

Article

Hybrid Mann Viscosity Implicit Iteration Methods for Triple Hierarchical Variational Inequalities, Systems of Variational Inequalities and Fixed Point Problems

Lu-Chuan Ceng ^{1,†}  and Qing Yuan ^{2,*,†} ¹ Department of Mathematics, Shanghai Normal University, Shanghai 200234, China; zenglc@hotmail.com² School of Mathematics and Statistics, Linyi University, Linyi 276000, China

* Correspondence: yuanqing@lyu.edu.cn

† These authors contributed equally to this work.

Received: 22 December 2018; Accepted: 29 January 2019; Published: 2 February 2019



Abstract: In the present work, we introduce a hybrid Mann viscosity-like implicit iteration to find solutions of a monotone classical variational inequality with a variational inequality constraint over the common solution set of a general system of variational inequalities and a problem of common fixed points of an asymptotically nonexpansive mapping and a countable of uniformly Lipschitzian pseudocontractive mappings in Hilbert spaces, which is called the triple hierarchical constrained variational inequality. Strong convergence of the proposed method to the unique solution of the problem is guaranteed under some suitable assumptions. As a sub-result, we provide an algorithm to solve problem of common fixed points of pseudocontractive, nonexpansive mappings, variational inequality problems and generalized mixed bifunction equilibrium problems in Hilbert spaces.

Keywords: hybrid Mann viscosity implicit iteration method; triple hierarchical constrained variational inequality; general system of variational inequalities; fixed point; asymptotically nonexpansive mapping; pseudocontractive mapping; strong convergence; Hilbert spaces

1. Introduction

We suppose that H is a real or complex Hilbert space and let H be with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We suppose that C is a convex nonempty closed set of H . We also suppose that P_C is the metric projection from H onto C . Since C is a convex nonempty closed set, we conclude that P_C is defined. Let T be a mapping on convex nonempty closed set C . Denote by $\text{Fix}(T)$ the set of fixed points of T , i.e., $\text{Fix}(T) = \{x \in C : (I - T)x = 0\}$. \rightarrow and \rightharpoonup present strong convergence and weak convergence, respectively. A mapping $T : C \rightarrow C$ is named to be asymptotically nonexpansive if there exists a sequence $\{\theta_n\} \subset [0, +\infty)$ with $\lim_{n \rightarrow \infty} \theta_n = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \theta_n \|x - y\|, \quad \forall n \geq 0, x, y \in C. \quad (1)$$

If $\theta \equiv 0$, then T is named to be nonexpansive, that is,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (2)$$

Suppose that A is a nonself mapping from convex nonempty closed set C to entire space H . The classical variational inequality (VI) is to find $x^* \in C$ such that

$$\langle \mu Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \quad (3)$$

where μ is some positive real number. We denote by $\text{VI}(C, A)$ the set of solutions of VI (3).

Assume that B_1 is a nonself mapping from convex nonempty closed set C to entire space H and B_2 is a nonself mapping from convex nonempty closed set C to entire space H , respectively. we study the system of approximating $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* - y^* + x^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* - x^* + y^*, x - y^* \rangle \geq 0, & \forall x \in C. \end{cases} \tag{4}$$

Here, μ_1 and μ_2 are two real numbers. The system (4) is named to be a general system of variational inequalities (GSVI). We note that the system (4) can be transformed into a problem of zero points $(I - T)x = 0$, that is, the fixed point of T as following

Lemma 1 ([1]). Fix $x^*, y^* \in C$, where (x^*, y^*) satisfies the system (4) if and only if

$$x^* \in \text{GSVI}(C, B_1, B_2),$$

where $\text{GSVI}(C, B_1, B_2)$ is the set of solutions of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, and $y^* = P_C(I - \mu_2 B_2)x^*$.

Recently, the variational inequality (3) and the system (4) have been intensively investigated by many authors via fixed-point methods; see [2–11] and the references therein. A mapping $f : C \rightarrow C$ is said to be a contraction on C if there exists a constant $\delta \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \delta \|x - y\|$ for all $x, y \in C$. A mapping $F : C \rightarrow H$ is called monotone if $\langle Fx - Fy, x - y \rangle \geq 0 \forall x, y \in C$. It is called η -strongly monotone if there exists a constant $\eta > 0$ such that $\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2 \forall x, y \in C$. Moreover, it is called α -inverse-strongly monotone (or α -cocoercive), if there exists a constant $\alpha > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \alpha \|Fx - Fy\|^2, \quad \forall x, y \in C.$$

Furthermore, let X be a real Banach space whose topological dual space is denoted with X^* . The normalized duality $J : X \rightarrow 2^{X^*}$ is defined through

$$J(x) = \{\psi \in X^* : \langle x, \psi \rangle = \|x\|^2 = \|\psi\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We suppose that T is a mapping. Its domain and range are denoted by $D(T)$ and range $R(T)$, respectively. It called pseudocontractive if

$$\|x - y\| \leq \|x - y + r((I - T)x - (I - T)y)\|, \quad \forall x, y \in D(T), \forall r > 0.$$

From a result of Kato [12], we know that the notion of pseudocontraction is equivalent to the following definition: There exists $j(x - y) \in J(x - y)$ such that

$$\langle j(x - y), Tx - Ty \rangle \leq \|x - y\|^2, \quad \forall x, y \in D(T).$$

It is well known that the class of pseudocontractive mappings, whose complementary operators are accretive, is an important and significant generation of nonexpansive mappings (see [13–19]). In 2011, Ceng et al. [20] introduced an implicit viscosity approximation method for computing approximate fixed points of pseudocontractive mapping T , and obtained the norm convergence of sequence $\{x_n\}$ generated by their implicit method to a fixed-point of T .

The main aim of this paper is to introduce and analyze a hybrid Mann viscosity implicit iteration method for solving a monotone variational inequality with a variational inequality constraint over the common solution set of the GSVI (4) for two inverse-strongly monotone mappings and a common fixed point problem (CFPP) of a countable family of uniformly Lipschitzian pseudocontractive mappings and an asymptotically nonexpansive mapping in Hilbert spaces, which is called the triple hierarchical constrained variational inequality (THCVI). Here, the hybrid Mann viscosity implicit iteration method

is based on the viscosity approximation method, Korpelevich extragradient method, Mann iteration method and hybrid steepest-descent method. With relatively weak assumptions, the authors prove the strong convergence analysis of the their method to the unique solution of the THCVI. As an application, we list an algorithm to solve problems of common fixed point of pseudocontractive and nonexpansive mappings, classical variational inequalities and generalized mixed equilibrium problems in Hilbert setting.

2. Preliminaries

In this subsection, we suppose H is a Hilbert space. Its inner product denoted by $\langle \cdot, \cdot \rangle$. We also suppose C is a convex nonempty closed set of H . Here, we list some basic concepts and facts. A nonself mapping F from convex nonempty closed set C to entire space H is said to be κ -Lipschitzian if there is a number $\kappa > 0$ with $\|F(x) - F(y)\| \leq \kappa\|x - y\| \forall x, y \in C$. In particular, if $\kappa = 1$, then the nonself mapping F is named to be a nonexpansive operator. A self mapping A on entire space H is name to be a strongly positive bounded linear operator if we have a number $\gamma > 0$ with

$$\frac{\langle Ax, x \rangle}{\gamma} \geq \|x\|^2, \quad \forall x \in H.$$

It is easy to see that the self mapping A is a γ -strongly monotone $\|A\|$ -Lipschitzian operator. Recall that a self mapping T on convex nonempty closed set C is named to be

- (a) a contraction if we have a number $\alpha \in (0, 1)$ with

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \forall x, y \in C;$$

- (b) a pseudocontraction if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C;$$

- (c) strong pseudocontraction if we have a number $\alpha \in (0, 1)$ with

$$\langle Tx - Ty, x - y \rangle \leq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

We use the following concept in the sequel.

Definition 1. Let $\{T_n\}_{n=0}^\infty$ be a mapping sequence of continuous self pseudocontractions on C . Then, $\{T_n\}_{n=0}^\infty$ is said to be a countable family of ℓ -uniformly Lipschitzian pseudocontractive self-mappings on C if we have a number $\ell > 0$ such that each T_n is ℓ -Lipschitz continuous.

Fix $x \in H$, there is a unique element in C , denoted by P_Cx , with

$$\|x - P_Cx\| \leq \|x - y\|, \quad \forall y \in C. \tag{5}$$

where P_C stands for a metric projection of entire space H onto convex nonempty closed set C . It is well known that P_C is a nonexpansive mapping with

$$\langle x - y, P_Cx - P_Cy \rangle \geq \|P_Cx - P_Cy\|^2, \quad \forall x, y \in H. \tag{6}$$

Nevertheless, P_Cx has the functions: $P_Cx \in C$ and

$$\langle x - P_Cx, y - P_Cx \rangle \leq 0, \tag{7}$$

$$\|x - y\|^2 - \|x - P_Cx\|^2 \geq \|y - P_Cx\|^2, \quad \forall x \in H, y \in C. \tag{8}$$

We also have

$$2\langle x - y, y \rangle + \|x - y\|^2 = \|x\|^2 - \|y\|^2. \tag{9}$$

We need the following propositions and lemmas for our main presentation.

Proposition 1 ([21]). We suppose C is a convex nonempty closed set of a Banach space X . We suppose S_0, S_1, \dots is an operator sequence on convex nonempty closed C . Let

$$\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1} x\| : x \in C\} < \infty.$$

It follows that $\{S_n y\}$ converges strongly to some point of C for each $y \in C$. Nevertheless, we let S be a mapping on convex nonempty closed C defined through $Sy = \lim_{n \rightarrow \infty} S_n y$ for all $y \in C$. Then $\lim_{n \rightarrow \infty} \sup\{\|Sx - S_n x\| : x \in C\} = 0$.

Proposition 2 ([22]). We suppose C is a convex nonempty closed set of a Banach space X . We also suppose T is a continuous and strong pseudocontraction on convex nonempty closed C . This shows the fact that T has a fixed point in C . Indeed, it is also unique.

The following lemma is trivial. In fact, it an immediate consequence of the subdifferential of $\frac{1}{2}\|\cdot\|^2$.

Lemma 2. We suppose H is a Hilbert space. In H , we have

$$\|x + y\|^2 - \|x\|^2 \leq 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Lemma 3 ([23]). We suppose $\{a_n\}$ is a number sequence such that

$$a_{n+1} \leq a_n + \lambda_n \gamma_n - \lambda_n a_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ are real numbers such that

(i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$; or, equivalently,

$$\prod_{n=0}^{\infty} (1 - \lambda_n) := \lim_{n \rightarrow \infty} \prod_{k=0}^n (1 - \lambda_k) = 0;$$

(ii) $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$ or $\sum_{n=0}^{\infty} |\lambda_n \gamma_n| < \infty$.

