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Some φ-Fixed Point Results for $(F,\varphi,\alpha\text{-}\psi)$-Contractive Type Mappings with Applications

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Abstract: In this paper, we introduce the notions of $(F,\varphi,\alpha\text{-}\psi)$-contractions and $(F,\varphi,\alpha\text{-}\psi)$-weak contractions in metric spaces and utilize the same to prove some existence and uniqueness φ-fixed point results. Some illustrative examples are given to demonstrate the usefulness and effectiveness of our results. As applications, we deduce some fixed point theorems in partial metric spaces besides proving an existence result on the solution of nonlinear differential equations. Our results extend, generalize and improve some relevant results of the existing literature.

Keywords: φ-fixed point; $\alpha\text{-}\psi$-contraction mappings; $(F,\varphi,\alpha\text{-}\psi)$-contraction mappings; $(F,\varphi,\alpha\text{-}\psi)$-weak contraction mappings; partial metric spaces

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1. Introduction

Metric fixed point theory is a very interesting and rapidly growing domain in mathematics. Especially, this theory has fruitful applications in various domains of sciences such as: Physics, Chemistry, Computer Siences, Economics and several others. The most important result of this theory is the celebrated contraction principle essentially due to Banach [1]. This principle states that every self contraction mapping T defined on a complete metric space (X,d) has a unique fixed point. Several authors extended and generalized this principle by considering various kind of control functions. With similar endeavourer, Samet et al. [2] introduced the class of (α,ψ)-contractions and utilized the same to prove some existence and uniqueness fixed point results, which generalize several well-known fixed point results particularly due to Banach [1], Berinde [3], Suzuki [4] and some others.

On the other hand, employing a control function $F:[0,\infty)^3 \to [0,\infty)$ satisfying suitable properties, Jleli et al. [5] introduced the concepts of φ-fixed point and (F,φ)-contraction mappings and utilized the same to prove some φ-fixed point results for such mappings in the setting of complete metric spaces and also deduced some fixed point results in the setting of complete partial metric spaces. In 2017, Kumrod et. al. [6] improved the notion of (F,φ)-contraction mappings by introducing the concept of (F,φ,ψ)-contraction mappings and established φ-fixed point results for such mappings which extend the corresponding results due to Jleli et al. [5]. Recently, Asadi. [7] improved the control functions F by replacing the continuity condition on F with a weaker one and proved similar results of Kumrod et al. [6].

Following this direction of research, in this paper, our attempted improvements are four-fold described in the following lines:

(1) to introduce the concepts of $(F,\varphi,\alpha\text{-}\psi)$-contraction mappings and $(F,\varphi,\alpha\text{-}\psi)$-weak contraction mappings in metric spaces;
(2) to establish some φ-fixed point theorems in metric spaces which generalize the corresponding results contained in [2,5–7] ;
(3) to deduce some fixed point theorems in the setting of partial metric spaces which extend the results contained in [5,6],
(4) to examine the existence of solution of a second order ordinary differential equation.

2. Mathematical Preliminaries

As the present exposition involves several definitions, technical terminologies and notions, firstly we proceed to present the relevant background material needed in the sequel.

Matthews [8] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks as follows.

Definition 1. *[8] Let X be a non-empty set. We say that the mapping $p : X \times X \to [0,\infty)$ is a partial metric on X if the following conditions are satisfied (for all $x, y, z \in X$):*

(P1) $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$;
(P2) $p(x,x) \leq p(x,y)$;
(P3) $p(x,y) = p(y,x)$;
(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,d) is called a partial metric space.

Clearly if $p(x,y) = 0$, then $p(x,x) = p(y,y) = p(x,y) = 0$ (due to (P2) and (P3)) so that $x = y$ (in view of (P1)). But if $x = y$, $p(x,y)$ may not be zero. On the other hand, if $p(x,x) = 0$, for each $x \in X$, then the partial metric space (X,p) is a metric space. This shows that how a partial metric differs from the standard metric. Several interesting examples of partial metric spaces which are not metric spaces can be found in [8]. For more details about partial metric space, we refer the reader to [9–11].

For a partial metric p on a non-empty set X, the function $d_p : X \times X \to [0,\infty)$ which is given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \text{ for all } x,y \in X, \tag{1}$$

remains a standard metric on X.

Lemma 1. *[8] Let (X,p) be a partial metric space. Then*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if $\{x_n\}$ is a Cauchy sequence in the metric space (X,d_p);*
(b) *if the metric space (X,d_p) is complete, then the partial metric space (X,p) is also complete and vice versa.*

Lemma 2. *[12] Let (X,p) be a partial metric space and $\varphi : X \to [0,\infty)$ a function defined by $\varphi(x) = p(x,x)$, for all $x \in X$. Then the φ is lower semi-continuous in (X,d_p).*

Next, let Ψ denote to the set of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$, for each $t > 0$, where ψ^n is the n-th iterate of ψ. In the literature, such functions are known as (c)-comparison functions or Bianchini–Grandolfi Gauge functions.

Remark 1. *Observe that $\sum_{k=1}^{\infty} \psi^n(t) < \infty$ implies $\lim_{n\to\infty} \psi^n(t) = 0$, for all $t \in (0,\infty)$.*

The following lemma is well known in the literature and holds for every Bianchini–Grandolfi Gauge function.

Lemma 3. *If $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) < t$, for all $t > 0$.*

Samet et al. [2] introduced the concepts of α-admissible and α-ψ-contraction mappings as follows:

Definition 2. *[2] Let X be a non-empty set, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is α-admissible if*

$$x, y \in X, \;\; \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 3. *[2] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an α-ψ-contraction with respect to the metric d if there exist two functions $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that (for all $x, y \in X$)*

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)).$$

By using these notions Samet et al. [2] established some fixed point results for such mappings in the context of complete metric spaces.

On the other hand, Jleli et al. [5] introduced yet another control function $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:

(F1) $\max\{a, b\} \leq F(a, b, c), \quad$ for all $\; a, b, c \in [0, \infty)$;
(F2) $F(0, 0, 0) = 0$;
(F3) F is continuous.

