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Interpolative Ćirić-Reich-Rus Type Contractions via the Branciari Distance

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Abstract: In this paper, we initiate the concept of interpolative Ćirić-Reich-Rus type contractions via the Branciari distance and prove some related fixed points results for such mappings. Moreover, an example is provided to show the useability of our obtained results.

Keywords: branciari distance; interpolative Ćirić-Reich-Rus type contraction; fixed point

MSC: 46T99; 47H10; 54H25

1. Introduction and Preliminaries


Theorem 1. Let $(X, \rho)$ be a complete metric space and $T$ be a self-mapping on $X$. Suppose there exists $\lambda \in [0, \frac{1}{2})$ such that

$$\rho (T \xi, T \eta) \leq \lambda [\rho (\xi, T \xi) + \rho (\eta, T \eta)]$$

for each $\xi, \eta \in X$. Then $T$ has a unique fixed point.

More information concerning Kannan fixed point theorem can be found in the early paper by Reich [4]. Denote by $\text{Fix}(T)$ the set of fixed points of a self-mapping $T$ on a non-empty set $X$. In 2018, Karapınar [5] considered Theorem 1 concerning interpolation theory. The main result in [5] via an interpolative Kannan type contraction is

Theorem 2 ([5]). Let $(X, \rho)$ be a complete metric space. Suppose that the self-mapping $T : X \rightarrow X$ is such that

$$\rho (T \xi, T \eta) \leq \lambda [\rho (\xi, T \xi)]^a \cdot [\rho (\eta, T \eta)]^{1-a},$$

where $\lambda \in (0, 1)$ and $a \in (0, 1)$, for all $\xi, \eta \in X \setminus \text{Fix}(T)$ with $\text{Fix}(T) = \{u \in X : Tu = u\}$. Then $T$ possesses a unique fixed point in $X$.

If $T : X \rightarrow X$ satisfies (1), $T$ is said to be an interpolative Kannan type contraction. Very recently, the authors in [6] pointed out a gap in [5], that is, the fixed point in Theorem 2 may be not unique. For more other details, see ([7,8]). On the other hand, the known fixed point of Reich [9] is stated as follows.
Theorem 3. Let \((X, \rho)\) be a complete metric space. If \(T : X \rightarrow X\) is such that
\[
\rho (T\xi, T\eta) \leq \lambda [\rho(\xi, \eta) + \rho(\xi, T\xi) + \rho(\eta, T\eta)],
\]
for all \(\xi, \eta \in X\), where \(\lambda \in \left[0, \frac{1}{3}\right]\), then \(T\) possesses a unique fixed point.

Note that this result was proved independently also by Ćirić and Rus. For this reason, whenever we mention Reich type contractions, we shall say “Ćirić-Reich-Rus type contractions.”

On the other hand, the concept of a Branciari distance space has been introduced by Branciari [10] where the triangular inequality is replaced by a quadrilateral one. For some known fixed point results in this setting, we may refer to [11–21]. In the sequel, \(\mathbb{N}\) will represent the set of all positive integer numbers. First, we recall some basic concepts and notations on Branciari distance (rectangular metric) spaces.

Definition 1. Let \(X\) be a non-empty set. Let \(d : X \times X \rightarrow [0, \infty)\) be a function such that for all \(\xi, \eta \in X\) and all distinct points \(u, v \in X\), each distinct from \(\xi\) and \(\eta\):
\[
\begin{align*}
(d1) & \quad d(\xi, \eta) = 0 \text{ if and only if } \xi = \eta \text{ (identification);} \\
(d2) & \quad d(\xi, \eta) = d(\eta, \xi) \quad \text{(symmetry);} \\
(d3) & \quad d(\xi, \eta) \leq d(\xi, u) + d(u, v) + d(v, \eta) \quad \text{(quadrilateral inequality).}
\end{align*}
\]
Then \(d\) is called a Branciari distance and the pair \((X, d)\) is called a Branciari distance space.