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 4 ([24]). We suppose T is a nonexpansive mapping defined on a convex nonempty subset C of a Hilbert space H . Let λ be a number in $(0, 1]$. We suppose F is a self κ -Lipschitzian and η -strongly monotone mapping on entire space H . Define the mapping $T^\lambda : C \rightarrow H$ through

$$T^\lambda x := Tx - \lambda \mu F(Tx), \quad \forall x \in C.$$

Then, T^λ is a contraction if $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda \tau) \|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 5. Let the mapping $A : C \rightarrow H$ be α -inverse-strongly monotone. Then, for a given $\lambda \geq 0$,

$$\|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \geq \|(I - \lambda A)x - (I - \lambda A)y\|^2.$$

In particular, if $0 \leq \lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive.

Proof.

$$\begin{aligned} & \|(I - \lambda A)y - (I - \lambda A)x\|^2 \\ &= \|\lambda(Ay - Ax)\|^2 - 2\langle \lambda(Ay - Ax), y - x \rangle + \|y - x\|^2 \\ &\leq \lambda^2\|Ay - Ax\|^2 - 2\lambda\alpha\|Ay - Ax\|^2 + \|y - x\|^2 \\ &= \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 + \|y - x\|^2. \end{aligned}$$

□

Utilizing Lemma 5, we immediately obtain the following lemma.

Lemma 6. We suppose the nonself mappings B_1, B_2 is α -inverse-strongly monotone and β -inverse-strongly monotone defined on convex nonempty closed subset C of entire space H , respectively. Let the self mapping G be defined as $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$. If $0 \leq \mu_1 \leq 2\alpha$ and $0 \leq \mu_2 \leq 2\beta$, then $G : C \rightarrow C$ is nonexpansive.

Lemma 7 ([25]). We suppose that X is a real Banach space with a weakly continuous duality and C is a convex nonempty closed set in X . Let T be a self mapping defined the set C and we also suppose it is asymptotically nonexpansive with a empty fixed-point set. Then, $T - I$ is demiclosed at zero, i.e., let $\{x_n\}$ be a sequence in set C converging weakly to some x , where x in C and the sequence $\{(I - T)x_n\}$ converges strongly to 0, then $(T - I)x = 0$, where I is the identity mapping of X .

Lemma 8 ([26]). We suppose C is a convex nonempty closed set in a Hilbert space H and A is a monotone and hemicontinuous nonself mapping defined on convex nonempty closed set C to H . Then, we have

- (i) $VI(C, A) = \{x^* \in C : \langle Ay, y - x^* \rangle \geq 0, \forall y \in C\}$;
- (ii) $VI(C, A) = \text{Fix}(P_C(I - \lambda A))$ for all $\lambda > 0$; and
- (iii) $VI(C, A)$ is singleton, if A is Lipschitz continuous strongly monotone.

3. Main Results

We suppose C is a convex nonempty closed set. Let the mappings A_1, B_i be nonself monotone mappings for $i = 1, 2$ from C to H . We also let T be a self asymptotically nonexpansive mapping. Suppose $\{S_n\}_{n=0}^\infty$ is a countable family of self mapping. We also assume it is ℓ -uniformly Lipschitzian pseudocontractive on set C . Consider the variational inequality for monotone mapping A_1 over the common solution set Ω of the GSVI (4) and the CFPP of $\{S_n\}_{n=0}^\infty$ and T :

$$\begin{aligned} & \text{Find } \bar{x} \in VI(\Omega, A_1) \\ & := \{\bar{x} \in \Omega : \langle A_1 \bar{x}, y - \bar{x} \rangle \geq 0 \forall y \in \Omega\}, \end{aligned}$$

where $\Omega := \bigcap_{n=0}^\infty \text{Fix}(S_n) \cap \text{GSVI}(C, B_1, B_2) \cap \text{Fix}(T) \neq \emptyset$.

This section introduces the following monotone variational inequality with the variational inequality constraint over the common solution set of the GSVI (4) and the CFPP of $\{S_n\}_{n=0}^\infty$ and T , which is called the triple hierarchical constrained variational inequality (THCVI):

Problem 1. Assume that

- (C1) $T : C \rightarrow C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$.
- (C2) $\{S_n\}_{n=0}^\infty$ is a countable family of ℓ -uniformly Lipschitzian pseudocontractive self-mappings on C .

- (C3) $B_1 : C \rightarrow H$ is an α -inverse-strongly monotone operator and $B_2 : C \rightarrow H$ is a β -inverse-strongly monotone operator.
- (C4) $\text{GSVI}(C, B_1, B_2) := \text{Fix}(G)$ where $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ for $\mu_1, \mu_2 > 0$.
- (C5) $\Omega := \bigcap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{GSVI}(C, B_1, B_2) \cap \text{Fix}(T) \neq \emptyset$.
- (C6) $\sum_{n=1}^{\infty} \sup_{x \in D} \|S_n x - S_{n-1} x\| < \infty$ for any bounded subset D of C .
- (C7) $S : C \rightarrow C$ is the mapping defined by $Sx = \lim_{n \rightarrow \infty} S_n x \forall x \in C$, such that

$$\text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n).$$

- (C8) $A_1 : C \rightarrow H$ is an ζ -inverse-strongly monotone operator and $A_2 : C \rightarrow H$ is a κ -Lipschitzian and η -strongly monotone operator.
- (C9) $f : C \rightarrow C$ is a contraction mapping with coefficient $\delta \in [0, 1)$.
- (C10) $\text{VI}(\Omega, A_1) \neq \emptyset$.

Then, the objective is to

$$\begin{aligned} &\text{find } x^* \in \text{VI}(\text{VI}(\Omega, A_1), A_2) \\ &:= \{x^* \in \text{VI}(\Omega, A_1) : \langle A_2 x^*, v - x^* \rangle \geq 0 \forall v \in \text{VI}(\Omega, A_1)\}. \end{aligned}$$

Since the original problem is a variational inequality problem, we therefore call it a triple hierarchical constrained variational inequality (THCVI). We introduce the following hybrid Mann viscosity implicit iteration method to find the solution of such a problem.

We show the main result of this paper, that is, the strong convergence analysis for Algorithm 1.

Algorithm 1: Hybrid Mann viscosity-like implicit iterative algorithm.

Step 0. Take $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\delta_n\}_{n=0}^{\infty}, \{\sigma_n\}_{n=0}^{\infty} \subset (0, \infty)$, and $\mu > 0$; arbitrarily choose $x_0 \in C$; and let $n := 0$.

Step 1. Given $x_n \in C$, compute $x_{n+1} \in C$ as

$$\begin{cases} u_n = \gamma_n x_n + (1 - \gamma_n) S_n u_n, \\ v_n = P_C(u_n - \mu_2 B_2 u_n), \\ z_n = P_C(v_n - \mu_1 B_1 v_n), \\ y_n = \sigma_n x_n + (1 - \sigma_n) P_C(I - \delta_n A_1) z_n, \\ x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) P_C(I - \alpha_n \mu A_2) T^n y_n. \end{cases} \tag{10}$$

Update $n := n + 1$ and go to Step 1.

Theorem 1. Assume that $\mu_1 \in (0, 2\alpha)$, $\mu_2 \in (0, 2\beta)$, and $\delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$ for $\mu \in (0, \frac{2\eta}{\kappa^2})$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\sigma_n\} \subset (0, 1]$ and $\{\delta_n\} \subset (0, 2\zeta]$ are the sequences such that

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
- (ii) $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} = 0$, $\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = 0$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.
- (iii) $0 < \liminf_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} \sigma_n < 1$ and $\sum_{n=0}^{\infty} |\sigma_{n+1} - \sigma_n| < \infty$.
- (iv) $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ and $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.
- (v) $\delta_n \leq \alpha_n$ and $\sum_{n=0}^{\infty} \|T^{n+1} y_n - T^n y_n\| < \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm 1 satisfies the following properties:

- (a) $\{x_n\}_{n=0}^{\infty}$ is bounded.
- (b) $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$, $\lim_{n \rightarrow \infty} \|x_n - Gx_n\| = 0$, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$.

(c) $\{x_n\}_{n=0}^\infty$ converges to the unique solution of Problem 1 if $\frac{\|x_n - y_n\|}{\delta_n} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. First, let us show that $P_{VI(\Omega, A_1)}(I - \mu A_2)$ is a contractive mapping. Indeed, by Lemma 4, we have

$$\begin{aligned} (1 - \tau)\|x - y\| &\geq \|(I - \mu A_2)x - (I - \mu A_2)y\| \\ &\geq \|P_{VI(\Omega, A_1)}(I - \mu A_2)x - P_{VI(\Omega, A_1)}(I - \mu A_2)y\|, \end{aligned}$$

for any $x, y \in C$, which implies that $P_{VI(\Omega, A_1)}(I - \mu A_2)$ is a contraction mapping. Banach’s Contraction Mapping Principle tell us that $P_{VI(\Omega, A_1)}(I - \mu A_2)$ has a fixed point and further it is unique. For example, $x^* \in C$, that is, $x^* = P_{VI(\Omega, A_1)}(I - \mu A_2)x^*$. Hence, by Lemma 8, we get

$$\{x^*\} = \text{Fix}(P_{VI(\Omega, A_1)}(I - \mu A_2)) = VI(VI(\Omega, A_1), A_2).$$

That is, Problem 1 has a unique solution. Taking into account that

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1,$$

we usually suppose $\{\gamma_n\} \subset [a, b] \subset (0, 1)$ for some $a, b \in (0, 1)$. Note that the mapping $G : C \rightarrow C$ is defined as $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, where $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Thus, by Lemma 6, we know that G is nonexpansive. It is easy to see that there exists an element $u_n \in C$ such that

$$u_n = \gamma_n x_n + (1 - \gamma_n)S_n u_n. \tag{11}$$

In fact, it is a unique element. Thus, we can consider the mapping

$$F_n x = \gamma_n x_n + (1 - \gamma_n)S_n x, \quad \forall x \in C.$$

Since $S_n : C \rightarrow C$ is a continuous pseudocontraction mapping, we deduce that all $x, y \in C$,

$$\langle F_n x - F_n y, x - y \rangle = (1 - \gamma_n)\langle S_n x - S_n y, x - y \rangle \leq (1 - \gamma_n)\|x - y\|^2.$$

In addition, from $\{\gamma_n\} \subset [a, b] \subset (0, 1)$ we get $0 < 1 - \gamma_n < 1$ for all $n \geq 0$. Thus, F_n is a continuous and strong pseudocontraction mapping of C into itself. By Proposition 2, we know that there exists a unique element $u_n \in C$, for each $n \geq 0$, satisfying (11). Thus, it can be readily seen that the hybrid Mann viscosity implicit iterative scheme (10) can be rewritten as

$$\left\{ \begin{array}{l} u_n = \gamma_n x_n + (1 - \gamma_n)S_n u_n, \\ z_n = Gu_n, \\ y_n = \sigma_n x_n + (1 - \sigma_n)P_C(I - \delta_n A_1)z_n, \\ x_{n+1} = (1 - \beta_n)P_C(I - \alpha_n \mu A_2)T^n y_n + \beta_n f(y_n), \quad \forall n \geq 0. \end{array} \right. \tag{12}$$

Next, we divide the rest of the proof into several steps.

