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Some Identities for the Two Variable Fubini Polynomials

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Abstract: In this paper, we perform a further investigation for the Fubini polynomials. By making use of the generating function methods and Padé approximation techniques, we establish some new identities for the two variable Fubini polynomials. Some special cases as well as immediate consequences of the main results presented here are also considered.

Keywords: Fubini polynomials; generating functions; Padé approximation; combinatorial identities

MSC: 11B83; 05A19

1. Introduction

The two variable Fubini polynomials are usually defined by means of the generating function (see, e.g., [1,2])

$$\frac{e^{xt}}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x, y) \frac{t^n}{n!}. \quad (1)$$

In particular, the case $x = 0$ in (1) gives the Fubini polynomials $F_n(y) = F_n(0, y)$, and the case $x = 0$ and $y = 1$ in (1) is called the ordered Bell numbers. Moreover, the case $y = -1/2$ in (1) gives the Euler polynomials $E_n(x) = F_n(x, -1/2)$.

Recently, some authors have been very interested in arithmetic properties for the Fubini polynomials. For example, Kim et al. [2] showed that the Fubini polynomials can be expressed by

$$F_n(y) = \sum_{k=0}^n S(n, k) k! y^k \quad (n \geq 0), \quad (2)$$

where $S(n, k)$ are the Stirling numbers of the second kind. After that, Kim et al. [3] introduced the ω -torsion Fubini polynomials defined by the generating function

$$\frac{e^{xt}}{1 - y^\omega(e^t - 1)^\omega} = \sum_{n=0}^{\infty} F_{n,\omega}(x, y) \frac{t^n}{n!}, \quad (3)$$

and used the fermionic p -adic integral on \mathbb{Z}_p described in [4] to give some similar symmetric identities for the ω -torsion Fubini polynomials to the ones stated in [5–7]. In particular, they showed that for non-negative integer n and positive integers ω_1, ω_2 such that $\omega_1 \equiv 1 \pmod{2}$ and $\omega_2 \equiv 1 \pmod{2}$,

$$\begin{aligned} & \sum_{i=0}^{\omega_1-1} \sum_{j=0}^{\omega_2 i} \binom{\omega_2 i}{j} y^{\omega_2 i} (-1)^j F_{n, \omega_1}(\omega_2 i - j, y) \\ &= \sum_{i=0}^{\omega_2-1} \sum_{j=0}^{\omega_1 i} \binom{\omega_1 i}{j} y^{\omega_1 i} (-1)^j F_{n, \omega_2}(\omega_1 i - j, y), \end{aligned} \tag{4}$$

by virtue of which they obtained another expression of the Fubini polynomials

$$F_n(y) = \sum_{k=0}^{\omega-1} \sum_{i=0}^k \binom{k}{i} (-1)^i F_{n, \omega}(k - i, y) y^k, \tag{5}$$

where n, ω are non-negative integers with $\omega \geq 1$. On the other hand, Zhao and Chen [8] explored the computational problem of the sums of products of the Fubini polynomials, and determined some explicit formulas involving a sequence of numbers. Zhang and Lin [9] proved Zhao and Chen’s [8] conjecture on the sequence of numbers, by virtue of which they established some congruences for the Fubini polynomials. Chen and Chen [10] further studied the computational problem of the sums of the products of the two variable Fubini polynomials, and discovered some similar formulas involving a new type second order non-linear recursive polynomials to the ones of Zhao and Chen [8]. For the latest developments in two variable higher-order Fubini polynomials, we refer the interested reader to [11–13].

Motivated by the above authors, we perform a further investigation for the Fubini polynomials in this paper. By making use of the generating function methods and Padé approximation techniques, we establish some new identities for the two variable Fubini polynomials. Some special cases as well as immediate consequences of the main results presented here are also considered.

2. Padé Approximants

It is clear that Padé approximants provide rational approximations to functions defined by a power series expansion, and have played important roles in many fields of mathematics, physics and engineering. We here recall the definition of Padé approximations to general series and their expression in the case of the exponential function.