Notice that in some sources, Branciari distance is called as “a rectangular metric” or “a generalized metric”. On the other hand, it was reported in [22] that the topology of standard metric and Branciari distance are not comparable.

Definition 2. Let \((X, d)\) be a Branciari distance space and \(\{\xi_n\}\) be a sequence in \(X\).
\(\text{(i)}\) A sequence \(\{\xi_n\}\) is convergent to point \(x \in X\) if \(\lim_{n \to \infty} d(\xi_n, x) = 0\).
\(\text{(ii)}\) A sequence \(\{\xi_n\}\) is said to be Cauchy if for every \(\varepsilon > 0\), there exists a positive integer \(N = N(\varepsilon)\) such that \(d(\xi_n, \xi_m) < \varepsilon\) for all \(n, m > N\).
\(\text{(iii)}\) We say that \((X, d)\) is complete if each Cauchy sequence in \(X\) is convergent.

Lemma 1. Let \((X, d)\) be a Branciari distance space. We say that a mapping \(T : X \rightarrow X\) is continuous at \(u \in X\), if we have \(T\xi_n \rightarrow Tu\) (in other words, \(\lim_{n \to \infty} d(T\xi_n, Tu) = 0\)) for any sequence \(\{\xi_n\}\) in \(X\) converges to \(u \in X\), that is, \(\xi_n \rightarrow u\).

The following proposition is useful in the sequel.

Proposition 1 ([23]). Suppose that \(\{\xi_n\}\) is a Cauchy sequence in a Branciari distance space such that
\[
\lim_{n \to \infty} d(\xi_n, u) = \lim_{n \to \infty} d(\xi_n, z) = 0,
\]
where \(u, z \in X\). Then \(u = z\).

In this paper, using the Branciari distance, we initiate the notion of interpolative Ćirić-Reich-Rus type contractions. We also present an example illustrating our approach.

2. Main Results
We start this section by introducing the notion of interpolative Ćirić-Reich-Rus type contractions.
We derive Theorem 4.

Let \( T \) be a Branciari distance space. A self-mapping \( T \) on \( X \) is called an interpolative Ćirić-Reich-Rus type contraction if there are \( \lambda \in [0, 1) \) and positive reals \( \alpha, \beta \) with \( \alpha + \beta < 1 \) such that

\[
d(T\xi, T\eta) \leq \lambda [d(\xi, \eta)]^\beta \cdot [d(T\xi, T\eta)]^\alpha \cdot [d(\eta, T\eta)]^{1-\alpha-\beta},
\]

for all \( \xi, \eta \in X \setminus \text{Fix}(T) \).

**Theorem 4.** Let \( T : X \to X \) be an interpolative Ćirić-Reich-Rus type contraction on a complete Branciari distance space \( (X, p) \), then \( T \) has a fixed point in \( X \).

**Proof.** We take an arbitrary point \( \xi_0 \in (X, p) \). Consider \( \{\xi_n\} \) by \( \xi_n = T^n(\xi_0) \) for each positive integer \( n \). If there exists \( n_0 \) such that \( \xi_{n_0} = T^{n_0}(\xi_0) \), then \( \xi_{n_0} \) is a fixed point of \( T \). It completes the proof. Throughout the proof, we assume that \( \xi_n \neq \xi_{n+1} \) for each \( n \geq 0 \).

**Step 1:** We shall prove that

\[
\lim_{n \to \infty} d(\xi_n, \xi_{n+1}) = 0.
\]  

By substituting the values \( \xi = \xi_n \) and \( \eta = \xi_{n-1} \) in (3), we find that

\[
d(\xi_{n+1}, \xi_n) = d(T\xi_n, T\xi_{n-1}) \leq \lambda [d(\xi_n, \xi_{n-1})]^\beta \cdot [d(T\xi_n, T\xi_{n-1})]^\alpha \cdot [d(\xi_{n-1}, T\xi_{n-1})]^{1-\alpha-\beta}
\]

\[
= \lambda [d(\xi_n, \xi_{n-1})]^\beta \cdot [d(\xi_{n-1}, \xi_{n+1})]^\alpha \cdot [d(\xi_{n-1}, T\xi_{n-1})]^{1-\alpha-\beta}
\]

\[
= \lambda [d(\xi_{n-1}, \xi_n)]^{1-\alpha} \cdot [d(\xi_n, \xi_{n+1})]^\alpha.
\]