Step 1. We claim that $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{v_n\}, \{T^n y_n\}$ and $\{A_2(T^n y_n)\}$ are bounded. Indeed, take an element $p \in \Omega = \bigcap_{n=0}^\infty \text{Fix}(S_n) \cap \text{GSVI}(C, B_1, B_2) \cap \text{Fix}(T)$ arbitrarily. Then, we have $S_n p = p$, $Gp = p$ and $Tp = p$. Since each $S_n : C \rightarrow C$ is a pseudocontraction mapping, it follows that

$$\begin{aligned} \|u_n - p\|^2 &= \gamma_n \langle x_n - p, u_n - p \rangle + (1 - \gamma_n)\langle S_n u_n - p, u_n - p \rangle \\ &\leq \gamma_n \|x_n - p\| \|u_n - p\| + (1 - \gamma_n)\|u_n - p\|^2, \end{aligned}$$

which hence yields

$$\|u_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 0. \tag{13}$$

Then, we get

$$\|z_n - p\| = \|Gu_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{14}$$

Since $1 > \limsup_{n \rightarrow \infty} \sigma_n \geq \liminf_{n \rightarrow \infty} \sigma_n > 0$, we reach $\{\sigma_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. In addition, since $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = 0$, we may assume, without loss of generality, that

$$\theta_n \leq \frac{\alpha_n(\tau - \delta)}{2} (\leq \alpha_n(\tau - \delta))$$

and $\beta_n \leq \alpha_n$ for all $n \geq 0$. Taking into account the ζ -inverse-strong monotonicity of A_1 with $\{\delta_n\} \subset (0, 2\zeta]$, we deduce from Lemma 5 and (14) that

$$\begin{aligned} \|y_n - p\| &\leq (1 - \sigma_n)\|P_C(I - \delta_n A_1)z_n - p\| + \sigma_n\|p - x_n\| \\ &\leq (1 - \sigma_n)\|(I - \delta_n A_1)z_n - (I - \delta_n A_1)p - \delta_n A_1 p\| + \sigma_n\|p - x_n\| \\ &\leq (1 - \sigma_n)(\|z_n - p\| + \delta_n\|A_1 p\|) + \sigma_n\|p - x_n\| \\ &\leq (1 - \sigma_n)\|x_n - p\| + \delta_n\|A_1 p\| + \sigma_n\|p - x_n\| \\ &= \|x_n - p\| + \delta_n\|A_1 p\|. \end{aligned} \tag{15}$$

Utilizing Lemma 4 and (15), we obtain from (12) that

$$\begin{aligned} &\|x_{n+1} - p\| \\ &\leq \beta_n\|f(y_n) - p\| + (1 - \beta_n)\|P_C(I - \alpha_n \mu A_2)T^n y_n - p\| \\ &\leq \alpha_n(\|f(y_n) - f(p)\| + \|p - f(p)\|) + \|(I - \alpha_n \mu A_2)T^n y_n - (I - \alpha_n \mu A_2)p - \alpha_n \mu A_2 p\| \\ &\leq \alpha_n(\delta\|p - y_n\| + \|p - f(p)\|) + (1 - \alpha_n \tau)\|T^n y_n - p\| + \alpha_n\|\mu A_2 p\| \\ &\leq \alpha_n(\delta\|p - y_n\| + \|p - f(p)\|) + (1 - \alpha_n \tau)(1 + \theta_n)\|y_n - p\| + \alpha_n\|\mu A_2 p\| \\ &\leq \alpha_n(\delta\|p - y_n\| + \|p - f(p)\|) + (1 - \alpha_n \tau + \theta_n)\|y_n - p\| + \alpha_n\|\mu A_2 p\| \\ &= [1 - \alpha_n(\tau - \delta) + \theta_n]\|y_n - p\| + \alpha_n(\|f(p) - p\| + \|\mu A_2 p\|) \\ &\leq [1 - \alpha_n(\tau - \delta) + \frac{\alpha_n(\tau - \delta)}{2}](\|x_n - p\| + \delta_n\|A_1 p\|) + \alpha_n(\|f(p) - p\| + \|\mu A_2 p\|) \\ &\leq [1 - \frac{\alpha_n(\tau - \delta)}{2}]\|x_n - p\| + \delta_n\|A_1 p\| + \alpha_n(\|f(p) - p\| + \|\mu A_2 p\|) \\ &\leq [1 - \frac{\alpha_n(\tau - \delta)}{2}]\|x_n - p\| + \alpha_n(\|A_1 p\| + \|\mu A_2 p\| + \|f(p) - p\|) \\ &= [1 - \frac{\alpha_n(\tau - \delta)}{2}]\|x_n - p\| + \frac{\alpha_n(\tau - \delta)}{2} \cdot \frac{2(\|A_1 p\| + \|\mu A_2 p\| + \|f(p) - p\|)}{\tau - \delta} \\ &\leq \max\{\|x_n - p\|, \frac{2(\|A_1 p\| + \|\mu A_2 p\| + \|f(p) - p\|)}{\tau - \delta}\}. \end{aligned}$$

By induction, we have

$$\|x_{n+1} - p\| \leq \max\{\|p - x_0\|, \frac{2(\|A_1 p\| + \|p - f(p)\| + \|\mu A_2 p\|)}{\tau - \delta}\}, \quad \forall n \geq 0.$$

It immediately follows that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}, \{z_n\}, \{u_n\}, \{T^n y_n\}$ and $\{A_2(T^n y_n)\}$ (due to (13)–(15) and the Lipschitz continuity of T and A_2). Taking into account that $\{S_n\}$ is ℓ -uniformly Lipschitzian on C , we know that

$$\|S_n u_n\| \leq \|S_n u_n - p\| + \|p\| \leq \ell\|u_n - p\| + \|p\|,$$

which implies that $\{S_n u_n\}$ is bounded. In addition, from Lemma 1 and $p \in \Omega \subset \text{GSVI}(C, B_1, B_2)$, it also follows that (p, q) is a solution of GSVI (4) where $q = P_C(I - \mu_2 B_2)p$. Note that $v_n = P_C(I - \mu_2 B_2)u_n$ for all $n \geq 0$. Then, by Lemma 5, we obtain

$$\begin{aligned} \|v_n\| &\leq \|v_n - q\| + \|q\| \\ &= \|P_C(I - \mu_2 B_2)u_n - P_C(I - \mu_2 B_2)p\| + \|q\| \\ &\leq \|(I - \mu_2 B_2)u_n - (I - \mu_2 B_2)p\| + \|q\| \\ &\leq \|q\| + \|p - u_n\|. \end{aligned}$$

This shows that $\{v_n\}$ is bounded.

Step 2. We claim that $\|x_{n+1} - x_n\| \rightarrow 0$ and $\|y_{n+1} - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Indeed, we set $p_n = P_C(I - \delta_n A_1)z_n$ and $q_n = P_C(I - \alpha_n \mu A_2)T^n y_n$. Then, from (12), we have

$$\begin{cases} u_n = \gamma_n x_n + (1 - \gamma_n)S_n u_n, \\ y_n = \sigma_n x_n + (1 - \sigma_n)p_n, \\ x_{n+1} = \beta_n f(y_n) + (1 - \beta_n)q_n. \end{cases}$$

Simple calculations show that

$$\begin{cases} u_n - u_{n-1} = \gamma_n(x_n - x_{n-1}) + (\gamma_n - \gamma_{n-1})(x_{n-1} - S_{n-1}u_{n-1}) + (1 - \gamma_n)(S_n u_n - S_{n-1}u_{n-1}), \\ y_n - y_{n-1} = \sigma_n(x_n - x_{n-1}) + (\sigma_n - \sigma_{n-1})(x_{n-1} - p_{n-1}) + (1 - \sigma_n)(p_n - p_{n-1}), \\ x_{n+1} - x_n = \beta_n(f(y_n) - f(y_{n-1})) + (\beta_n - \beta_{n-1})(f(y_{n-1}) - q_{n-1}) + (1 - \beta_n)(q_n - q_{n-1}). \end{cases} \tag{16}$$

It follows that

$$\begin{aligned} & \|u_n - u_{n-1}\|^2 \\ &= \gamma_n \langle x_n - x_{n-1}, u_n - u_{n-1} \rangle + (1 - \gamma_n) \langle S_n u_n - S_{n-1}u_{n-1}, u_n - u_{n-1} \rangle \\ & \quad + (\gamma_n - \gamma_{n-1}) \langle x_{n-1} - S_{n-1}u_{n-1}, u_n - u_{n-1} \rangle \\ &= \gamma_n \langle x_n - x_{n-1}, u_n - u_{n-1} \rangle + (1 - \gamma_n) [\langle S_n u_n - S_{n-1}u_{n-1}, u_n - u_{n-1} \rangle \\ & \quad + \langle S_{n-1}u_n - S_{n-1}u_{n-1}, u_n - u_{n-1} \rangle] + (\gamma_n - \gamma_{n-1}) \langle x_{n-1} - S_{n-1}u_{n-1}, u_n - u_{n-1} \rangle \\ &\leq \gamma_n \|x_{n-1} - x_n\| \|u_n - u_{n-1}\| + (1 - \gamma_n) [\|S_n u_n - S_{n-1}u_{n-1}\| \|u_n - u_{n-1}\| \\ & \quad + \|u_{n-1} - u_n\|^2] + |\gamma_n - \gamma_{n-1}| \|x_{n-1} - S_{n-1}u_{n-1}\| \|u_n - u_{n-1}\|, \end{aligned}$$

which hence yields

$$\|u_n - u_{n-1}\| \leq \gamma_n \|x_{n-1} - x_n\| + (1 - \gamma_n) [\|S_n u_n - S_{n-1}u_{n-1}\| + \|u_{n-1} - u_n\|] + |\gamma_n - \gamma_{n-1}| \|x_{n-1} - S_{n-1}u_{n-1}\|.$$

This immediately leads to

$$\begin{aligned} \|u_n - u_{n-1}\| &\leq \|x_{n-1} - x_n\| + \frac{1 - \gamma_n}{\gamma_n} \|S_n u_n - S_{n-1}u_{n-1}\| + |\gamma_n - \gamma_{n-1}| \frac{\|x_{n-1} - S_{n-1}u_{n-1}\|}{\gamma_n} \\ &\leq \|x_n - x_{n-1}\| + \frac{1}{a} \|S_n u_n - S_{n-1}u_{n-1}\| + |\gamma_n - \gamma_{n-1}| \frac{\|x_{n-1} - S_{n-1}u_{n-1}\|}{a}. \end{aligned} \tag{17}$$