In the sequel we denote by $\mathcal{F}$ the set of all functions F satisfying the conditions (F1)–(F3). The authors in [5], also introduced the concepts of φ-fixed point and (F, φ)-contraction mappings as follows:

Definition 4. *[5] Let X be a non-empty set, $T : X \to X$ and $\varphi : X \to [0, \infty)$ a given function. An element $x^* \in X$ is said to be φ-fixed point of the mapping $T : X \to X$ if and only if it is a fixed point of T and $\varphi(x^*) = 0$.*

Definition 5. *[5] Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ a given function. A mapping $T : X \to X$ is said to be an (F, φ)-contraction mapping if there exists $F \in \mathcal{F}$ and $k \in [0, 1)$ such that (for all $x, y \in X$)*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq kF(d(x, y), \varphi(x), \varphi(y)).$$

Based on these definitions, Jleli et al. [5] proved some φ-fixed point results for such mappings in the setting of complete metric spaces.

Kumrod et. al. [6] generalized the notion of (F, φ)-contraction mappings by introducing the concept of (F, φ, ψ)-contraction mappings and established φ-fixed point results for such mappings which extended the results of Jleli et al. [5].

Definition 6. *[6] Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ a given function. A mapping $T : X \to X$ is said to be an (F, φ, ψ)-contraction mapping if there exists $F \in \mathcal{F}$ and a continuous function $\psi \in \Psi$ such that (for all $x, y \in X$)*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \psi(F(d(x, y), \varphi(x), \varphi(y))).$$

Recently, Asadi. [7] proved similar results of Kumrod et al. [6] by replacing the condition (F3) with the following condition:

(F3′) $\limsup_{n \to \infty} F(a_n, b_n, 0) \leq F(a, b, 0)$, when $a_n \to a$ and $b_n \to b$ as $n \to \infty$.

The class of all functions F satisfying conditions (F1), (F2) and (F3′) is denoted by $\mathcal{F}_{\mathcal{M}}$. Observe that $\mathcal{F} \subset \mathcal{F}_{\mathcal{M}}$ but the converse is not true. The following functions F belong to $\mathcal{F}_{\mathcal{M}}$ but not in $\mathcal{F}$ as such functions are not continuous:

1. $F(a, b, c) = a + b + [c]$;
2. $F(a, b, c) = \max\{a, b\} + [c]$.

3. Main Results

First, we introduce the notion of $(F,\varphi,\alpha\text{-}\psi)$-contraction mappings as follows.

Definition 7. *Let (X,d) be a metric space, $F\in\mathcal{F}_{\mathcal{M}}$ and $T:X\to X$. We say that T is an $(F,\varphi,\alpha\text{-}\psi)$-contraction mapping if there exist three functions $\alpha:X\times X\to[0,\infty)$, $\varphi:X\to[0,\infty)$ and $\psi\in\Psi$ such that (for all $x,y\in X$)*

$$\alpha(x,y)F(d(Tx,Ty),\varphi(Tx),\varphi(Ty))\leq\psi(F(d(x,y),\varphi(x),\varphi(y))).$$

Remark 2. *By choosing the essential functions α, F and ψ suitably in Definition 7, one can deduce many contractions which substantiates that $(F,\varphi,\alpha\text{-}\psi)$-contraction unifies several kind of contractions existing in the literature.*

(a) *By setting $F(a,b,c)=a+b+c$, for all $a,b,c\in[0,\infty)$ and $\varphi(x)=0$, for all $x\in X$, we deduce α-ψ-contraction given in [2].*

(b) *With $\alpha(x,y)=1$, for all $x,y\in X$, $F\in\mathcal{F}$ and $\psi(t)=kt$, (for all $t\geq 0$ and for some $k\in[0,1)$), we obtain (F,φ)-contraction contained in [5].*

(c) *By choosing $F\in\mathcal{F}$, $\alpha(x,y)=1$, $y=Tx$, for all $x,y\in X$ with $\psi(t)=kt$, (for all $t\geq 0$ and some $k\in[0,1)$), we obtain graphic (F,φ)-contraction given in [5].*

(d) *Taking $\alpha(x,y)=1$, for all $x,y\in X$ and $F\in\mathcal{F}$, we obtain (F,φ,ψ)-contraction given in [6].*

(e) *Putting $\alpha(x,y)=1$, for all $x,y\in X$ and $F\in\mathcal{F}_{\mathcal{M}}$, we obtain (F,φ,ψ)-contraction given in [7].*

Now, we state and prove our first main result of this section which runs as follows.

Theorem 1. *Let (X,d) be a complete metric space, $\varphi:X\to[0,\infty)$ a lower semi-continuous function, $\psi\in\Psi$, $F\in\mathcal{F}_{\mathcal{M}}$, $\alpha:X\times X\to[0,\infty]$ and $T:X\to X$ be an $(F,\varphi,\alpha\text{-}\psi)$-contraction mapping. Assume that the following conditions are satisfied:*

(i) *T is α-admissible;*

(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;*

(iii) *T is continuous, or alternately,*

$(iii)'$ *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$, for all n and $\lim_{n\to\infty}x_n=x\in X$, then $\alpha(x_n,x)\geq 1$, for all n.*

Then T has a φ-fixed point.