We derive

\[
[d(\xi_n, \xi_{n+1})]^{1-\alpha} \leq \lambda [d(\xi_{n-1}, \xi_n)]^{1-\alpha}.
\]  

So, we conclude that

\[
d(\xi_n, \xi_{n+1}) \leq d(\xi_{n-1}, \xi_n), \quad \text{for all } n \geq 1.
\]  

That is, \( \{d(\xi_{n-1}, \xi_n)\} \) is a non-increasing sequence with non-negative terms. Eventually, there is a nonnegative constant \( \ell \) such that \( \lim_{n \to \infty} d(\xi_{n-1}, \xi_n) = \ell \). Note that \( \ell \geq 0 \). Indeed, from (6), we deduce that

\[
d(\xi_n, \xi_{n+1}) \leq \lambda^d(\xi_{n-1}, \xi_n) \leq \lambda^n d(\xi_0, \xi_1).
\]

Regarding \( \lambda < 1 \), and by taking \( n \to \infty \) in the inequality (8), we deduce that \( \ell = 0 \).

**Step 2:** We shall also show that

\[
\lim_{n \to \infty} d(\xi_n, \xi_{n+2}) = 0.
\]

Using (3), (7) and the quadrilateral inequality, we have
\[ d(\xi_{n+2}, \xi_n) = d(T\xi_{n+1}, T\xi_n) \leq \lambda \left[ d(\xi_{n+1}, \xi_n) \right]^\beta \left[ d(T\xi_{n+1}, T\xi_n) \right]^a \cdot \left[ d(\xi_{n-1}, T\xi_{n-1}) \right]^{1-a-\beta} \]
\[ = \lambda \left[ d(\xi_{n+1}, \xi_n) \right]^\beta \cdot \left[ d(\xi_{n+1}, \xi_n) \right]^a \cdot \left[ d(\xi_{n-1}, \xi_n) \right]^{1-a-\beta} \]
\[ \leq \lambda \left[ d(\xi_{n+1}, \xi_n) \right]^\beta \cdot \left[ d(\xi_{n-1}, \xi_n) \right]^{1-a-\beta} \]
\[ \leq \lambda \left[ d(\xi_{n+1}, \xi_n) \right]^\beta \cdot \left[ d(\xi_n, \xi_{n+1}) \right]^{1-a-\beta} \]
\[ \leq \lambda \left[ d(\xi_{n+2}, \xi_n) + d(\xi_{n+1}, \xi_n) \right]^\beta \cdot \left[ d(\xi_{n-1}, \xi_n) \right]^{1-a-\beta} \]
\[ \leq \lambda \left[ d(\xi_{n+2}, \xi_n) + 2d(\xi_n, \xi_{n-1}) \right]^\beta \cdot \left[ d(\xi_{n-1}, \xi_n) \right]^{1-a-\beta} \]
\[ \leq \lambda \left[ d(\xi_{n+2}, \xi_n) + 2d(\xi_n, \xi_{n-1}) \right] \cdot \left[ d(\xi_{n-1}, \xi_n) \right]^{1-a-\beta} \]

We deduce that
\[ (1 - \lambda)d(\xi_{n+2}, \xi_n) \leq 2\lambda d(\xi_{n-1}, \xi_n), \quad \text{for all } n \geq 1. \]

Therefore,
\[ d(\xi_{n+2}, \xi_n) \leq \frac{2\lambda}{1-\lambda} d(\xi_{n-1}, \xi_n), \quad \text{for all } n \geq 1. \]  \hfill (10)

Letting \( n \to \infty \) in (10) and using (4), we get (9), which completes the proof of step 2.