Putting $D = \{u_n : n \geq 0\}$, we know that D is a bounded subset of C . Then, by the assumption, we get $\sum_{n=1}^\infty \sup_{x \in D} \|S_n x - S_{n-1}x\| < \infty$. Noticing $\|S_n u_n - S_{n-1}u_{n-1}\| \leq \sup_{x \in D} \|S_n x - S_{n-1}x\| \forall n \geq 1$, we have

$$\sum_{n=1}^\infty \|S_n u_n - S_{n-1}u_{n-1}\| < \infty. \tag{18}$$

In addition, from $p_n = P_C(I - \delta_n A_1)z_n$ and $\{\delta_n\} \subset (0, 2\zeta]$, we observe that

$$\begin{aligned} \|p_n - p_{n-1}\| &\leq \|(I - \delta_n A_1)z_n - (I - \delta_{n-1} A_1)z_{n-1}\| \\ &= \|(I - \delta_n A_1)z_n - (I - \delta_n A_1)z_{n-1} - (\delta_n - \delta_{n-1})A_1 z_{n-1}\| \\ &\leq \|(I - \delta_n A_1)z_n - (I - \delta_n A_1)z_{n-1}\| + |\delta_n - \delta_{n-1}| \|A_1 z_{n-1}\| \\ &\leq \|z_n - z_{n-1}\| + |\delta_n - \delta_{n-1}| \|A_1 z_{n-1}\| \\ &\leq \|u_n - u_{n-1}\| + M_0 |\delta_n - \delta_{n-1}|, \end{aligned} \tag{19}$$

where $\sup_{n \geq 0} \|A_1 z_n\| \leq M_0$ for some $M_0 > 0$. Thus, from (16), (17) and (19), we get

$$\begin{aligned}
 & \|y_n - y_{n-1}\| \\
 & \leq \sigma_n \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|x_{n-1} - p_{n-1}\| + (1 - \sigma_n) \|p_n - p_{n-1}\| \\
 & \leq \sigma_n \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|x_{n-1} - p_{n-1}\| + (1 - \sigma_n) (\|u_n - u_{n-1}\| + M_0 |\delta_n - \delta_{n-1}|) \\
 & \leq \sigma_n \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|x_{n-1} - p_{n-1}\| + (1 - \sigma_n) (\|x_n - x_{n-1}\| \\
 & \quad + \frac{1}{a} \|S_n u_n - S_{n-1} u_n\| + |\gamma_n - \gamma_{n-1}| \frac{\|x_{n-1} - S_{n-1} u_{n-1}\|}{a} + M_0 |\delta_n - \delta_{n-1}|) \\
 & \leq \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|x_{n-1} - p_{n-1}\| + \frac{1}{a} \|S_n u_n - S_{n-1} u_n\| \\
 & \quad + |\gamma_n - \gamma_{n-1}| \frac{\|x_{n-1} - S_{n-1} u_{n-1}\|}{a} + M_0 |\delta_n - \delta_{n-1}| \\
 & \leq \|x_n - x_{n-1}\| + M (|\sigma_n - \sigma_{n-1}| + \|S_n u_n - S_{n-1} u_n\| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|),
 \end{aligned} \tag{20}$$

where $\sup_{n \geq 0} \{\|x_n - p_n\| + \frac{1}{a} + \frac{\|x_n - S_n u_n\|}{a} + M_0\} \leq M$ for some $M > 0$.

Furthermore, from $q_n = P_C(I - \alpha_n \mu A_2) T^n y_n$ and Lemma 4, we note that

$$\begin{aligned}
 & \|q_n - q_{n-1}\| \\
 & \leq \|(I - \alpha_n \mu A_2) T^n y_n - (I - \alpha_{n-1} \mu A_2) T^{n-1} y_{n-1}\| \\
 & = \|(I - \alpha_n \mu A_2) T^n y_n - (I - \alpha_n \mu A_2) T^{n-1} y_{n-1} - (\alpha_n - \alpha_{n-1}) \mu A_2 T^{n-1} y_{n-1}\| \\
 & \leq (1 - \alpha_n \tau) \|T^n y_n - T^n y_{n-1} + T^n y_{n-1} - T^{n-1} y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|\mu A_2 T^{n-1} y_{n-1}\| \\
 & \leq (1 - \alpha_n \tau) (\|y_n - y_{n-1}\| + \|T^n y_{n-1} - T^{n-1} y_{n-1}\|) + |\alpha_n - \alpha_{n-1}| \|\mu A_2 T^{n-1} y_{n-1}\| \\
 & \leq (1 - \alpha_n \tau + \theta_n) \|y_n - y_{n-1}\| + \|T^n y_{n-1} - T^{n-1} y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|\mu A_2 T^{n-1} y_{n-1}\|.
 \end{aligned} \tag{21}$$

Hence, from (16), (20) and (21), we get

$$\begin{aligned}
 & \|x_{n+1} - x_n\| \\
 & \leq \beta_n \|f(y_n) - f(y_{n-1})\| + |\beta_n - \beta_{n-1}| \|f(y_{n-1}) - q_{n-1}\| + (1 - \beta_n) \|q_n - q_{n-1}\| \\
 & \leq \alpha_n \delta \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \|f(y_{n-1}) - q_{n-1}\| + (1 - \alpha_n \tau + \theta_n) \|y_n - y_{n-1}\| \\
 & \quad + \|T^n y_{n-1} - T^{n-1} y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|\mu A_2 T^{n-1} y_{n-1}\| \\
 & \leq [1 - \alpha_n (\tau - \delta) + \frac{\alpha_n (\tau - \delta)}{2}] \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \|f(y_{n-1}) - q_{n-1}\| \\
 & \quad + \|T^n y_{n-1} - T^{n-1} y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|\mu A_2 T^{n-1} y_{n-1}\| \\
 & \leq (1 - \frac{\alpha_n (\tau - \delta)}{2}) (\|x_n - x_{n-1}\| + M (|\sigma_n - \sigma_{n-1}| + \|S_n u_n - S_{n-1} u_n\| \\
 & \quad + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|)) + |\beta_n - \beta_{n-1}| \|f(y_{n-1}) - q_{n-1}\| \\
 & \quad + \|T^n y_{n-1} - T^{n-1} y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|\mu A_2 T^{n-1} y_{n-1}\| \\
 & \leq (1 - \frac{\alpha_n (\tau - \delta)}{2}) \|x_n - x_{n-1}\| + M_1 (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \\
 & \quad + |\delta_n - \delta_{n-1}| + |\sigma_n - \sigma_{n-1}| + \|S_n u_n - S_{n-1} u_n\|) + \|T^n y_{n-1} - T^{n-1} y_{n-1}\|,
 \end{aligned} \tag{22}$$

where $\sup_{n \geq 1} \{M + \|f(y_{n-1}) - q_{n-1}\| + \|\mu A_2 T^{n-1} y_{n-1}\|\} \leq M_1$ for some $M_1 > 0$. From (18) and Conditions (i)–(v), we know that $\sum_{n=0}^{\infty} \frac{\alpha_n (\tau - \delta)}{2} = \infty$ and

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \{M_1 (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}| \\
 & \quad + |\sigma_n - \sigma_{n-1}| + \|S_n u_n - S_{n-1} u_n\|) + \|T^n y_{n-1} - T^{n-1} y_{n-1}\|\} < \infty.
 \end{aligned}$$

Consequently, applying Lemma 3 to (22), we obtain that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{23}$$

In terms of (18), (23) and Conditions (ii)–(iv), we deduce from (20) that

$$\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0.$$

Step 3. We claim that $\|x_n - Gx_n\| \rightarrow 0$ as $n \rightarrow \infty$. Indeed, noticing $q_n = P_C(I - \alpha_n \mu A_2)T^n y_n$ for all $n \geq 0$, we obtain from (7) that for each $p \in \Omega$,

$$\langle (I - \alpha_n \mu A_2)T^n y_n - P_C(I - \alpha_n \mu A_2)T^n y_n, p - q_n \rangle \leq 0,$$

which hence leads to

$$\begin{aligned} \|q_n - p\|^2 &= \langle P_C(I - \alpha_n \mu A_2)T^n y_n - (I - \alpha_n \mu A_2)T^n y_n, q_n - p \rangle \\ &\quad + \langle (I - \alpha_n \mu A_2)T^n y_n - p, q_n - p \rangle \\ &\leq \langle (I - \alpha_n \mu A_2)T^n y_n - p, q_n - p \rangle \\ &= \langle (I - \alpha_n \mu A_2)T^n y_n - (I - \alpha_n \mu A_2)p, q_n - p \rangle - \alpha_n \langle \mu A_2 p, q_n - p \rangle \\ &\leq \frac{1}{2}(1 - \alpha_n \tau)^2 \|T^n y_n - p\|^2 + \frac{1}{2} \|q_n - p\|^2 - \alpha_n \langle \mu A_2 p, q_n - p \rangle. \end{aligned}$$

It follows from (1) that

$$\begin{aligned} \|q_n - p\|^2 &\leq (1 - \alpha_n \tau)(1 + \theta_n)^2 \|y_n - p\|^2 - 2\alpha_n \langle \mu A_2 p, q_n - p \rangle \\ &\leq (1 - \alpha_n \tau) \|y_n - p\|^2 + \theta_n(2 + \theta_n) \|y_n - p\|^2 - 2\alpha_n \langle \mu A_2 p, q_n - p \rangle. \end{aligned} \tag{24}$$