Proof. In view of the condition (ii), there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$. Define a sequence $\{x_n\}$ in X by

$$x_{n+1}=Tx_n,\ \text{for all } n\in\mathbb{N}\cup\{0\}.$$

As T is α-admissible, we have

$$\alpha(x_0,x_1)=\alpha(x_0,Tx_0)\geq 1\implies\alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\geq 1.$$

By induction, we get

$$\alpha(x_n,x_{n+1})\geq 1,\quad\text{for all } n\in\mathbb{N}. \tag{2}$$

Using (2) and the fact that T is $(F,\varphi,\alpha\text{-}\psi)$-contraction, we obtain

$$\begin{aligned}F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1}))&\leq\alpha(x_{n-1},x_n)F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1}))\\&\leq\psi(F(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n))),\ \text{for all } n\in\mathbb{N}.\end{aligned}$$

By induction, we get

$$F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1}))\leq\psi^n(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1))),\ \text{ for all } n\in\mathbb{N}.$$

This together with (F1) imply that

$$\max\{d(x_n, x_{n+1}), \varphi(x_n)\} \le \psi^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))), \text{ for all } n \in \mathbb{N}. \tag{3}$$

From (3), we obtain

$$d(x_n, x_{n+1}) \le \psi^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))). \tag{4}$$

Now, we assert that $\{x_n\}$ is a Cauchy sequence. To prove our assertion, let $m, n \in \mathbb{N}$ such that $m > n$. On using (4) and triangle inequality, we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\le \psi^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) + \psi^{n+1}(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) \\ &+ ... + \psi^{m-1}(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) \\ &= \sum_{i=1}^{m-1} \psi^i(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))) - \sum_{j=1}^{n-1} \psi^j(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))). \end{aligned} \tag{5}$$

In view of Remark 1 and (5), we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence, $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete metric space, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{6}$$

Now, we need to show that $\varphi(x^*) = 0$. Using (3), we have

$$\varphi(x_n) \le \psi^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))).$$

Letting $n \to \infty$ in the above inequality and using Remark 1, we obtain

$$\lim_{n\to\infty} \varphi(x_n) = 0. \tag{7}$$

Since φ is lower semi-continuous, it follows from (6) and (7) that $0 \le \varphi(x^*) \le \lim_{n\to\infty} \inf \varphi(x_n) = 0$. Hence,

$$\varphi(x^*) = 0. \tag{8}$$

Now, assume that (iii) holds, i.e, T is continuous mapping, it follows that $\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0$. By the uniqueness of the limit, we get $Tx^* = x^*$. Therefore, x^* is a φ-fixed point of T.

Alternatively, assume that $(iii)'$ holds. In view of (2) and (6), we have $\alpha(x_n, x^*) \ge 1$, for all $n \in \mathbb{N}$. As T is (F,φ,α-ψ)-contraction, using (F1) and Lemma 3, we get

$$\begin{aligned} d(x_{n+1}, Tx^*) &\le max\{d(x_{n+1}, Tx^*), \varphi(x_{n+1})\} \\ &\le F(d(x_{n+1}, Tx^*), \varphi(x_{n+1}), \varphi(Tx^*)) \\ &\le \alpha(x_n, x^*)F(d(x_{n+1}, Tx^*), \varphi(x_{n+1}), \varphi(Tx^*)) \\ &\le \psi(F(d(x_n, x^*), \varphi(x_n), \varphi(x^*))) \\ &< F(d(x_n, x^*), \varphi(x_n), \varphi(x^*)) \\ &= F(d(x_n, x^*), \varphi(x_n), 0). \end{aligned}$$

On taking $\limsup_{n\to\infty}$ of both sides of the above inequality, using (F2) and (F3′), one gets

$$\limsup_{n\to\infty} d(x_{n+1}, Tx^*) \leq \limsup_{n\to\infty} F(d(x_n, x^*), \varphi(x_n), 0) \leq F(0,0,0) = 0,$$

which implies that (in view of (6))

$$d(x^*, Tx^*) = 0. \tag{9}$$

Therefore, from (8) and (9), we get that x^* is a φ-fixed point of T. This completes the proof. □

Remark 3. *Notice that, in view of Remark 2 ((a) and (c)), the results ([2] Theorems 2.1 and 2.2) and ([5] Theorem 2.2 (ii)) can be deduced from Theorem 1.*

To support our result, we give three illustrative examples. Precisely, we show that Theorem 1 can be used to cover these examples while the result due to Jleli et al. [5], Kumrod et. al. [6] and Asadi [7] are not applicable.

First, we support our result by the following example in which the mapping T is continuous.

Example 1. *Let $X = [0, \infty)$ endowed with the usual metric $d(x, y) = |x - y|$, for all $x, y \in [0, \infty)$. It is clear that (X, d) is a complete metric space. Consider the mapping $T : X \to X$ defined by*

$$Tx = \begin{cases} 2x - \frac{7}{4}, & x > 1, \\ \frac{x}{2(x+1)}, & 0 \leq x \leq 1. \end{cases}$$

Obviously, T is continuous and neither non-expansive nor expansive. Now, we define two essential functions $F : [0, \infty)^3 \to [0, \infty)$ and $\varphi : X \to [0, \infty)$ by

$$F(a, b, c) = a + b + c, \text{ for all } a, b, c \in [0, \infty) \text{ and } \varphi(x) = x, \text{ for all } x \in X.$$

It is obvious that $F \in \mathcal{F}_{\mathcal{M}}$ and φ is lower simi-continuous function.

Now, consider the function: $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. This implies that $x, y \in [0, 1]$ and by the definitions of T and α, we have

$$Tx = \frac{x}{2(x+1)} \in [0, 1], \quad Ty = \frac{y}{2(y+1)} \in [0, 1] \text{ and } \alpha(Tx, Ty) = 1.$$

Hence, T is an α-admissible mapping. Moreover, $1 \in X$ and $\alpha(1, T1) = \alpha(1, \frac{1}{4}) = 1$.

Finally, we show that T is an $(F,\varphi,\alpha\text{-}\psi)$-contraction mapping. Let $x, y \in X$.

If $x, y \in [0, 1]$, then (as $\alpha(x, y) = 1$)

$$\begin{aligned} \alpha(x, y)(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) &= \left| \frac{x}{2(x+1)} - \frac{y}{2(y+1)} \right| + \frac{x}{2(x+1)} + \frac{y}{2(y+1)} \\ &\leq \frac{1}{2}|x - y| + \frac{x}{2} + \frac{y}{2} \\ &= \frac{1}{2}(d(x, y) + \varphi(x) + \varphi(y)). \end{aligned}$$

The other cases are obvious (as $\alpha(x, y) = 0$). Hence, T is an $(F,\varphi,\alpha\text{-}\psi)$-contraction mapping with $\psi(t) = t/2$, for all $t \geq 0$. Therefore, all the hypotheses of Theorem 1 are satisfied. Hence, T has a φ-fixed point (namely $x = 0$).