**Step 3:** We shall prove that \( \xi_n \neq \xi_m \) for all \( n \neq m \).

Suppose that \( \xi_n = \xi_m \) for some \( n > m \), so we have \( \xi_{n+1} = T\xi_n = T\xi_m = \xi_{m+1} \).

By continuing in this direction, we obtain \( \xi_{n+k} = \xi_{m+k} \) for all \( k \in \mathbb{N} \). By (5) and (7), we have
\[ 0 < d(\xi_m, \xi_{m+1}) = d(T\xi_n, T\xi_n) \]
\[ \leq \lambda \left[ d(\xi_{n-1}, \xi_n) \right]^{1-a} \cdot \left[ d(\xi_n, \xi_{n+1}) \right]^a \]
\[ \leq \lambda \left[ d(\xi_{n-1}, \xi_n) \right] \]
\[ < d(\xi_{n-1}, \xi_n) < d(\xi_{n-2}, \xi_{n-1}) \]
\[ < \cdots < d(\xi_m, \xi_{m+1}), \]

which is a contradiction. Thus, in that follows, we can assume that \( \xi_n \neq \xi_m \) for all \( n \neq m \).

**Step 4:** We shall prove that \( \{\xi_n\} \) is a Cauchy sequence, that is, \( \lim_{n \to \infty} d(\xi_n, \xi_{n+p}) = 0 \) for all \( p \in \mathbb{N} \).

The cases \( p = 1 \) and \( p = 2 \) are proved in step 1 and step 2, respectively. Now, take \( p \geq 3 \) arbitrary.

We distinguish two cases:

Case (1). Let \( p = 2m \) where \( m \geq 2 \). By quadrilateral inequality, using (8), we find
We conclude that \( \mathbf{Mathematics} \)

Finally, we get

Case (2): Let \( \xi_n \), \( \xi_{n+2} \)

\[
d(\xi_n, \xi_{n+2m}) \leq d(\xi_n, \xi_{n+2}) + d(\xi_{n+2}, \xi_{n+3}) + d(\xi_{n+3}, \xi_{n+2m})
\]

\[
\leq d(\xi_n, \xi_{n+2}) + d(\xi_{n+2}, \xi_{n+3}) + d(\xi_{n+3}, \xi_{n+4}) + d(\xi_{n+4}, \xi_{n+5}) + d(\xi_{n+5}, \xi_{n+2m})
\]

\[
\leq d(\xi_n, \xi_{n+2}) + d(\xi_{n+2}, \xi_{n+3}) + d(\xi_{n+3}, \xi_{n+4}) + d(\xi_{n+4}, \xi_{n+5}) + \cdots + d(\xi_{n+2m-1}, \xi_{n+2m})
\]

\[
= d(\xi_n, \xi_{n+2}) + \sum_{k=n+2}^{n+2m-1} d(\xi_k, \xi_{k+1})
\]

\[
\leq d(\xi_n, \xi_{n+2}) + \sum_{k=n+2}^{n+2m-1} \lambda^k d(\xi_0, \xi_1)
\]

\[
\leq d(\xi_n, \xi_{n+2}) + d(\xi_0, \xi_1) \sum_{k=n+2}^{\infty} \lambda^k
\]

\[
= d(\xi_n, \xi_{n+2}) + \frac{\lambda^{n+2}}{1-\lambda} d(\xi_0, \xi_1).
\]

Obviously,

\[
\lim_{n \to \infty} d(\xi_n, \xi_{n+2m}) = 0.
\]