From (15) and (24), we get

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\leq \beta_n \|f(y_n) - f(p)\|^2 + (1 - \beta_n) \|q_n - p\|^2 + 2\beta_n \langle f(p) - p, x_{n+1} - p \rangle \\ &\leq \alpha_n \delta \|y_n - p\|^2 + (1 - \alpha_n \tau) \|y_n - p\|^2 + \theta_n(2 + \theta_n) \|y_n - p\|^2 \\ &\quad - 2\alpha_n \langle \mu A_2 p, q_n - p \rangle + 2\beta_n \langle f(p) - p, x_{n+1} - p \rangle \\ &= [1 - \alpha_n(\tau - \delta)] \|y_n - p\|^2 + \theta_n(2 + \theta_n) \|y_n - p\|^2 - 2\alpha_n \langle \mu A_2 p, q_n - p \rangle \\ &\quad + 2\beta_n \langle f(p) - p, x_{n+1} - p \rangle \\ &\leq [1 - \alpha_n(\tau - \delta)] [\sigma_n \|x_n - p\| + (1 - \sigma_n)(\|z_n - p\| + \delta_n \|A_1 p\|)]^2 \\ &\quad + \theta_n(2 + \theta_n) \|y_n - p\|^2 - 2\alpha_n \langle \mu A_2 p, q_n - p \rangle + 2\beta_n \langle f(p) - p, x_{n+1} - p \rangle \\ &\leq [1 - \alpha_n(\tau - \delta)] [\sigma_n \|x_n - p\|^2 + (1 - \sigma_n)(\|z_n - p\| + \delta_n \|A_1 p\|)^2] \\ &\quad + \theta_n(2 + \theta_n) \|y_n - p\|^2 - 2\alpha_n \langle \mu A_2 p, q_n - p \rangle + 2\beta_n \langle f(p) - p, x_{n+1} - p \rangle \\ &\leq [1 - \alpha_n(\tau - \delta)] [\sigma_n \|x_n - p\|^2 + (1 - \sigma_n) \|z_n - p\|^2 + \delta_n \|A_1 p\| (2\|z_n - p\| + \delta_n \|A_1 p\|)] \\ &\quad + \theta_n(2 + \theta_n) \|y_n - p\|^2 - 2\alpha_n \langle \mu A_2 p, q_n - p \rangle + 2\beta_n \langle f(p) - p, x_{n+1} - p \rangle \\ &\leq [1 - \alpha_n(\tau - \delta)] [\sigma_n \|x_n - p\|^2 + (1 - \sigma_n) \|z_n - p\|^2] + \alpha_n \|A_1 p\| (2\|z_n - p\| + \alpha_n \|A_1 p\|) \\ &\quad + \theta_n(2 + \theta_n) \|y_n - p\|^2 + 2\alpha_n (\| \mu A_2 p \| \|q_n - p\| + \|f(p) - p\| \|x_{n+1} - p\|). \end{aligned} \tag{25}$$

We now note that $q = P_C(p - \mu_2 B_2 p)$, $v_n = P_C(u_n - \mu_2 B_2 u_n)$ and $z_n = P_C(v_n - \mu_1 B_1 v_n)$. Then $z_n = Gu_n$. By Lemma 5, we have

$$\begin{aligned} \|v_n - q\|^2 &= \|P_C(u_n - \mu_2 B_2 u_n) - P_C(p - \mu_2 B_2 p)\|^2 \\ &\leq \|u_n - p - \mu_2(B_2 u_n - B_2 p)\|^2 \\ &\leq \|u_n - p\|^2 - \mu_2(2\beta - \mu_2) \|B_2 u_n - B_2 p\|^2 \end{aligned} \tag{26}$$

and

$$\begin{aligned} \|z_n - p\|^2 &= \|P_C(v_n - \mu_1 B_1 v_n) - P_C(q - \mu_1 B_1 q)\|^2 \\ &\leq \|v_n - q - \mu_1(B_1 v_n - B_1 q)\|^2 \\ &\leq \|v_n - q\|^2 - \mu_1(2\alpha - \mu_1) \|B_1 v_n - B_1 q\|^2. \end{aligned} \tag{27}$$

Substituting (26) for (27), we obtain from (13) that

$$\begin{aligned} \|z_n - p\|^2 &\leq \|u_n - p\|^2 - \mu_2(2\beta - \mu_2) \|B_2 u_n - B_2 p\|^2 - \mu_1(2\alpha - \mu_1) \|B_1 v_n - B_1 q\|^2 \\ &\leq \|p - x_n\|^2 - \mu_2(2\beta - \mu_2) \|B_2 u_n - B_2 p\|^2 - \mu_1(2\alpha - \mu_1) \|B_1 v_n - B_1 q\|^2. \end{aligned} \tag{28}$$

Combining (25) and (28), we get

$$\begin{aligned}
 & \|x_{n+1} - p\|^2 \\
 & \leq [1 - \alpha_n(\tau - \delta)][\sigma_n\|p - x_n\|^2 + (1 - \sigma_n)\|p - z_n\|^2] + \alpha_n\|A_1p\|(2\|p - z_n\| + \alpha_n\|A_1p\|) \\
 & \quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\
 & \leq [1 - \alpha_n(\tau - \delta)]\{\sigma_n\|p - x_n\|^2 + (1 - \sigma_n)[\|p - x_n\|^2 - \mu_2(2\beta - \mu_2)\|B_2u_n - B_2p\|^2 \\
 & \quad - \mu_1(2\alpha - \mu_1)\|B_1v_n - B_1q\|^2]\} + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\
 & \quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|p - q_n\| + \|p - f(p)\|\|x_{n+1} - p\|) \\
 & \leq [1 - \alpha_n(\tau - \delta)]\|x_n - p\|^2 - [1 - \alpha_n(\tau - \delta)](1 - \sigma_n)[\mu_2(2\beta - \mu_2)\|B_2u_n - B_2p\|^2 \\
 & \quad + \mu_1(2\alpha - \mu_1)\|B_1v_n - B_1q\|^2] + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\
 & \quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\
 & \leq \|x_n - p\|^2 - [1 - \alpha_n(\tau - \delta)](1 - \sigma_n)[\mu_2(2\beta - \mu_2)\|B_2u_n - B_2p\|^2 \\
 & \quad + \mu_1(2\alpha - \mu_1)\|B_1v_n - B_1q\|^2] + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\
 & \quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|),
 \end{aligned}$$

which immediately yields

$$\begin{aligned}
 & [1 - \alpha_n(\tau - \delta)](1 - \sigma_n)[\mu_2(2\beta - \mu_2)\|B_2u_n - B_2p\|^2 + \mu_1(2\alpha - \mu_1)\|B_1v_n - B_1q\|^2] \\
 & \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\
 & \quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\
 & \leq (\|p - x_n\| + \|p - x_{n+1}\|)\|x_n - x_{n+1}\| + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\
 & \quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|).
 \end{aligned}$$

Since $\liminf_{n \rightarrow \infty} (1 - \sigma_n) > 0$ (due to Condition (iii)), $\mu_1 \in (0, 2\alpha)$, $\mu_2 \in (0, 2\beta)$, $\lim_{n \rightarrow \infty} \theta_n = 0$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we obtain from (23) that

$$\lim_{n \rightarrow \infty} \|B_2u_n - B_2p\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|B_1v_n - B_1q\| = 0. \tag{29}$$

Additionally, from (6) and (9), we have

$$\begin{aligned}
 \|v_n - q\|^2 & = \|P_C(u_n - \mu_2 B_2u_n) - P_C(p - \mu_2 B_2p)\|^2 \\
 & \leq \langle u_n - \mu_2 B_2u_n - (p - \mu_2 B_2p), v_n - q \rangle \\
 & = \langle u_n - p, v_n - q \rangle + \mu_2 \langle B_2p - B_2u_n, v_n - q \rangle \\
 & \leq \frac{1}{2} [\|u_n - p\|^2 + \|v_n - q\|^2 - \|u_n - v_n - (p - q)\|^2] + \mu_2 \|B_2p - B_2u_n\| \|v_n - q\|,
 \end{aligned}$$

which implies that

$$\|p - v_n\|^2 \leq \|p - u_n\|^2 - \|u_n - v_n - (p - q)\|^2 + 2\mu_2 \|B_2p - B_2u_n\| \|v_n - q\|. \tag{30}$$

In the same way, we derive

$$\begin{aligned}
 \|p - z_n\|^2 & = \|P_C(v_n - \mu_1 B_1v_n) - P_C(q - \mu_1 B_1q)\|^2 \\
 & \leq \langle v_n - \mu_1 B_1v_n - (q - \mu_1 B_1q), z_n - p \rangle \\
 & = \langle v_n - q, z_n - p \rangle + \mu_1 \langle B_1q - B_1v_n, z_n - p \rangle \\
 & \leq \frac{1}{2} [\|v_n - q\|^2 + \|z_n - p\|^2 - \|v_n - z_n + (p - q)\|^2] + \mu_1 \|B_1q - B_1v_n\| \|z_n - p\|,
 \end{aligned}$$

which implies that

$$\|z_n - p\|^2 \leq \|v_n - q\|^2 - \|v_n - z_n + (p - q)\|^2 + 2\mu_1 \|B_1q - B_1v_n\| \|z_n - p\|. \tag{31}$$

Substituting (3) for (31), we deduce from (13) that

$$\begin{aligned} \|p - z_n\|^2 &\leq \|p - u_n\|^2 - \|u_n - v_n - (p - q)\|^2 - \|v_n - z_n + (p - q)\|^2 \\ &\quad + 2\mu_2\|B_2p - B_2u_n\|\|v_n - q\| + 2\mu_1\|B_1q - B_1v_n\|\|z_n - p\| \\ &\leq \|p - x_n\|^2 - \|u_n - v_n - (p - q)\|^2 - \|v_n - z_n + (p - q)\|^2 \\ &\quad + 2\mu_1\|B_1q - B_1v_n\|\|z_n - p\| + 2\mu_2\|B_2p - B_2u_n\|\|v_n - q\|. \end{aligned} \tag{32}$$

Combining (25) and (32), we have

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\leq [1 - \alpha_n(\tau - \delta)][\sigma_n\|x_n - p\|^2 + (1 - \sigma_n)\|z_n - p\|^2] + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\ &\quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\ &\leq [1 - \alpha_n(\tau - \delta)]\{\sigma_n\|x_n - p\|^2 + (1 - \sigma_n)[\|x_n - p\|^2 - \|u_n - v_n - (p - q)\|^2 \\ &\quad - \|v_n - z_n + (p - q)\|^2 + 2\mu_1\|B_1q - B_1v_n\|\|z_n - p\| + 2\mu_2\|B_2p - B_2u_n\|\|v_n - q\|]\} \\ &\quad + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) + \theta_n(2 + \theta_n)\|y_n - p\|^2 \\ &\quad + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\ &\leq [1 - \alpha_n(\tau - \delta)]\|x_n - p\|^2 - [1 - \alpha_n(\tau - \delta)](1 - \sigma_n)[\|u_n - v_n - (p - q)\|^2 \\ &\quad + \|v_n - z_n + (p - q)\|^2] + 2\mu_1\|B_1q - B_1v_n\|\|z_n - p\| + 2\mu_2\|B_2p - B_2u_n\|\|v_n - q\| \\ &\quad + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) + \theta_n(2 + \theta_n)\|y_n - p\|^2 \\ &\quad + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\ &\leq \|x_n - p\|^2 - [1 - \alpha_n(\tau - \delta)](1 - \sigma_n)[\|u_n - v_n - (p - q)\|^2 + \|v_n - z_n + (p - q)\|^2] \\ &\quad + 2\mu_1\|B_1q - B_1v_n\|\|z_n - p\| + 2\mu_2\|B_2p - B_2u_n\|\|v_n - q\| + \alpha_n\|A_1p\|(2\|z_n - p\| \\ &\quad + \alpha_n\|A_1p\|) + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|), \end{aligned}$$

which hence yields

$$\begin{aligned} &[1 - \alpha_n(\tau - \delta)](1 - \sigma_n)[\|u_n - v_n - (p - q)\|^2 + \|v_n - z_n + (p - q)\|^2] \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\mu_1\|B_1q - B_1v_n\|\|z_n - p\| \\ &\quad + 2\mu_2\|B_2p - B_2u_n\|\|v_n - q\| + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\ &\quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\ &\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + 2\mu_1\|B_1q - B_1v_n\|\|z_n - p\| \\ &\quad + 2\mu_2\|B_2p - B_2u_n\|\|v_n - q\| + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\ &\quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|). \end{aligned}$$