Notice that the results due to Jleli et al. [5], Kumrod et. al. [6] and Asadi [7] cannot be applied in the context of Example 1 for earlier defined F. Indeed, for any $\psi \in \Psi$ (in view of Lemma 3), we have

$$d(T1,T2)+\varphi(T1)+\varphi(T2)=\frac{9}{2}>4=d(1,2)+\varphi(1)+\varphi(2)>\psi(d(1,2)+\varphi(1)+\varphi(2)).$$

Next, we support our result by two more examples in which the mapping T is not continuous.

Example 2. *Let $X=[0,\infty)$ endowed with the usual metric $d(x,y)=|x-y|$, for all $x,y\in[0,\infty)$. Then (X,d) is a complete metric space. Define $T:X\to X$ by*

$$Tx=\begin{cases} 3x-\frac{8}{3}, & x>1,\\ \frac{x}{4}, & 0\le x\le 1.\end{cases}$$

Obviously T is not continuous at $x=1$. Define two essential functions $F:[0,\infty)^3\to[0,\infty)$ and $\varphi: X\to[0,\infty)$ by

$$F(a,b,c)=a+b+c,\ \text{for all } a,b,c\in[0,\infty) \text{ and } \varphi(x)=x,\ \text{for all } x\in X.$$

Clearly $F\in\mathcal{F}_{\mathcal{M}}$ and φ is lower simi-continuous function.
Now, define $\alpha: X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases} 1, & \text{if } x,y\in[0,1],\\ 0, & \text{otherwise}.\end{cases}$$

Let $x,y\in X$ such that $\alpha(x,y)\ge 1$. This implies that $x,y\in[0,1]$ and by the definition of T and α, we have

$$Tx=\frac{x}{4}\in[0,1],\quad Ty=\frac{y}{4}\in[0,1] \text{ and } \alpha(Tx,Ty)=1.$$

Therefore, T is an α-admissible mapping. Observe that $\alpha(1,T1)=1$. Furthermore, let $\{x_n\}\subseteq X$ be any sequence such that $\alpha(x_n,x_{n+1})\ge 1$, for all n and $x_n\to x\in X$ as $n\to\infty$. Then by the definition of α, we have $x_n\in[0,1]$, for all n and $x\in[0,1]$. Therefore, $\alpha(x_n,x)=1$. for all n.

Next, we show that T is an (F,φ,α-ψ)-contraction mapping. Let $x,y\in X$.
If $x,y\in[0,1]$, then (as $\alpha(x,y)=1$)

$$\begin{aligned}\alpha(x,y)(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)) &= \left|\frac{x}{4}-\frac{y}{4}\right|+\frac{x}{4}+\frac{y}{4}\\ &=\frac{1}{4}(d(x,y)+\varphi(x)+\varphi(y))\\ &\le\frac{1}{3}(d(x,y)+\varphi(x)+\varphi(y)).\end{aligned}$$

The other cases are obvious (as $\alpha(x,y)=0$). Hence, T is an (F,φ,α-ψ)-contraction mapping with $\psi(t)=t/3$, for all $t\ge 0$. Therefore, all the hypotheses of Theorem 1 are satisfied. Hence, T has a φ-fixed point (namely $x=0$).

Observe that the results due to Jleli et al. [5], Kumrod et. al. [6] and Asadi [7] are not applicable in the context of Example 2 for earlier defined F. Indeed, for any $\psi\in\Psi$ (in view of Lemma 3), we have

$$d(T1,T2)+\varphi(T1)+\varphi(T2)=\frac{20}{3}>4=d(1,2)+\varphi(1)+\varphi(2)>\psi(d(1,2)+\varphi(1)+\varphi(2)).$$

Example 3. *Let $X=[1,\infty)$ endowed with the usual metric $d(x,y)=|x-y|$, for all $x,y\in[1,\infty)$. It is clear that (X,d) is a complete metric space. Consider the mapping $T:X\to X$ defined by*

$$Tx = \begin{cases} 3x - 5, & x > 2, \\ \frac{x+2}{2}, & 1 \le x \le 2. \end{cases}$$

Obviously T is not continuous at $x = 2$. Now, define two essential functions $F : [0,\infty)^3 \to [0,\infty)$ and $\varphi : X \to [0,\infty)$ by

$$F(a,b,c) = \max\{a,b\} + [c], \text{ for all } a,b,c \in [0,\infty) \text{ and } \varphi(x) = \begin{cases} x-2, & x > 2, \\ 0, & 1 \le x \le 2, \end{cases}$$

where $[c]$ is the integer of c. It is obvious that $F \in \mathcal{F}_{\mathcal{M}}$ and φ is lower semi-continuous function.

Next, define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in [1,2], \\ 0, & \text{otherwise.} \end{cases}$$

Let $x,y \in X$ such that $\alpha(x,y) \ge 1$. This implies that $x,y \in [1,2]$ and by the definitions of T and α, we have

$$Tx = \frac{x+2}{2} \in [1,2], \quad Ty = \frac{y+2}{2} \in [1,2] \text{ and } \alpha(Tx,Ty) = 1 \ge 1.$$

Therefore, T is an α-admissible mapping. Observe that $2 \in X$ and $\alpha(2,T2) = 1$. Furthermore, let $\{x_n\} \subseteq X$ be any sequence such that $\alpha(x_n, x_{n+1})$, for all n and $x_n \to x \in X$ as $n \to \infty$. Then by the definition of α, we have $x_n \in [1,2]$, for all n and $x \in [1,2]$. Therefore, $\alpha(x_n,x) = 1$, for all n.

Finally, we show that T is an (F,φ,α-ψ)-contraction mapping. Let $x,y \in X$.
If $x,y \in [1,2]$, then (as $\alpha(x,y) = 1$)

$$\begin{aligned} \alpha(x,y)(\max\{d(Tx,Ty),\varphi(Tx)\} + [\varphi(Ty)]) &= \max\left\{\left|\frac{x+2}{2} - \frac{y+2}{2}\right|, 0\right\} + 0 \\ &= \frac{1}{2}(\max\{d(x,y),\varphi(x)\} + [\varphi(y)]). \end{aligned}$$

The other cases are obvious (as $\alpha(x,y) = 0$). Hence, T is an (F,φ,α-ψ)-contraction mapping with $\psi(t) = t/2$, for all $t \ge 0$. Therefore, all hypotheses of Theorem 1 are satisfied. Thus, T has a φ-fixed point (namely $x = 2$).