Let m, n be non-negative integers and let \mathcal{P}_k be the set of all polynomials of degree $\leq k$. Considering a function f with a Taylor expansion

$$f(t) = \sum_{k=0}^{\infty} c_k t^k \tag{6}$$

in a neighborhood of the origin, a Padé form of type (m, n) is a pair (P, Q) such that

$$P = \sum_{k=0}^m p_k t^k \in \mathcal{P}_m, \quad Q = \sum_{k=0}^n q_k t^k \in \mathcal{P}_n \quad (Q \neq 0), \tag{7}$$

and

$$Qf - P = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \rightarrow 0. \tag{8}$$

It is well known that every Padé form of type (m, n) for $f(t)$ always exists and satisfies the same rational function, and the uniquely determined rational function P/Q is called the Padé approximant of type (m, n) for $f(t)$. For nonnegative integers m, n , the Padé approximant of type (m, n) for the exponential function e^t is the unique rational function (see, e.g., [14–16])

$$R_{m,n}(t) = \frac{P_m(t)}{Q_n(t)} \quad (P_m \in \mathcal{P}_m, Q_n \in \mathcal{P}_n, Q_n(0) = 1), \tag{9}$$

with the property

$$e^t - R_{m,n}(t) = \mathcal{O}(t^{m+n+1}) \quad \text{as } t \rightarrow 0. \tag{10}$$

In fact, the explicit formulas for P_m and Q_n can be expressed as follows (see, e.g., [17,18]):

$$P_m(t) = \sum_{k=0}^m \frac{(m+n-k)! \cdot m!}{(m+n)! \cdot (m-k)!} \cdot \frac{t^k}{k!}, \tag{11}$$

$$Q_n(t) = \sum_{k=0}^n \frac{(m+n-k)! \cdot n!}{(m+n)! \cdot (n-k)!} \cdot \frac{(-t)^k}{k!}, \tag{12}$$

and

$$Q_n(t)e^t - P_m(t) = (-1)^n \frac{t^{m+n+1}}{(m+n)!} \int_0^1 x^n (1-x)^m e^{xt} dx, \tag{13}$$

where $P_m(t)$ and $Q_n(t)$ is called the Padé numerator and denominator of type (m, n) for e^t , respectively.

We shall make use of the above properties of Padé approximants to establish some new identities for the two variable Fubini polynomials in next section.

3. The Statement of Results

It is easily seen from (1) that

$$-ye^t \sum_{j=0}^{\infty} F_j(x, y) \frac{t^j}{j!} + (y+1) \sum_{j=0}^{\infty} F_j(x, y) \frac{t^j}{j!} = e^{xt}. \tag{14}$$

If we denote the right hand side of (13) by $S_{m,n}(t)$ then we have

$$e^t = \frac{P_m(t) + S_{m,n}(t)}{Q_n(t)}. \tag{15}$$

By applying (15) to the left hand side of (14), we discover

$$\begin{aligned} & -yP_m(t) \sum_{j=0}^{\infty} F_j(x, y) \frac{t^j}{j!} - yS_{m,n}(t) \sum_{j=0}^{\infty} F_j(x, y) \frac{t^j}{j!} \\ & + (y+1)Q_n(t) \sum_{j=0}^{\infty} F_j(x, y) \frac{t^j}{j!} = Q_n(t)e^{xt}. \end{aligned} \tag{16}$$

We now apply the exponential series $e^{xt} = \sum_{k=0}^{\infty} x^k t^k / k!$ to the right hand side of (16). With the help of the Beta function, we get

$$\begin{aligned} S_{m,n}(t) &= (-1)^n \frac{t^{m+n+1}}{(m+n)!} \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 x^{n+k} (1-x)^m dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^n m! \cdot (n+k)!}{(m+n)! \cdot (m+n+k+1)!} \cdot \frac{t^{m+n+k+1}}{k!}. \end{aligned} \tag{17}$$

Let $p_{m,n;k}, q_{m,n;k}, s_{m,n;k}$ be the coefficients of the polynomials $P_m(t), Q_n(t), S_{m,n}(t)$ given by

$$P_m(t) = \sum_{k=0}^m p_{m,n;k} t^k, \quad Q_n(t) = \sum_{k=0}^n q_{m,n;k} t^k, \quad S_{m,n}(t) = \sum_{k=0}^{\infty} s_{m,n;k} t^{m+n+k+1}. \tag{18}$$