Since $\liminf_{n \rightarrow \infty} (1 - \sigma_n) > 0$ (due to Condition (iii)), $\lim_{n \rightarrow \infty} \theta_n = 0$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we conclude from (23) and (29) that

$$\lim_{n \rightarrow \infty} \|u_n - v_n - (p - q)\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|v_n - z_n + (p - q)\| = 0. \tag{33}$$

It follows that

$$\|u_n - z_n\| \leq \|u_n - v_n - (p - q)\| + \|v_n - z_n + (p - q)\| \rightarrow 0 \quad (n \rightarrow \infty).$$

That is,

$$\lim_{n \rightarrow \infty} \|u_n - Gu_n\| = \lim_{n \rightarrow \infty} \|u_n - z_n\| = 0. \tag{34}$$

In addition, according to (12), we have

$$\begin{aligned} \|p - u_n\|^2 &= \gamma_n \langle p - x_n, p - u_n \rangle + (1 - \gamma_n) \langle S_n u_n - p, u_n - p \rangle \\ &\leq \gamma_n \langle x_n - p, u_n - p \rangle + (1 - \gamma_n) \|u_n - p\|^2, \end{aligned}$$

which, together with (9), yields

$$\begin{aligned} \|p - u_n\|^2 &\leq \langle x_n - p, u_n - p \rangle \\ &= \frac{1}{2}[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2]. \end{aligned}$$

This immediately implies that

$$\|p - u_n\|^2 \leq \|p - x_n\|^2 - \|u_n - x_n\|^2,$$

which together with (14) and (26), yields

$$\begin{aligned} &\|p - x_{n+1}\|^2 \\ &\leq [1 - \alpha_n(\tau - \delta)][\sigma_n\|p - x_n\|^2 + (1 - \sigma_n)\|p - u_n\|^2] + \alpha_n\|A_1p\|(2\|p - z_n\| + \alpha_n\|A_1p\|) \\ &\quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\ &\leq [1 - \alpha_n(\tau - \delta)]\{\sigma_n\|x_n - p\|^2 + (1 - \sigma_n)[\|x_n - p\|^2 - \|x_n - u_n\|^2]\} \\ &\quad + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) + \theta_n(2 + \theta_n)\|y_n - p\|^2 \\ &\quad + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\ &\leq [1 - \alpha_n(\tau - \delta)]\|x_n - p\|^2 - [1 - \alpha_n(\tau - \delta)](1 - \sigma_n)\|x_n - u_n\|^2 \\ &\quad + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) + \theta_n(2 + \theta_n)\|y_n - p\|^2 \\ &\quad + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\ &\leq \|x_n - p\|^2 - [1 - \alpha_n(\tau - \delta)](1 - \sigma_n)\|x_n - u_n\|^2 + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\ &\quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|). \end{aligned}$$

Hence, we have

$$\begin{aligned} &[1 - \alpha_n(\tau - \delta)](1 - \sigma_n)\|x_n - u_n\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\ &\quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|) \\ &\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n\|A_1p\|(2\|z_n - p\| + \alpha_n\|A_1p\|) \\ &\quad + \theta_n(2 + \theta_n)\|y_n - p\|^2 + 2\alpha_n(\|\mu A_2p\|\|q_n - p\| + \|f(p) - p\|\|x_{n+1} - p\|). \end{aligned}$$

Since $\liminf_{n \rightarrow \infty} (1 - \sigma_n) > 0$ (due to Condition (iii)), $\lim_{n \rightarrow \infty} \theta_n = 0$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we obtain from (23) that

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \tag{35}$$

In addition, observe that

$$\|x_n - z_n\| \leq \|x_n - u_n\| + \|u_n - Gu_n\|,$$

$$\|x_n - Gx_n\| \leq \|x_n - z_n\| + \|Gu_n - Gx_n\| \leq \|x_n - z_n\| + \|u_n - x_n\|,$$

and

$$\|x_n - y_n\| \leq (1 - \sigma_n)\|x_n - (I - \delta_n A_1)z_n\| \leq \|x_n - z_n\| + \alpha_n\|A_1z_n\|.$$

Then, from (34) and (35), it follows that

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0, \quad \lim_{n \rightarrow \infty} \|x_n - Gx_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \tag{36}$$

Step 4. We claim that $\|x_n - S_n x_n\| \rightarrow 0$, $\|x_n - q_n\| \rightarrow 0$ and $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$. Indeed, combining (11) and (25), we obtain that

$$\|S_n u_n - u_n\| = \frac{\gamma_n}{1 - \gamma_n} \|x_n - u_n\| \leq \frac{b}{1 - b} \|x_n - u_n\| \rightarrow 0 \quad (n \rightarrow \infty).$$

That is,

$$\lim_{n \rightarrow \infty} \|S_n u_n - u_n\| = 0. \tag{37}$$

Since $\{S_n\}_{n=0}^\infty$ is ℓ -uniformly Lipschitzian on C , we deduce from (35) and (37) that

$$\begin{aligned} \|S_n x_n - x_n\| &\leq \|S_n x_n - S_n u_n\| + \|S_n u_n - u_n\| + \|u_n - x_n\| \\ &\leq \ell \|x_n - u_n\| + \|S_n u_n - u_n\| + \|u_n - x_n\| \\ &= (\ell + 1) \|x_n - u_n\| + \|S_n u_n - u_n\| \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

That is,

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \tag{38}$$

In addition, we observe that

$$\begin{aligned} \|x_n - T^n y_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^n y_n\| \\ &\leq \|x_n - x_{n+1}\| + \beta_n \|f(y_n) - T^n y_n\| + (1 - \beta_n) \|(I - \alpha_n \mu A_2) T^n y_n - T^n y_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n (\|f(y_n) - T^n y_n\| + \|\mu A_2(T^n y_n)\|). \end{aligned}$$

Hence, we get

$$\begin{aligned} \|y_n - T^n y_n\| &\leq \|y_n - x_n\| + \|x_n - T^n y_n\| \\ &\leq \|y_n - x_n\| + \|x_n - x_{n+1}\| + \alpha_n (\|f(y_n) - T^n y_n\| + \|\mu A_2(T^n y_n)\|). \end{aligned}$$

Consequently, from (23), (36) and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we obtain that

$$\lim_{n \rightarrow \infty} \|x_n - T^n y_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|y_n - T^n y_n\| = 0. \tag{39}$$

Thus, it follows that

$$\begin{aligned} \|x_n - q_n\| &\leq \|x_n - (I - \alpha_n \mu A_2) T^n y_n\| \\ &\leq \|x_n - T^n y_n\| + \alpha_n \|\mu A_2(T^n y_n)\| \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

That is,

$$\lim_{n \rightarrow \infty} \|x_n - q_n\| = 0. \tag{40}$$

We also note that

$$\begin{aligned} \|y_n - T y_n\| &\leq \|y_n - T^n y_n\| + \|T^n y_n - T^{n+1} y_n\| + \|T^{n+1} y_n - T y_n\| \\ &\leq \|y_n - T^n y_n\| + \|T^n y_n - T^{n+1} y_n\| + (1 + \theta_1) \|T^n y_n - y_n\| \\ &= \|T^n y_n - T^{n+1} y_n\| + (2 + \theta_1) \|T^n y_n - y_n\|. \end{aligned}$$

By Condition (v) and (39), we get

$$\lim_{n \rightarrow \infty} \|y_n - T y_n\| = 0.$$

In addition, noticing that

$$\begin{aligned} \|x_n - T x_n\| &\leq \|x_n - y_n\| + \|y_n - T y_n\| + \|T y_n - T x_n\| \\ &\leq \|y_n - T y_n\| + (2 + \theta_1) \|x_n - y_n\|, \end{aligned}$$

we deduce from (36) that

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0. \tag{41}$$

Step 5. We claim that $\|x_n - \bar{S} x_n\| \rightarrow 0$ as $n \rightarrow \infty$ where $\bar{S} := (2I - S)^{-1}$. Indeed, first, let us show that $S : C \rightarrow C$ is pseudocontractive and ℓ -Lipschitzian such that $\lim_{n \rightarrow \infty} \|S x_n - x_n\| = 0$

where $Sx = \lim_{n \rightarrow \infty} S_n x \ \forall x \in C$. Observe that for all $x, y \in C$, $\lim_{n \rightarrow \infty} \|S_n x - Sx\| = 0$ and $\lim_{n \rightarrow \infty} \|S_n y - Sy\| = 0$. Since each S_n is pseudocontractive, we get

$$\langle Sx - Sy, x - y \rangle = \lim_{n \rightarrow \infty} \langle S_n x - S_n y, x - y \rangle \leq \|x - y\|^2.$$

This means that S is pseudocontractive. Noting that $\{S_n\}_{n=0}^\infty$ is ℓ -uniformly Lipschitzian on C , we have

$$\|Sx - Sy\| = \lim_{n \rightarrow \infty} \|S_n x - S_n y\| \leq \ell \|x - y\|, \quad \forall x, y \in C.$$

This means that S is ℓ -Lipschitzian. Taking into account the boundedness of $\{x_n\}$ and putting $D = \overline{\text{conv}}\{x_n : n \geq 0\}$ (the closure of convex hull of the set $\{x_n : n \geq 0\}$), by Assumption (C6) we have $\sum_{n=1}^\infty \sup_{x \in D} \|S_n x - S_{n-1} x\| < \infty$. Hence, by Proposition 1, we get

$$\limsup_{n \rightarrow \infty} \sup_{x \in D} \|S_n x - Sx\| = 0,$$

which immediately yields

$$\lim_{n \rightarrow \infty} \|S_n x_n - Sx_n\| = 0. \tag{42}$$

Thus, combining (38) with (42), we have

$$\|x_n - Sx_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \rightarrow 0 \quad (n \rightarrow \infty).$$

That is,

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0. \tag{43}$$

Now, let us show that. if we define $\bar{S} := (2I - S)^{-1}$, then $\bar{S} : C \rightarrow C$ is nonexpansive, $\text{Fix}(\bar{S}) = \text{Fix}(S) = \bigcap_{n=0}^\infty \text{Fix}(S_n)$ and $\lim_{n \rightarrow \infty} \|x_n - \bar{S}x_n\| = 0$. Indeed, put $\bar{S} := (2I - S)^{-1}$, where I is the identity mapping of H . Then, it is known that \bar{S} is nonexpansive and the fixed point set $\text{Fix}(\bar{S}) = \text{Fix}(S) = \bigcap_{n=0}^\infty \text{Fix}(S_n)$. From (43), it follows that

$$\begin{aligned} \|x_n - \bar{S}x_n\| &= \|\bar{S}\bar{S}^{-1}x_n - \bar{S}x_n\| \\ &\leq \|\bar{S}^{-1}x_n - x_n\| \\ &= \|(2I - S)x_n - x_n\| = \|x_n - Sx_n\| \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

That is,

$$\lim_{n \rightarrow \infty} \|x_n - \bar{S}x_n\| = 0. \tag{44}$$

Step 6. We claim that

$$\limsup_{n \rightarrow \infty} \langle A_2 x^*, x^* - q_n \rangle \leq 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \langle A_1 x^*, x^* - z_n \rangle \leq 0, \tag{45}$$

where $\{x^*\} = \text{VI}(\text{VI}(\Omega, A_1), A_2)$. Indeed, we fix sequence $\{q_{n_i}\}$ of $\{q_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle A_2 x^*, x^* - q_n \rangle = \lim_{i \rightarrow \infty} \langle A_2 x^*, x^* - q_{n_i} \rangle.$$

Since $\{q_n\}$ is a bounded sequence in C , we may assume, without loss of generality, that $q_{n_i} \rightarrow \bar{x} \in C$. Since $\lim_{n \rightarrow \infty} \|x_n - q_n\| = 0$ (due to (40)), it follows from $q_{n_i} \rightarrow \bar{x}$ that $x_{n_i} \rightarrow \bar{x}$.