Note that the results due to Jleli et al. [5], Kumrod et. al. [6] and Asadi [7] are not applicable in the context of Example 3 for such defined F. Indeed, for any $\psi \in \Psi$ (in view of Lemma 3), we have

$$\begin{aligned} \max\{d(T1,T3),\varphi(T1)\} + [\varphi(T3)] = \frac{9}{2} > 3 &= \max\{d(1,3),\varphi(1)\} + [\varphi(3)] \\ &> \psi(\max\{d(1,3),\varphi(1)\} + [\varphi(3)]). \end{aligned}$$

The following theorem ensures the uniqueness of the φ-fixed point.

Theorem 2. *In addition to the hypotheses of Theorem 1, suppose that the following condition is satisfied:*

(iv) *for all $x,y \in X$, there exists $z \in X$ such that $\alpha(x,z) \ge 1$ and $\alpha(y,z) \ge 1$.*

Then the φ-fixed point of T is unique.

Proof. In view of Theorem 1, T has φ-fixed point. Assume that x_1^* and x_2^* are two φ-fixed points of T, that is,

$$Tx_1^* = x_1^*, \; Tx_2^* = x_2^* \text{ and } \varphi(x_1^*) = \varphi(x_2^*) = 0.$$

From the condition (iv), there exists $z \in X$ such that

$$\alpha(x_1^*, z) \geq 1 \quad \text{and} \quad \alpha(x_2^*, z) \geq 1. \tag{10}$$

Using α-admissibility of T and (10), we obtain

$$\alpha(x_1^*, T^n z) \geq 1 \quad \text{and} \quad \alpha(x_2^*, T^n z) \geq 1, \ \forall\, n \in \mathbb{N}. \tag{11}$$

Using the $(F,\varphi,\alpha\text{-}\psi)$-contractivity assumption and (11), we have

$$\begin{aligned} F(d(x_1^*, T^n z), 0, \varphi(T^n z)) &= F(d(Tx_1^*, T(T^{n-1}z)), \varphi(Tx_1^*), \varphi(T(T^{n-1}z))) \\ &\leq \alpha(x_1^*, T^{n-1}z) F(d(Tx_1^*, T(T^{n-1}z)), \varphi(Tx_1^*), \varphi(T(T^{n-1}z))) \\ &\leq \psi(F(d(x_1^*, T^{n-1}z), 0, \varphi(T^{n-1}z))). \end{aligned}$$

This inductively implies that

$$F(d(x_1^*, T^n z), 0, \varphi(T^n z)) \leq \psi^n(F(d(x_1^*, z), 0, \varphi(z))), \ \forall\, n \in \mathbb{N},$$

which together with the condition (F1) imply that

$$\max\{d(x_1^*, T^n z), 0\} \leq \psi^n(F(d(x_1^*, z), 0, \varphi(z))),$$

or

$$d(x_1^*, T^n z) \leq \psi^n(F(d(x_1^*, z), 0, \varphi(z))).$$

Letting $n \to \infty$, we obtain $\lim_{n\to\infty} d(x_1^*, T^n z) = 0$.
Similarly, one can prove that $\lim_{n\to\infty} d(x_2^*, T^n z) = 0$. The uniqueness of the limit gives rise $x_1^* = x_2^*$. Hence, the φ-fixed point of T is unique. This completes the proof. □

Remark 4. *In view of Remark 2 ((a), (b), (c) and (e)) several relevant results such as: ([2], Theorem 2.3) ([5], Theorem 2.1 (ii)), ([6], Theorem 2.5 (ii)) and ([7], Theorem 2.5 (ii)) can be deduced from Theorem 2.*

Next, we generalize the contractive condition in Definition 7 and prove the another main result in this work.

Definition 8. *Let (X, d) be a metric space, $F \in \mathcal{F}_{\mathcal{M}}$ and $T : X \to X$. We say that T is an $(F,\varphi,\alpha\text{-}\psi)$-weak contraction mapping if there exist three functions $\alpha : X \times X \to [0, \infty)$, $\varphi : X \to [0, \infty)$, $\psi \in \Psi$ and $L \geq 0$ such that (for all $x, y \in X$)*

$$\begin{aligned} \alpha(x,y)F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leq\ & \psi(F(d(x,y), \varphi(x), \varphi(y)) \\ &+ L[F(M(x,y), \varphi(y), \varphi(Tx)) - F(0, \varphi(y), \varphi(Tx))], \end{aligned}$$

where $M(x,y) = \min\{d(x,Tx), d(y,Ty), d(y,Tx)\}$.

Remark 5. *By choosing the essential functions α, F and ψ suitably in Definition 8, one can deduce many contractions which substantiates that $(F,\varphi,\alpha\text{-}\psi)$-weak contraction unifies several kind of contractions existing in the literature.*

(a) On setting $\alpha(x,y) = 1$,, for all $x, y \in X$, $F \in \mathcal{F}$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0,1)$, we obtain (F, φ)-weak contractions given in [5].
(b) Taking $\alpha(x,y) = 1$, for all $x, y \in X$ and $F \in \mathcal{F}$, we obtain (F, φ, ψ)-weak contractions given in [6].
(c) Putting $\alpha(x,y) = 1$, for all $x, y \in X$ and $F \in \mathcal{F}_{\mathcal{M}}$, we obtain (F, φ, ψ)-weak contractions given in [7].

Theorem 3. *Let (X, d) be a complete metric space, $\varphi : X \to [0, \infty)$ a lower semi-continuous function, $\psi \in \Psi$, $F \in \mathcal{F}_{\mathcal{M}}$, $\alpha : X \times X \to [0, \infty]$ and $T : X \to X$ be an (F,φ,α-ψ)-weak contraction mapping. Assume that the following conditions are satisfied:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous, or alternately,*
$(iii)'$ *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $\lim_{n\to\infty} x_n = x \in X$, then $\alpha(x_n, x) \geq 1$, for all n.*

Then T has a φ-fixed point.