Obviously, we know from (11), (12) and (17) that

$$p_{m,n;k} = \frac{m! \cdot (m+n-k)!}{k! \cdot (m+n)! \cdot (m-k)!}, \quad q_{m,n;k} = \frac{(-1)^k n! \cdot (m+n-k)!}{k! \cdot (m+n)! \cdot (n-k)!} \tag{19}$$

and

$$s_{m,n;k} = \frac{(-1)^n m! \cdot (n+k)!}{k! \cdot (m+n)! \cdot (m+n+k+1)!} \tag{20}$$

It follows from (16) and (18) that

$$\begin{aligned} & -y \sum_{k=0}^m p_{m,n;k} t^k \sum_{j=0}^{\infty} F_j(x,y) \frac{t^j}{j!} - y \sum_{k=0}^{\infty} s_{m,n;k} t^{m+n+k+1} \sum_{j=0}^{\infty} F_j(x,y) \frac{t^j}{j!} \\ & + (y+1) \sum_{k=0}^n q_{m,n;k} t^k \sum_{j=0}^{\infty} F_j(x,y) \frac{t^j}{j!} = \sum_{k=0}^n q_{m,n;k} t^k \sum_{j=0}^{\infty} \frac{x^j t^j}{j!}, \end{aligned}$$

which together with the Cauchy product yields

$$\begin{aligned} & -y \sum_{l=0}^{\infty} \left(\sum_{\substack{k+j=l \\ k,j \geq 0}} p_{m,n;k} \frac{F_j(x,y)}{j!} \right) t^l - y \sum_{l=0}^{\infty} \left(\sum_{\substack{k+j=l-m-n-1 \\ k,j \geq 0}} s_{m,n;k} \frac{F_j(x,y)}{j!} \right) t^l \\ & + (y+1) \sum_{l=0}^{\infty} \left(\sum_{\substack{k+j=l \\ k,j \geq 0}} q_{m,n;k} \frac{F_j(x,y)}{j!} \right) t^l = \sum_{l=0}^{\infty} \left(\sum_{\substack{k+j=l \\ k,j \geq 0}} q_{m,n;k} \frac{x^j}{j!} \right) t^l. \end{aligned} \tag{21}$$

If we compare the coefficients of t^l in (21), we get that for non-negative integer l with $0 \leq l \leq m+n$,

$$-y \sum_{\substack{k+j=l \\ k,j \geq 0}} p_{m,n;k} \frac{F_j(x,y)}{j!} + (y+1) \sum_{\substack{k+j=l \\ k,j \geq 0}} q_{m,n;k} \frac{F_j(x,y)}{j!} = \sum_{\substack{k+j=l \\ k,j \geq 0}} q_{m,n;k} \frac{x^j}{j!} \tag{22}$$

Thus, by applying (19) to (22), we have

$$\begin{aligned} & -y \sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{m}{k} (m+n-k)! \frac{F_j(x,y)}{j!} + (y+1) \sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{n}{k} (m+n-k)! (-1)^k \frac{F_j(x,y)}{j!} \\ & = \sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{n}{k} (m+n-k)! (-1)^k \frac{x^j}{j!}. \end{aligned} \tag{23}$$

It follows from (23) that we state the following result.

Theorem 1. Let m, n, l be non-negative integers with $0 \leq l \leq m+n$. Then

$$\begin{aligned} & -y \sum_{k=0}^l \binom{m}{k} (m+n-k)! \frac{F_{l-k}(x,y)}{(l-k)!} + (y+1) \sum_{k=0}^l \binom{n}{k} (m+n-k)! (-1)^k \frac{F_{l-k}(x,y)}{(l-k)!} \\ & = \sum_{k=0}^l \binom{n}{k} (m+n-k)! (-1)^k \frac{x^{l-k}}{(l-k)!}. \end{aligned} \tag{24}$$