Note that G and \bar{S} are nonexpansive and that T is asymptotically nonexpansive. Since $(I - G)x_n \rightarrow 0$, $(I - T)x_n \rightarrow 0$ and $(I - \bar{S})x_n \rightarrow 0$ (due to (36), (41) and (44)), by Lemma 7 we have that $\bar{x} \in \text{Fix}(G) = \text{GSVI}(C, B_1, B_2)$, $\bar{x} \in \text{Fix}(T)$ and $\bar{x} \in \text{Fix}(\bar{S}) = \bigcap_{n=0}^\infty \text{Fix}(S_n)$. Then, $\bar{x} \in \Omega =$

$\bigcap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{GSVI}(C, B_1, B_2) \cap \text{Fix}(T)$. We claim that $\bar{x} \in \text{VI}(\Omega, A_1)$. In fact, let $y \in \Omega$ be fixed arbitrarily. Then, it follows from (12), (14) and the ζ -inverse-strong monotonicity of A_1 that

$$\begin{aligned} \|y_n - y\|^2 &\leq \sigma_n \|x_n - y\|^2 + (1 - \sigma_n) \|P_C(z_n - \delta_n A_1 z_n) - P_C y\|^2 \\ &\leq \sigma_n \|x_n - y\|^2 + (1 - \sigma_n) \|(z_n - y) - \delta_n A_1 z_n\|^2 \\ &= \sigma_n \|x_n - y\|^2 + (1 - \sigma_n) [\|z_n - y\|^2 + 2\delta_n \langle A_1 z_n, y - z_n \rangle + \delta_n^2 \|A_1 z_n\|^2] \\ &\leq \|x_n - y\|^2 + (1 - \sigma_n) [2\delta_n \langle A_1 y, y - z_n \rangle + \delta_n^2 \|A_1 z_n\|^2], \end{aligned}$$

which together with $\{\sigma_n\} \subset [c, d]$, implies that for all $n \geq 0$,

$$\begin{aligned} 0 &\leq \frac{1}{\delta_n(1-\sigma_n)} (\|x_n - y\|^2 - \|y_n - y\|^2) + 2\langle A_1 y, y - z_n \rangle + \frac{\delta_n}{1-\sigma_n} \|A_1 z_n\|^2 \\ &\leq (\|x_n - y\| + \|y_n - y\|) \frac{\|x_n - y_n\|}{\delta_n(1-d)} + 2\langle A_1 y, y - z_n \rangle + \frac{\delta_n}{1-d} \|A_1 z_n\|^2. \end{aligned}$$

From (36) it is easy to see that $x_{n_i} \rightarrow \bar{x}$ leads to $z_{n_i} \rightarrow \bar{x}$. Since $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\|x_n - y_n\| = o(\delta_n)$ (due to the assumption), we have

$$\begin{aligned} 0 &\leq \liminf_{n \rightarrow \infty} \{ (\|x_n - y\| + \|y_n - y\|) \frac{\|x_n - y_n\|}{\delta_n(1-d)} + 2\langle A_1 y, y - z_n \rangle + \frac{\delta_n}{1-d} \|A_1 z_n\|^2 \} \\ &= \liminf_{n \rightarrow \infty} 2\langle A_1 y, y - z_n \rangle \leq \lim_{i \rightarrow \infty} 2\langle A_1 y, y - z_{n_i} \rangle = 2\langle A_1 y, y - \bar{x} \rangle. \end{aligned}$$

It follows that

$$\langle A_1 y, y - \bar{x} \rangle \geq 0, \quad \forall y \in \Omega.$$

Accordingly, Lemma 8 and the ζ -inverse-strong monotonicity of A_1 ensure that

$$\langle A_1 \bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in \Omega;$$

that is, $\bar{x} \in \text{VI}(\Omega, A_1)$. Consequently, from $\{x^*\} = \text{VI}(\text{VI}(\Omega, A_1), A_2)$, we have

$$\limsup_{n \rightarrow \infty} \langle A_2 x^*, x^* - q_n \rangle = \lim_{i \rightarrow \infty} \langle A_2 x^*, x^* - q_{n_i} \rangle = \langle A_2 x^*, x^* - \bar{x} \rangle \leq 0.$$

On the other hand, we choose a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle A_1 x^*, x^* - z_n \rangle = \lim_{k \rightarrow \infty} \langle A_1 x^*, x^* - z_{n_k} \rangle.$$

Since $\{z_n\}$ is a bounded sequence in C , we may assume, without loss of generality, that $z_{n_k} \rightarrow \hat{x} \in C$. From (36), it is easy to see that $z_{n_k} \rightarrow \hat{x}$ yields $x_{n_k} \rightarrow \hat{x}$. By the same arguments as in the proof of $\bar{x} \in \Omega$, we have $\hat{x} \in \Omega$. From $x^* \in \text{VI}(\Omega, A_1)$, we get

$$\limsup_{n \rightarrow \infty} \langle A_1 x^*, x^* - z_n \rangle = \lim_{k \rightarrow \infty} \langle A_1 x^*, x^* - z_{n_k} \rangle = \langle A_1 x^*, x^* - \hat{x} \rangle \leq 0. \tag{46}$$

Therefore, the inequalities in (45) hold.

Step 7. We claim that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Indeed, putting $p = x^*$ in (14) and at Lines 5–6 in (25), we obtain that $\|z_n - x^*\| \leq \|x_n - x^*\|$ and

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq [1 - \alpha_n(\tau - \delta)] \|y_n - x^*\|^2 + \theta_n(2 + \theta_n) \|y_n - x^*\|^2 \\ &\quad - 2\alpha_n \langle \mu A_2 x^*, q_n - x^* \rangle + 2\beta_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle. \end{aligned} \tag{47}$$

From (12) and the ζ -inverse-strong monotonicity of A_1 , it follows that

$$\begin{aligned} \|y_n - x^*\|^2 &\leq \sigma_n \|x_n - x^*\|^2 + (1 - \sigma_n) \|P_C(z_n - \delta_n A_1 z_n) - x^*\|^2 \\ &\leq \sigma_n \|x_n - x^*\|^2 + (1 - \sigma_n) \|(z_n - x^*) - \delta_n A_1 z_n\|^2 \\ &= \sigma_n \|x_n - x^*\|^2 + (1 - \sigma_n) [\|z_n - x^*\|^2 + 2\delta_n \langle A_1 z_n, x^* - z_n \rangle + \delta_n^2 \|A_1 z_n\|^2] \\ &\leq \sigma_n \|x_n - x^*\|^2 + (1 - \sigma_n) [\|x_n - x^*\|^2 + 2\delta_n \langle A_1 x^*, x^* - z_n \rangle + \delta_n^2 \|A_1 z_n\|^2] \\ &= \|x_n - x^*\|^2 + (1 - \sigma_n) [2\delta_n \langle A_1 x^*, x^* - z_n \rangle + \delta_n^2 \|A_1 z_n\|^2]. \end{aligned} \tag{48}$$

Thus, in terms of (47) and (48), we get

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\leq [1 - \alpha_n(\tau - \delta)] \|y_n - x^*\|^2 + \theta_n(2 + \theta_n) \|y_n - x^*\|^2 + 2\alpha_n \langle \mu A_2 x^*, x^* - q_n \rangle \\ &\quad + 2\beta_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\leq [1 - \alpha_n(\tau - \delta)] \{ \|x_n - x^*\|^2 + (1 - \sigma_n) [2\delta_n \langle A_1 x^*, x^* - z_n \rangle + \delta_n^2 \|A_1 z_n\|^2] \} \\ &\quad + \theta_n(2 + \theta_n) \|y_n - x^*\|^2 + 2\alpha_n \langle \mu A_2 x^*, x^* - q_n \rangle + 2\beta_n \|f(x^*) - x^*\| \|x_{n+1} - x^*\| \\ &\leq [1 - \alpha_n(\tau - \delta)] \|x_n - x^*\|^2 + (1 - \alpha_n(\tau - \delta)) (1 - \sigma_n) 2\delta_n \langle A_1 x^*, x^* - z_n \rangle + \alpha_n^2 \|A_1 z_n\|^2 \\ &\quad + \theta_n(2 + \theta_n) \|y_n - x^*\|^2 + 2\alpha_n \langle \mu A_2 x^*, x^* - q_n \rangle + 2\beta_n \|f(x^*) - x^*\| \|x_{n+1} - x^*\| \\ &= [1 - \alpha_n(\tau - \delta)] \|x_n - x^*\|^2 + \alpha_n(\tau - \delta) \left\{ \frac{(1 - \alpha_n(\tau - \delta))(1 - \sigma_n) 2\delta_n}{\alpha_n(\tau - \delta)} \langle A_1 x^*, x^* - z_n \rangle + \frac{\alpha_n \|A_1 z_n\|^2}{\tau - \delta} \right. \\ &\quad \left. + \frac{\theta_n(2 + \theta_n) \|y_n - x^*\|^2}{\alpha_n(\tau - \delta)} + \frac{2}{\tau - \delta} \langle \mu A_2 x^*, x^* - q_n \rangle + \frac{2\beta_n \|f(x^*) - x^*\| \|x_{n+1} - x^*\|}{\alpha_n(\tau - \delta)} \right\}. \end{aligned} \tag{49}$$