Proof. In view of the condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n, \text{ for all } n \in \mathbb{N}.$$

As T is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \implies \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Using (12) and the fact that T is (F,φ,α-ψ)-weak contraction, we obtain

$$\begin{aligned} F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) &\leq \alpha(x_{n-1}, x_n) F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) \\ &\leq \psi(F(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))) \\ &\quad + L[F(d(x_n, x_n), \varphi(x_n), \varphi(x_n)) - F(0, \varphi(x_n), \varphi(x_n))] \\ &= \psi(F(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

By induction, we get

$$F(d(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) \leq \psi^n(F(d(x_0, x_1), \varphi(x_0), \varphi(x_1))).$$

The rest of the proof follows in the lines of the proof of Theorem 1. □

Remark 6. *In view of Remark 5 ((a), (b) and (c)) the results ([5] Theorem 2.3 (ii)), ([6] Theorem 2.9 (ii)) and ([7] Theorem 2.13 (ii)) can be deduced from Theorem 3.*

4. Results on partial metric spaces

In this section, we employ Theorems 2 and 3 to deduce some new results in the setting of partial metric spaces.

Let $\mathbb{G}$ be the family of all functions $G : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

$(G1)$ $\max\{a, b\} \leq G(a + b + c)$, $\forall\, a, b, c \in [0, \infty)$;
$(G2)$ $G(0) = 0$;
$(G3)$ $\lim_{n\to\infty} \sup G(a_n) \leq G(a)$, when $a_n \to a$ as $n \to \infty$.

Example 4. *Define $G : [0, \infty) \to [0, \infty)$ by*

$$G(a) = a, \quad \forall\, a \in [0, \infty).$$

It is easy to see that the conditions $(G1)$–$(G3)$ are satisfied. Hence, $G \in \mathbb{G}$.

Using Theorem 2, we deduce the following fixed point result in the setting of partial metric spaces.

Theorem 4. *Let (X, p) be a complete partial metric space, $G \in \mathbb{G}$ and $T : X \to X$ an α-admissible mapping such that*

$$\alpha(x,y)G(p(Tx,Ty)) \leq \psi(G(p(x,y))), \tag{13}$$

for all $x, y \in X$, where $\psi \in \Psi$. Suppose that the following conditions are fulfilled:

- (i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- (ii) *$\psi(2t) = 2\psi(t)$ and $G(2t) = 2G(t)$, for all $t \in [0,\infty)$;*
- (iii) *for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$;*
- (iv) *T is continuous, or alternately,*
- $(iv)'$ *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $\lim_{n\to\infty} x_n = x \in X$, then $\alpha(x_n, x) \geq 1$, for all n.*

Then T has a unique fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

Proof. Consider the metric d_p given in (1). Then (X, d_p) forms a complete metric space (due to Lemma 1). Define a mapping $\varphi : X \to [0,\infty)$ by $\varphi(x) = p(x,x)$, for all $x \in X$. Then φ is lower semi-continuous function (in view of Lemma 2). Now, using (13) and the condition (ii), we obtain that (for all $x, y \in X$)

$$\alpha(x,y)G(d_p(Tx,Ty) + p(Tx,Tx) + p(Ty,Ty)) \leq \psi(G(d_p(x,y) + p(x,x) + p(y,y))),$$

or

$$\alpha(x,y)G(d_p(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(G(d_p(x,y) + \varphi(x) + \varphi(y))). \tag{14}$$

Next, define $F : [0,\infty)^3 \to [0,\infty]$ by $F(a,b,c) = G(a+b+c)$, for all $a,b,c \in [0,\infty)$. Observe that from $(G1)$, we have $\max\{a,b\} \leq G(a+b+c) = F(a,b,c)$. Also, $(G2)$ implies that $F(0,0,0) = 0$. Furthermore, if $\{a_n\} \to a$ and $\{b_n\} \to b$ as $n \to \infty$, then (in view of $(G3)$)

$$\limsup_{n\to\infty} F(a_n, b_n, 0) = \limsup_{n\to\infty} G((a_n + b_n + 0)) \leq G(a+b) = F(a,b,0).$$

Therefore, $F \in \mathcal{F}_{\mathcal{M}}$. Using (14) and the definition of F, we have (for all $x, y \in X$)

$$\alpha(x,y)F(d_p(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leq \psi(F(d_p(x,y), \varphi(x), \varphi(y))).$$

Thus, all the hypotheses of Theorem 2 are satisfied and, hence, T has a φ-fixed point (say x^*) which yields that x^* is a fixed point of T and $p(x^*, x^*) = 0$. This completes the proof. □

Taking G (as defined in Example 4) in Theorem 4, we obtain motivated version of the main result of Samet et. al. [2] in the setting of partial metric spaces.

Corollary 1. *Let (X, p) be a complete partial metric space and $T : X \to X$ an α-admissible mapping such that*

$$\alpha(x,y)p(Tx,Ty) \leq \psi(p(x,y)),$$

for all $x, y \in X$, where $\psi \in \Psi$. Suppose that the following conditions are fulfilled:

- (i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- (ii) *$\psi(2t) = 2\psi(t)$, for all $t \in [0,\infty)$;*
- (iii) *for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$;*
- (iv) *T is continuous, or alternately,*
- $(iv)'$ *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $\lim_{n\to\infty} x_n = x \in X$, then $\alpha(x_n, x) \geq 1$, for all n.*

Then T has a unique fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

Putting $\alpha(x, y) = 1$, for all $x, y \in X$ in Theorem 4, we obtain the following result which is new to the existing literature.