We now discuss some special cases of Theorem 1. By taking $l = m + n$ in (24), we get that for non-negative integers m, n ,

$$\begin{aligned}
 & -y \sum_{k=0}^m \binom{m}{k} F_{m+n-k}(x, y) + (y + 1) \sum_{k=0}^n \binom{n}{k} (-1)^k F_{m+n-k}(x, y) \\
 & = \sum_{k=0}^n \binom{n}{k} (-1)^k x^{m+n-k}.
 \end{aligned}
 \tag{25}$$

In particular, if we take $n = 0$ in (25) then

$$-y \sum_{k=0}^m \binom{m}{k} F_{m-k}(x, y) + (y + 1) F_m(x, y) = x^m \quad (m \geq 0),
 \tag{26}$$

and if we take $m = 0$ in (25), in light of the binomial theorem, we have

$$-y F_n(x, y) + (y + 1) \sum_{k=0}^n \binom{n}{k} (-1)^k F_{n-k}(x, y) = (x - 1)^n \quad (n \geq 0).
 \tag{27}$$

We next consider the case $l \geq m + n + 1$ in (21). It is easily seen from (21) that for non-negative integers m, n and positive integer l with $l \geq m + n + 1$,

$$\begin{aligned}
 & -y \sum_{\substack{k+j=l \\ k,j \geq 0}} p_{m,n;k} \frac{F_j(x, y)}{j!} - y \sum_{\substack{k+j=l-m-n-1 \\ k,j \geq 0}} s_{m,n;k} \frac{F_j(x, y)}{j!} \\
 & + (y + 1) \sum_{\substack{k+j=l \\ k,j \geq 0}} q_{m,n;k} \frac{F_j(x, y)}{j!} = \sum_{\substack{k+j=l \\ k,j \geq 0}} q_{m,n;k} \frac{x^j}{j!}.
 \end{aligned}
 \tag{28}$$

By applying (19) and (20) to (28), we get

$$\begin{aligned}
 & -y \sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{m}{k} (m + n - k)! \frac{F_j(x, y)}{j!} \\
 & - (-1)^n y \sum_{\substack{k+j=l-m-n-1 \\ k,j \geq 0}} \frac{m! \cdot (n + k)!}{k! \cdot (m + n + k + 1)!} \cdot \frac{F_j(x, y)}{j!} \\
 & + (y + 1) \sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{n}{k} (m + n - k)! (-1)^k \frac{F_j(x, y)}{j!} \\
 & = \sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{n}{k} (m + n - k)! (-1)^k \frac{x^j}{j!}.
 \end{aligned}
 \tag{29}$$

Thus, by taking $l = m + n + r$ in (29), we discover the following result.

Theorem 2. Let m, n be non-negative integers. Then, for positive integer r ,

$$\begin{aligned}
 & -y \sum_{k=0}^m \binom{m}{k} (m+n-k)! \frac{F_{m+n+r-k}(x, y)}{(m+n+r-k)!} \\
 & + (y+1) \sum_{k=0}^n \binom{n}{k} (m+n-k)! (-1)^k \frac{F_{m+n+r-k}(x, y)}{(m+n+r-k)!} \\
 & = \frac{(-1)^n}{(r-1)!} y \sum_{k=0}^{r-1} \frac{\binom{r-1}{k}}{\binom{m+n+k}{m}} \cdot \frac{F_{r-1-k}(x, y)}{m+n+k+1} \\
 & + \sum_{k=0}^n \binom{n}{k} (m+n-k)! (-1)^k \frac{x^{m+n+r-k}}{(m+n+r-k)!}.
 \end{aligned} \tag{30}$$

