\alpha) = \frac{k(1-\alpha)^2}{\rho - k\alpha} + \frac{kt(1-\alpha)^2}{\rho - k\alpha - t + 1} + (k-1)(1-\alpha) + t\alpha.$$

Then we obtain that

$$(\rho - k\alpha)(\rho - k\alpha - n + k + 2)(\rho - n\alpha + 1) - k(1-\alpha)(\rho - k\alpha \\ -\alpha + 1)(\rho - n\alpha + \alpha + 1) + k(1-\alpha)^3(n-k-1) = 0.$$

Thus, our proof is finished. □

Corollary 1. *Let G be a graph of order n having vertex/edge connectivity k, where* $1 \leq k \leq n-2$, *the maximum adjacency spectral radius is the largest root of the* $f(\lambda) = \lambda^3 - (n-3)\lambda^2 - (n+k-2)\lambda + k(n-k-2) = 0$.

Proof. By Theorem 2, let $\alpha = 0$, then $f(\lambda) = \lambda^3 - (n-3)\lambda^2 - (n+k-2)\lambda + k(n-k-2) = 0$. It is obvious since $A_0 = A(G)$. □

By letting the special values for α, we have the following corollary.

Corollary 2. *Let G be a graph of order n having vertex/edge connectivity k, where* $1 \leq k \leq n-2$, *the signless Laplacian spectral radius* $\lambda_1 = \frac{2n+k-4+\sqrt{(2n-k-4)^2+8k}}{2}$.

Proof. By Theorem 2, let $\alpha = \frac{1}{2}$, then $f(\lambda) = \lambda^3 - \frac{1}{2}(3n+k-6)\lambda^2 + (\frac{1}{4}(n-4)(2n+3k) + k + 2)\lambda - \frac{1}{4}k(n^2 - 5n + 6) = 0$. It is obvious since $2A_{\frac{1}{2}} = D + Q$. Thus,

$$\begin{aligned} 8f(\lambda) &= 8[\lambda^3 - \frac{1}{2}(3n+k-6)\lambda^2 + (\frac{1}{4}(n-4)(2n+3k) + k + 2)\lambda \\ &\quad -\frac{1}{4}k(n^2 - 5n + 6)] \\ &= (2\lambda)^3 - (3n+k-6)(2\lambda)^2 + ((n-4)(2n+3k) + 4k + 8)(2\lambda) \\ &\quad -2k(n^2 - 5n + 6) \\ &= (\lambda_1)^3 - (3n+k-6)(\lambda_1)^2 + ((n-4)(2n+3k) + 4k + 8)(\lambda_1) \\ &\quad -2k(n^2 - 5n + 6). \end{aligned}$$

Let $\lambda_1 = 2\lambda$ and

$$\begin{aligned} F(\lambda_1) &= (\lambda_1)^3 - (3n+k-6)(\lambda_1)^2 + ((n-4)(2n+3k) + 4k + 8)(\lambda_1) \\ &\quad -2k(n^2 - 5n + 6) = 0. \end{aligned}$$

Then we get:

$$\lambda_1 = \frac{2n+k-4+\sqrt{(2n-k-4)^2+8k}}{2}.$$

□

The above result is the same as [6].

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