Case (2): Let \( p = 2m + 1 \) where \( m \geq 1 \). By quadrilateral inequality, using (8), we find

\[
d(\xi_n, \xi_{n+2m+1}) \leq d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+2}) + d(\xi_{n+2}, \xi_{n+2m+1})
\]

\[
\leq d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+2}) + d(\xi_{n+2}, \xi_{n+3}) + d(\xi_{n+3}, \xi_{n+4}) + d(\xi_{n+4}, \xi_{n+2m+1})
\]

\[
\leq d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+2}) + d(\xi_{n+2}, \xi_{n+3}) + \cdots + d(\xi_{n+2m}, \xi_{n+2m+1})
\]

\[
= \sum_{k=n}^{n+2m} d(\xi_k, \xi_{k+1}) \leq \sum_{k=n}^{n+2m} \lambda^k d(\xi_0, \xi_1)
\]

\[
\leq d(\xi_0, \xi_1) \sum_{k=n}^{\infty} \lambda^k
\]

\[
= \frac{\lambda^n}{1-\lambda} d(\xi_0, \xi_1) \to 0 \text{ as } n \to \infty.
\]

Finally, we get

\[
\lim_{n \to \infty} d(\xi_n, \xi_{n+p}) = 0 \quad \text{uniformly in } p.
\]

We conclude that \( \{\xi_n\} \) is a Cauchy sequence in \((X,d)\). Since \((X,d)\) is complete, there exists \( \xi \in X \) such that

\[
\lim_{n \to \infty} d(\xi_n, \xi) = 0. \quad (11)
\]
We shall show that $\xi$ is a fixed point of $T$. We argue by contradiction by assuming that $\xi \neq T\xi$. Recall that $\xi_n \neq T\xi_n$ for each $n \geq 0$. By letting $\xi = \xi_n$ and $\eta = \xi$ in (3), we determine that

$$d(\xi_{n+1}, T\xi) = d(T\xi_n, T\xi) \leq \lambda [d(\xi_n, \xi)]^\beta \cdot [d(\xi_n, T\xi_n)]^\alpha \cdot [d(\xi, T\xi)]^{1-\alpha-\beta}. \quad (12)$$

Letting $n \to \infty$ in the inequality (12), we find $\lim_{n \to \infty} d(\xi_n, T\xi) = 0$. By Proposition 1, we conclude that $T\xi = \xi$, which contradicts our last assumption. Thus $\xi = T\xi$, and so $\xi$ is a fixed point of $T$. \qed

The following example illustrates Theorem 4.

**Example 1.** Let $X = \{0, 1, 2, 3\}$ be a set endowed with the Branciari distance $\rho$ given as

<table>
<thead>
<tr>
<th>$\rho(\xi, \eta)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.1</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
<td>1</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.9</td>
<td>0.7</td>
<td>0.2</td>
<td>0</td>
</tr>
</tbody>
</table>

Consider the self-mapping $T$ on $X$ as $T: \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$. We have $\rho(1, 2) > \rho(1, 3) + \rho(3, 2)$, so $\rho$ is not a metric. Let $\xi, \eta \in X \setminus \text{Fix}(T)$. Then $(\xi, \eta) \in \{(1, 1), (2, 2), (1, 2), (2, 1)\}$. By choosing $\lambda \in [0.4, 1)$, $\alpha = 0.6$ and $\beta = 0.3$, it is obvious that the self-mapping $T$ is an interpolative Cirić-Reich-Rus type contraction. Here, $T$ has two fixed points, which are 0 and 3.

On the other hand, the inequality (2) does not hold for $x = 0$ and $y = 3$ (by taking the classical metric $d(x, y) = |x - y|$). That is, Theorem 3 is not applicable.

In what follows, we introduce the concept of interpolative Kannan type contractions.

**Definition 4.** Let $(X, d)$ be a Branciari distance space. A self-mapping $T$ on $X$ is called an interpolative Kannan type contraction, if there are constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(T\xi, T\eta) \leq \lambda [d(\xi, T\xi)]^\alpha \cdot [d(\eta, T\eta)]^{1-\alpha}, \quad (13)$$

for all $\xi, \eta \in X \setminus \text{Fix}(T)$.

**Theorem 5.** Let $T: X \to X$ be an interpolative Kannan type contraction on a complete Branciari distance space $(X, p)$, then $T$ has a fixed point in $X$.

We skip the proof since it is similar to the proof of Theorem 4.

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