It becomes obvious that the case $r = 1$ in Theorem 2 gives that for non-negative integers m, n ,

$$\begin{aligned}
 & -y \sum_{k=0}^m \binom{m}{k} \frac{F_{m+n+1-k}(x, y)}{m+n+1-k} + (y+1) \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{F_{m+n+1-k}(x, y)}{m+n+1-k} \\
 & = (-1)^n y \frac{m! \cdot n!}{(m+n+1)!} + \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{x^{m+n+1-k}}{m+n+1-k},
 \end{aligned} \tag{31}$$

where we have used $F_0(x, y) = 1$. It is worthy noticing that if we make the operation $\partial/\partial x$ on the both sides of (1) then we have

$$\frac{\partial}{\partial x} F_{n+1}(x, y) = (n+1)F_n(x, y) \quad (n \geq 0), \tag{32}$$

which implies that for non-negative integers n, k ,

$$\begin{aligned}
 \frac{\partial^k}{\partial x^k} F_{n+k}(x, y) & = (n+k)(n+k-1) \cdots (n+1)F_n(x, y) \\
 & = k! \cdot \binom{n+k}{k} F_n(x, y).
 \end{aligned} \tag{33}$$

By making the operation $\partial/\partial x$ on the both sides of (31), in light of (32), one can obtain (25) immediately.

We next present some analogous results to Theorems 1 and 2. Obviously, from (1) we have

$$e^{-zt} \sum_{j=0}^{\infty} F_j(x+z, y) \frac{t^j}{j!} = \frac{e^{xt}}{1-y(e^t-1)} = \sum_{j=0}^{\infty} F_j(x, y) \frac{t^j}{j!}. \tag{34}$$

If we apply (15) to the left hand side of (34) then we have

$$\begin{aligned}
 & P_m(-zt) \sum_{j=0}^{\infty} F_j(x+z, y) \frac{t^j}{j!} + S_{m,n}(-zt) \sum_{j=0}^{\infty} F_j(x+z, y) \frac{t^j}{j!} \\
 & = Q_n(-zt) \sum_{j=0}^{\infty} F_j(x, y) \frac{t^j}{j!}.
 \end{aligned} \tag{35}$$

It follows from (18) and (35) that

$$\begin{aligned}
 & \sum_{k=0}^m p_{m,n;k}(-zt)^k \sum_{j=0}^{\infty} F_j(x+z, y) \frac{t^j}{j!} + \sum_{k=0}^{\infty} s_{m,n;k}(-zt)^{m+n+k+1} \sum_{j=0}^{\infty} F_j(x+z, y) \frac{t^j}{j!} \\
 & = \sum_{k=0}^n q_{m,n;k}(-zt)^k \sum_{j=0}^{\infty} F_j(x, y) \frac{t^j}{j!},
 \end{aligned}$$

which together with the Cauchy product yields

$$\begin{aligned} & \sum_{l=0}^{\infty} \left(\sum_{\substack{k+j=l \\ k,j \geq 0}} p_{m,n;k} (-z)^k \frac{F_j(x+z,y)}{j!} \right) t^l \\ & + \sum_{l=0}^{\infty} \left(\sum_{\substack{k+j=l-m-n-1 \\ k,j \geq 0}} s_{m,n;k} (-z)^{m+n+k+1} \frac{F_j(x+z,y)}{j!} \right) t^l \\ & = \sum_{l=0}^{\infty} \left(\sum_{\substack{k+j=l \\ k,j \geq 0}} q_{m,n;k} (-z)^k \frac{F_j(x,y)}{j!} \right) t^l. \end{aligned} \tag{36}$$

By comparing the coefficients of t^l in (36), in view of (19), we get that for non-negative integers m, n, l with $0 \leq l \leq m + n$,

$$\sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{m}{k} (m+n-k)! (-z)^k \frac{F_j(x+z,y)}{j!} = \sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{n}{k} (m+n-k)! z^k \frac{F_j(x,y)}{j!}. \tag{37}$$

So from (37), we state the following result.

Theorem 3. Let m, n, l be non-negative integers with $0 \leq l \leq m + n$. Then

$$\sum_{k=0}^l \binom{m}{k} (m+n-k)! (-z)^k \frac{F_{l-k}(x+z,y)}{(l-k)!} = \sum_{k=0}^l \binom{n}{k} (m+n-k)! z^k \frac{F_{l-k}(x,y)}{(l-k)!}. \tag{38}$$