Corollary 2. *Let (X, p) be a complete partial metric space, $G \in \mathbb{G}$, $\psi \in \Psi$ and $T : X \to X$. Assume that the following conditions are satisfied:*

(i) $\psi(2t) = 2\psi(t)$ *and* $G(2t) = 2G(t)$*, for all* $t \in [0, \infty)$*;*
(ii) *the mapping T satisfies*

$$G(p(Tx, Ty)) \leq \psi(G(p(x, y))), \text{ for all } x, y \in X.$$

Then T has a unique fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

By setting G as in Example 4 in Corollary 2, we obtain directly the following result which was proved by Kumrod et. al. [6].

Corollary 3. *Let (X, p) be a complete partial metric space, $\psi \in \Psi$ and $T : X \to X$. Assume that the following conditions are satisfied:*

(i) $\psi(2t) = 2\psi(t)$*, for all* $t \in [0, \infty)$*.*
(ii) *the mapping T satisfies*

$$p(Tx, Ty) \leq \psi(p(x, y)), \text{ for all } x, y \in X.$$

Then T has a unique fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

By choosing G as in Example 4 and $\psi(t) = kt$, for all $t \in [0, \infty)$ and some $k \in [0, 1)$ in Corollary 2, we obtain directly the following result which was proved by Matthews [8]

Corollary 4. *Let (X, p) be a complete partial metric space and $T : X \to X$. Assume there exists $k \in (0, 1)$ such that (for all $x, y \in X$)*

$$p(Tx, Ty) \leq k(p(x, y)).$$

Then T has a unique fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

Next, from Theorem 3, we deduce the following result in the setting of partial metric spaces.

Theorem 5. *Let (X, p) be a complete partial metric space, $G \in \mathbb{G}$ and $T : X \to X$ an α-admissible mappings such that*

$$\alpha(x, y)G(p(Tx, Ty)) \leq \psi(G(p(x, y))) + L\left[G(N(x, y)) - G\left(\frac{p(y, y) + p(Tx, Tx)}{2}\right)\right], \tag{15}$$

for all $x, y \in X$, where $\psi \in \Psi$, $L \geq 0$ and $N(x, y) = \min\{p(x, Tx), p(y, Ty), p(y, Tx)\}$. Suppose that the following conditions are fulfilled:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(ii) $\psi(2t) = 2\psi(t)$ *and* $G(2t) = 2G(t)$*, for all* $t \in [0, \infty)$*;*
(iii) *T is continuous, or alternately,*
(iii)′ *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $\lim_{n\to\infty} x_n = x \in X$, then $\alpha(x_n, x) \geq 1$, for all n.*

Then T has a fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

Proof. Consider the metric d_p given in (1). In view of Lemma 1 the metric space (X, d_p) is complete. Define a mapping $\varphi : X \to [0, \infty)$ by $\varphi(x) = p(x, x)$, for all $x \in X$. Then φ is lower semi-continuous (duo to Lemma 2). Now, using (15) and the condition (ii), we obtain (for all $x, y \in X$)

$$\alpha(x,y)G(2p(Tx,Ty)) \leq \psi(G(2p(x,y))) + L[G(2N(x,y)) - G(p(y,y) + p(Tx,Tx))],$$

or

$$\begin{aligned}\alpha(x,y)(G(d_p(Tx,Ty) + \varphi(Tx) + \varphi(Ty))) \leq \psi(G(d_p(x,y) + \varphi(x) + \varphi(y)))\\ + L[G(M(x,y) + \varphi(y) + \varphi(Tx))\\ - G(\varphi(y) + \varphi(Tx))],\end{aligned} \tag{16}$$

where $M(x,y) = \min\{d_p(x,Tx), d_p(y,Ty), d_p(y,Tx)\}$.

Next, define $F : [0,\infty)^3 \to [0,\infty]$ by $F(a,b,c) = G(a+b+c)$, for all $a,b,c \in [0,\infty)$. Then, $F \in \mathcal{F}_{\mathcal{M}}$ (see proof of Theorem 3). Using (16) and the definition of F, we have (for all $x, y \in X$)

$$\begin{aligned}\alpha(x,y)(F(d_p(Tx,Ty), \varphi(Tx), \varphi(Ty))) \leq \psi(F(d_p(x,y), \varphi(x), \varphi(y)))\\ + L[F(M(x,y), \varphi(y), \varphi(Tx)) - F(0, \varphi(y), \varphi(Tx))].\end{aligned}$$

Therefore, all the hypotheses of Theorem 3 are satisfied and, hence, T has a φ-fixed point (say x^*) which yields that x^* is a fixed point of T and $p(x^*, x^*) = 0$. This completes the proof. □

Taking G as in Example 4 in Theorem 5, we obtain the following result.

Corollary 5. *Let (X, p) be a complete partial metric space and $T : X \to X$ be an α-admissible mappings such that*

$$\alpha(x,y)p(Tx,Ty) \leq \psi(p(x,y)) + L\left[N(x,y) - \frac{p(y,y) + p(Tx,Tx)}{2}\right],$$

for all $x, y \in X$, where $\psi \in \Psi$, $L \geq 0$ and $N(x,y) = \min\{p(x,Tx), p(y,Ty), p(y,Tx)\}$. Suppose that the following conditions are satisfied:

- (i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- (ii) *$\psi(2t) = 2\psi(t)$, for all $t \in [0,\infty)$;*
- (iii) *T is continuous, or alternately,*
- $(iii)'$ *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $\lim_{n\to\infty} x_n = x \in X$, then $\alpha(x_n, x) \geq 1$, for all n.*

Then T has a fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

Putting $\alpha(x,y) = 1$, for all $x, y \in X$ in Theorem 5, we obtain the following result which is new to the existing literature.

Corollary 6. *Let (X, p) be a complete partial metric space, $G \in \mathbb{G}$, $\psi \in \Psi$ and $T : X \to X$. Assume that the following conditions are satisfied:*

- (i) *$\psi(2t) = 2\psi(t)$ and $G(2t) = 2G(t)$, for all $t \in [0,\infty)$;*
- (ii) *the mapping T satisfies (for all $x, y \in X$)*

$$G(p(Tx,Ty)) \leq \psi(G(p(x,y))) + L\left[G(N(x,y)) - G\left(\frac{p(y,y) + p(Tx,Tx)}{2}\right)\right],$$

where $N(x,y) = \min\{p(x,Tx), p(y,Ty), p(y,Tx)\}$ and $L \geq 0$. Then T has a fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

Taking G as in Example 4 in Corollary 6, we obtain directly the following result which was proved by Kumrod et. al. [6].