It follows that we show some special cases of Theorem 3. By taking $l = m + n$ in (38), we get that for non-negative integers m, n ,

$$\sum_{k=0}^m \binom{m}{k} (-z)^k F_{m+n-k}(x+z,y) = \sum_{k=0}^n \binom{n}{k} z^k F_{m+n-k}(x,y),$$

which can be rewritten as

$$\sum_{k=0}^m \binom{m}{k} (-z)^{m-k} F_{n+k}(x+z,y) = \sum_{k=0}^n \binom{n}{k} z^{n-k} F_{m+k}(x,y). \tag{39}$$

If we replace m by $m + r$ and n by $n + r$ in (39) and make the operation $\partial^r / \partial x^r$, in view of (33), we obtain that for non-negative integers m, n, r ,

$$\begin{aligned} & \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+r+k}{r} (-z)^{m+r-k} F_{n+k}(x+z,y) \\ & = \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+r+k}{r} z^{n+r-k} F_{m+k}(x,y). \end{aligned} \tag{40}$$

Taking $y = -1/2$ in (40) gives that for non-negative integers m, n, r ,

$$\begin{aligned} & \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+r+k}{r} (-z)^{m+r-k} E_{n+k}(x+z) \\ & = \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+r+k}{r} z^{n+r-k} E_{m+k}(x). \end{aligned} \tag{41}$$

Since the Euler polynomials satisfy the symmetric relation $E_n(1 - x) = (-1)^n E_n(x)$ for non-negative integer n , so by taking $x + y + z = 1$ in (41), we discover that for non-negative integers m, n, r ,

$$\begin{aligned} & (-1)^m \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+r+k}{r} z^{m+r-k} E_{n+k}(y) \\ &= (-1)^{n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+r+k}{r} z^{n+r-k} E_{m+k}(x), \end{aligned} \tag{42}$$

which appeared in ([19], Corollary 1.2). In particular, the case $r = 0$ in (42) was first discovered by Sun [20].

We next consider the case $l \geq m + n + 1$ in (36). It is easily seen from (36) that for non-negative integers m, n and positive integer l with $l \geq m + n + 1$,

$$\begin{aligned} & \sum_{\substack{k+j=l \\ k,j \geq 0}} p_{m,n;k} (-z)^k \frac{F_j(x+z, y)}{j!} \\ &+ \sum_{\substack{k+j=l-m-n-1 \\ k,j \geq 0}} s_{m,n;k} (-z)^{m+n+k+1} \frac{F_j(x+z, y)}{j!} \\ &= \sum_{\substack{k+j=l \\ k,j \geq 0}} q_{m,n;k} (-z)^k \frac{F_j(x, y)}{j!}. \end{aligned} \tag{43}$$

By applying (19) and (20) to (43), we have

$$\begin{aligned} & \sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{m}{k} (m+n-k)! (-z)^k \frac{F_j(x+z, y)}{j!} \\ &+ (-1)^n \sum_{\substack{k+j=l-m-n-1 \\ k,j \geq 0}} \frac{m! \cdot (n+k)!}{k! \cdot (m+n+k+1)!} (-z)^{m+n+k+1} \frac{F_j(x+z, y)}{j!} \\ &= \sum_{\substack{k+j=l \\ k,j \geq 0}} \binom{n}{k} (m+n-k)! z^k \frac{F_j(x, y)}{j!}. \end{aligned} \tag{44}$$

Hence, by taking $l = m + n + r$ in (44), we get that for non-negative integers m, n and positive integer r ,

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} (m+n-k)! (-z)^k \frac{F_{m+n+r-k}(x+z, y)}{(m+n+r-k)!} \\ &+ \frac{(-1)^n}{(r-1)!} (-z)^{m+n+1} \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{m! \cdot (n+k)!}{(m+n+k+1)!} (-z)^k E_{r-1-k}(x+z, y) \\ &= \sum_{k=0}^n \binom{n}{k} (m+n-k)! z^k \frac{F_{m+n+r-k}(x, y)}{(m+n+r-k)!}. \end{aligned} \tag{45}$$

Since the two variable Fubini polynomials satisfy the addition theorem (see, e.g., [2,3])

$$F_n(x+z, y) = \sum_{k=0}^n \binom{n}{k} F_{n-k}(x, y) z^k \quad (n \geq 0), \tag{46}$$

so from (46) and the property of the Beta function, we have

$$\begin{aligned} & \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{m! \cdot (n+k)!}{(m+n+k+1)!} (-z)^k F_{r-1-k}(x+z, y) \\ &= \sum_{k=0}^{r-1} \binom{r-1}{k} (-z)^k F_{r-1-k}(x+z, y) \int_0^1 t^m (1-t)^{n+k} dt \\ &= \int_0^1 F_{r-1}(x+zt, y) t^m (1-t)^n dt. \end{aligned} \tag{47}$$

Thus, applying (45) to (47) gives the following result.

Theorem 4. *Let m, n be non-negative integers. Then, for positive integer r ,*

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} (m+n-k)! (-z)^k \frac{F_{m+n+r-k}(x+z, y)}{(m+n+r-k)!} \\ &+ \frac{(-1)^{m+1}}{(r-1)!} z^{m+n+1} \int_0^1 F_{r-1}(x+zt, y) t^m (1-t)^n dt \\ &= \sum_{k=0}^n \binom{n}{k} (m+n-k)! z^k \frac{F_{m+n+r-k}(x, y)}{(m+n+r-k)!}. \end{aligned} \tag{48}$$

If we take $y = -1/2$ in Theorem 4 then we get that for non-negative integers m, n and positive integer r ,

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} (m+n-k)! (-z)^k \frac{E_{m+n+r-k}(x+z)}{(m+n+r-k)!} \\ &+ \frac{(-1)^{m+1}}{(r-1)!} z^{m+n+1} \int_0^1 E_{r-1}(x+zt) t^m (1-t)^n dt \\ &= \sum_{k=0}^n \binom{n}{k} (m+n-k)! z^k \frac{E_{m+n+r-k}(x)}{(m+n+r-k)!}. \end{aligned} \tag{49}$$

In particular, if we take $r = 1$ in (49), in view of $E_n(1-x) = (-1)^n E_n(x)$ for non-negative integer n , we get that for non-negative integers m, n and $x + y + z = 1$,

$$\begin{aligned} & (-1)^n \sum_{k=0}^m \binom{m}{k} z^k \frac{E_{m+n+1-k}(y)}{m+n+1-k} \\ &+ (-1)^m \sum_{k=0}^n \binom{n}{k} z^k \frac{E_{m+n+1-k}(x)}{m+n+1-k} = -z^{m+n+1} \frac{m! \cdot n!}{(m+n+1)!}, \end{aligned} \tag{50}$$

which was firstly discovered by Sun [20], and also appeared in ([19], Corollary 1.4). For some similar results to (42) and (49), one is referred to [21,22].

Remark 1. *We mention that the corresponding result stated in ([23], Theorem 3.2) has a misprint: $L_{r-1-k}^{(\alpha+k)}(x)$ should be $L_{r-1-k}^{(\alpha+m+n+k+1)}(x)$, which leads to the corresponding misprints, for example, $-k$ appearing in the binomial coefficients in the second sum of the left hand side of ([23], Equations (3.10), (3.11), (3.23) and (3.24)) should be $-(m+n+k+1)$; $L_{l-m-n-k-1}^{(\alpha+k)}(x)$ in ([23], Equation (3.26)) should be $L_{l-m-n-k-1}^{(\alpha+m+n+k+1)}(x)$; $L_{r-1-k}^{(\alpha+k)}(x)$ in ([23], Equation (3.28)) should be $L_{r-1-k}^{(\alpha+n+k+1)}(x)$. The second author expresses his sincere gratitude to Professor Marek Vandas for pointing out a misprint in ([23], Theorem 3.2) in a recent private communication.*

4. Conclusions

In this paper, we use the generating function methods and Padé approximation techniques to establish some new identities for the two variable Fubini polynomials. Moreover, we discuss

some special cases as well as immediate consequences of our main results. It turns out that some known results are obtained as special cases. The methods shown in this paper may be applied to many other families of special polynomials, for example, one could consider the n -dimensional case instead, in particular, the readers could connect the two dimensional analysis with the one-dimensional polynomials.

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