Corollary 7. *Let (X, p) be a complete partial metric space, $\psi \in \Psi$ and $T : X \to X$. Assume that the following conditions are satisfied:*

(i) $\psi(2t) = 2\psi(t)$, *for all* $t \in [0, \infty)$.

(ii) *the mapping T satisfies (for all $x, y \in X$)*

$$p(Tx, Ty) \leq \psi(p(x, y)) + L\left[N(x, y) - \frac{p(y, y) + p(Tx, Tx)}{2}\right],$$

where $N(x, y) = \min\{p(x, Tx), p(y, Ty), p(y, Tx)\}$ and $L \geq 0$. Then T has a fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

Setting G as in Example 4 and $\psi(t) = kt$, for all $t \in [0, \infty)$ and some $k \in [0, 1)$ in Corollary 6, we obtain directly the following result which was proved by Kumrod et. al. [6].

Corollary 8. *Let (X, p) be a complete partial metric space and $T : X \to X$. Assume that the following condition is satisfied (for all $x, y \in X$)*

$$p(Tx, Ty) \leq kp(x, y) + L\left[p(y, Tx) - \frac{p(y, y) + p(Tx, Tx)}{2}\right],$$

for some $k \in [0, 1)$, where $N(x, y) = \min\{p(x, Tx), p(y, Ty), p(y, Tx)\}$ and $L \geq 0$. Then T has a fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$.

5. Application to Nonlinear Ordinary Differential Equations

In this section, we investigate the existence of a solution to the following boundary value problem of second order ordinary differential equation:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t, x(t)), \ t \in [0, 1]; \\ \qquad x(0) = x(1) = 0, \end{cases} \tag{17}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X = C[0, 1]$ be the space of all continuous functions defined on $[0, 1]$. Define a metric $d : X \times X \to \mathbb{R}$ on X by (for all $x, y \in X$)

$$d(x, y) = \|x - y\|_\infty = \max_{t \in [0,1]} |x(t) - y(t)|.$$

It is known that (X, d) is a complete metric space. The green function associated to (17) is defined by

$$G(s, t) = \begin{cases} t(1 - s), \ 0 \leq t \leq s \leq 1; \\ s(1 - t), \ 0 \leq s \leq t \leq 1. \end{cases}$$

Let $T : X \to X$ and $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given functions. Assume that the following conditions are satisfied:

(i) $|f(t, a) - f(t, b)| \leq \max_{a,b \in \mathbb{R}} |a - b|$, for all $t \in [0, 1]$, $a, b \in \mathbb{R}$ with $\xi(a, b) \geq 0$;

(ii) there exists $x_0 \in X$ such that for all $t \in [0, 1]$, we have $\xi(x_0(t), Tx_0(t)) \geq 0$;

(iii) if $\xi(x(t), y(t)) \geq 0$, for all $x, y \in X$ and $t \in [0, 1]$, then $\xi(Tx(t), Ty(t)) \geq 0$;

(iv) if $\{x_n\}$ is a sequence in X such that $\xi(x_n, x_{n+1}) \geq 0$, for all n and $x_n \to x \in X$, then $\xi(x_n, x) \geq 0$, for all n.

Theorem 6. *Assume that the conditions* (i)*–*(iv) *are satisfied. Then* (17) *has a solution in* $C^2[0,1]$.

Proof. Observe that $x \in X$ is a solution of (17) if and only if $x \in X$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s)f(s,x(s))ds, \quad \forall x \in X.$$

Define $T : X \to X$ by:

$$T(x(t)) = \phi(t) + \int_0^T G(t,s)f(s,x(s))ds, \quad \forall x \in X.$$

Let $x, y \in X$ such that $\xi(x(t), y(t) \geq 0$, for all $t \in [0,1]$. Using (i), we get

$$\begin{aligned} |T(x(t)) - T(y(t))| &= \left| \int_0^1 G(t,s)f(s,x(s))ds - \int_0^1 G(t,s)f(s,y(s))ds \right| \\ &\leq \int_0^1 G(t,s)\,|f(s,x(s)) - f(s,y(s))|\,ds \\ &\leq \int_0^1 G(t,s) \max_{s\in[0,1]} |x(s) - y(s)|ds \\ &\leq \left(\sup_{t\in[0,1]} \int_0^1 G(t,s)ds \right) \cdot \|x-y\|_\infty \end{aligned}$$

As $\int_0^1 G(t,s)ds = -\frac{t^2}{2} + \frac{t}{2}$, for all $t \in [0,1]$, $\sup_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{8}$, we obtain

$$\|Tx - Ty\|_\infty \leq \frac{1}{8}\|x-y\|_\infty.$$

Now, consider the two functions $F : [0,\infty)^3 \to [0,\infty)$ given by $F(a,b,c) = a+b+c$, for all $a,b,c \in [0,\infty)$ and $\varphi : X \to [0,\infty)$ which is given by $\varphi(x) = 0$, for all $x \in X$, then we have

$$\|Tx - Ty\|_\infty + \varphi(Tx) + \varphi(Ty) \leq \frac{1}{8}(\|x-y\|_\infty + \varphi(x) + \varphi(y)),$$

or

$$F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leq \frac{1}{8}F(d(x,y), \varphi(x), \varphi(y)).$$

Next, define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t), y(t)) \geq 0,\ t \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Then (for all $x, y \in X$,), we obtain

$$\alpha(x,y)F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leq \psi(F(d(x,y), \varphi(x), \varphi(y))).$$

with $\psi(u) = \frac{1}{8}u$. Hence T is an $(F, \varphi, \alpha\text{-}\psi)$-contraction mapping. Using the condition (iii) it is easy to verify that T is α-admissible. The condition (ii) shows that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. These together with condition (iv) show that all the hypothesis of Theorem 1 are satisfied and, hence, T has a φ-fixed point in X which is a solution of (17). This concludes the proof. □

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