It can be readily seen that (45) guarantees that

$$\limsup_{n \rightarrow \infty} \frac{(1 - \alpha_n(\tau - \delta))(1 - \sigma_n) 2\delta_n}{\alpha_n(\tau - \delta)} \langle A_1 x^*, x^* - z_n \rangle \leq 0$$

and

$$\limsup_{n \rightarrow \infty} \frac{2}{\tau - \delta} \langle \mu A_2 x^*, x^* - q_n \rangle \leq 0.$$

In fact, from $\limsup_{n \rightarrow \infty} \langle A_1 x^*, x^* - z_n \rangle \leq 0$, it follows that for any given $\varepsilon > 0$ there exists an integer $n_0 \geq 1$ such that $\langle A_1 x^*, x^* - z_n \rangle \leq \varepsilon, \forall n \geq n_0$. Then, from $\delta_n \leq \alpha_n$, we get

$$\begin{aligned} \frac{(1 - \alpha_n(\tau - \delta))(1 - \sigma_n) 2\delta_n}{\alpha_n(\tau - \delta)} \langle A_1 x^*, x^* - z_n \rangle &\leq \frac{(1 - \alpha_n(\tau - \delta))(1 - \sigma_n) 2\delta_n}{\alpha_n(\tau - \delta)} \varepsilon \\ &\leq \frac{2}{\tau - \delta} \varepsilon, \quad \forall n \geq n_0, \end{aligned}$$

which hence yields

$$\limsup_{n \rightarrow \infty} \frac{(1 - \alpha_n(\tau - \delta))(1 - \sigma_n) 2\delta_n}{\alpha_n(\tau - \delta)} \langle A_1 x^*, x^* - z_n \rangle \leq \frac{2}{\tau - \delta} \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we get $\limsup_{n \rightarrow \infty} \frac{(1 - \alpha_n(\tau - \delta))(1 - \sigma_n) 2\delta_n}{\alpha_n(\tau - \delta)} \langle A_1 x^*, x^* - z_n \rangle \leq 0$.

Since $\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = 0$ (due to Conditions (i) and (ii)), we deduce that $\sum_{n=0}^{\infty} \alpha_n(\tau - \delta) = \infty$ and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left\{ \frac{(1 - \alpha_n(\tau - \delta))(1 - \sigma_n) 2\delta_n}{\alpha_n(\tau - \delta)} \langle A_1 x^*, x^* - z_n \rangle + \frac{\alpha_n \|A_1 z_n\|^2}{\tau - \delta} \right. \\ \left. + \frac{\theta_n(2 + \theta_n) \|y_n - x^*\|^2}{\alpha_n(\tau - \delta)} + \frac{2}{\tau - \delta} \langle \mu A_2 x^*, x^* - q_n \rangle + \frac{2\beta_n \|f(x^*) - x^*\| \|x_{n+1} - x^*\|}{\alpha_n(\tau - \delta)} \right\} \leq 0. \end{aligned}$$

We can apply Lemma 3 to the relation (49) and conclude that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. This completes the proof. \square

The following results can be obtained by Theorem 1 easily, and hence we omit the details.

Corollary 1. We suppose C is a convex nonempty closed set of a real Hilbert space H and $f : C \rightarrow C$ is a contraction with the parameter $\delta \in [0, 1)$. Let A_1 be a ζ -inverse-strongly monotone nonself mapping on C and A_2 be a strongly positive bounded linear self operator one H with the parameter $\gamma > 0$, where $\delta < \tau := 1 - \sqrt{1 - \mu(2\gamma - \mu\|A_2\|^2)} \in (0, 1]$, $0 < \mu < \frac{2\gamma}{\|A_2\|^2}$. Let the mappings $B_1, B_2 : C \rightarrow H$ be α -inverse-strongly monotone and β -inverse-strongly monotone, respectively. Let T be an asymptotically nonexpansive self mapping on set C with a sequence $\{\theta_n\}$. Let $\{S_n\}_{n=0}^\infty$ be a countable family of ℓ -uniformly Lipschitzian pseudocontractive self-mappings on C satisfying the assumptions in Problem 1. For any given $x_0 \in C$, we suppose $\{x_n\}$ is a vector sequence through

$$\begin{cases} u_n = \frac{x_n + S_n u_n}{2}, \\ v_n = P_C(u_n - \mu_2 B_2 u_n), \\ z_n = P_C(v_n - \mu_1 B_1 v_n), \\ y_n = \sigma_n x_n + (1 - \sigma_n) P_C(I - \delta_n A_1) z_n, \\ x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) P_C(I - \alpha_n \mu A_2) T^n y_n, \quad \forall n \geq 0, \end{cases} \tag{50}$$

where $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\sigma_n\} \subset (0, 1]$ and $\{\delta_n\} \subset (0, 2\zeta]$ are the sequences as in Theorem 1. Then, the sequence $\{x_n\}_{n=0}^\infty$ generated by (50) satisfies the following properties:

- (a) $\{x_n\}_{n=0}^\infty$ is bounded.
- (b) $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$, $\lim_{n \rightarrow \infty} \|x_n - Gx_n\| = 0$, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$.
- (c) $\{x_n\}_{n=0}^\infty$ reaches to the unique solution of Problem 1 if $\frac{\|x_n - y_n\|}{\delta_n} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Since the linear bounded operator $A_2 : H \rightarrow H$ is positive and strong with the parameter $\gamma > 0$, we know that A_2 is κ -Lipschitzian and η -strongly monotone where $\kappa = \|A_2\|$ and $\eta = \gamma$. In this case, we obtain that $0 < \mu < \frac{2\eta}{\kappa^2} = \frac{2\gamma}{\|A_2\|^2}$, and

$$\delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} = 1 - \sqrt{1 - \mu(2\gamma - \mu\|A_2\|^2)} \in (0, 1].$$

Therefore, utilizing Theorem 1, we derive the desired result. \square

Corollary 2. We suppose C is a convex nonempty closed set of a real Hilbert space H . Let $f : C \rightarrow C$ be a contraction with the parameter $\delta \in [0, 1)$. Let $A_1 : C \rightarrow H$ be a ζ -inverse-strongly monotone mapping and $A_2 : C \rightarrow H$ be κ -Lipschitzian and η -strongly monotone with the parameters $\kappa, \eta > 0$, where $\delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$, $0 < \mu < \frac{2\eta}{\kappa^2}$. We suppose the nonself mappings $B_1, B_2 : C \rightarrow H$ are α -inverse-strongly monotone and β -inverse-strongly monotone, respectively. Let $T : C \rightarrow C$ be a nonexpansive mapping and $\{S_n\}_{n=0}^\infty$ be a countable family of ℓ -uniformly Lipschitzian pseudocontractive self-mappings on C satisfying the assumptions in Problem 1. For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} u_n = \gamma_n x_n + (1 - \gamma_n) S_n u_n, \\ v_n = P_C(u_n - \mu_2 B_2 u_n), \\ z_n = P_C(v_n - \mu_1 B_1 v_n), \\ y_n = \sigma_n x_n + (1 - \sigma_n) P_C(I - \delta_n A_1) z_n, \\ x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) P_C(I - \alpha_n \mu A_2) T y_n, \quad \forall n \geq 0, \end{cases} \tag{51}$$

where $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\sigma_n\} \subset (0, 1]$ and $\{\delta_n\} \subset (0, 2\zeta]$ are the sequences as in Theorem 1. Then, the sequence $\{x_n\}_{n=0}^\infty$ generated by (51) satisfies the following properties:

- (a) $\{x_n\}_{n=0}^\infty$ is bounded.
- (b) $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$, $\lim_{n \rightarrow \infty} \|x_n - Gx_n\| = 0$, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$.
- (c) $\{x_n\}_{n=0}^\infty$ reaches to the unique solution of Problem 1 if $\frac{\|x_n - y_n\|}{\delta_n} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Since T is a nonexpansive self mapping defined on set C , T is, of course, an asymptotically nonexpansive mapping with the parameter sequence $\{\theta_n\}$, where $\theta_n = 0 \forall n \geq 0$. Therefore, utilizing the similar argument process to that of Theorem 1, we obtain the desired result. \square

4. Applications to Finite Generalized Mixed Equilibria

We suppose set C is convex nonempty closed and a mapping T with fixed points is named as a attracting nonexpansive mapping if it is nonexpansive and satisfies:

$$\|Tx - p\| < \|x - p\| \quad \text{for all } x \notin \text{Fix}(T) \text{ and } p \in \text{Fix}(T).$$

Lemma 9 ([27]). *Let X be a strictly convex space, T_1 be an attracting nonexpansive mapping and T_2 be a nonexpansive mapping. We suppose they have common fixed points. Then, $\text{Fix}(T_1 T_2) = \text{Fix}(T_2 T_1) = \text{Fix}(T_1) \cap \text{Fix}(T_2)$.*

Let $A : C \rightarrow H$ be nonself mapping, $\varphi : C \rightarrow \mathbf{R}$ be a single-valued real function, and $\Theta : C \times C \rightarrow \mathbf{R}$ be a bifunction to \mathbf{R} . The mixed generalized equilibrium problem (MGEP) is to find $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x \rangle \geq 0 + \varphi(y) - \varphi(x), \quad \forall y \in C. \tag{52}$$

We borrow the collection of solutions of MGEP (52) by $\text{MGEP}(\Theta, \varphi, A)$. The GMEP (52) is quite useful in the sense that it includes many problems, namely, vector optimization problems, minimax problems, classical variational inequalities, Nash equilibrium problems in noncooperative games and others. For different aspects and solution methods, we refer to [28–38] and the references therein.

In particular, if $\varphi = 0$, then MGEP (52) become the generalized equilibrium problem (GEP) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{53}$$

The collection of solutions of GEP is used by $\text{GEP}(\Theta, A)$.

If $A = 0$, then MGEP (52) become the mixed equilibrium problem (MEP). which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

The collection of solutions of MEP is used by $\text{MEP}(\Theta, \varphi)$.

If $\varphi = 0$ and $A = 0$, then MGEP (52) become to the equilibrium problem (EP) (see Blum and Oettli [30]), which will approximate $x \in C$ with

$$\Theta(x, y) \geq 0, \quad \forall y \in C.$$

The collection of solutions of EP is used by $\text{EP}(\Theta)$.

Here, we list some elementary conclusions for the MEP.

It is first used in [38] that $\Theta : C \times C \rightarrow \mathbf{R}$ is a bifunction and $\varphi : C \rightarrow \mathbf{R}$ is a convex lower semicontinuous function restricted to the following items

(A1) $\forall x \in C, \Theta(x, x) \equiv 0$.

(A2) Θ has the monotonicity, i.e., $\forall x, y \in C, \Theta(x, y) + \Theta(y, x) \leq 0$.

(A3)

$$\limsup_{t \rightarrow 0^+} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y).$$

(A4) $\forall x \in C, \Theta(x, \cdot)$ is lower semicontinuous convex.

(B1) $\forall x \in H$ and $\forall r > 0$, we fix a set $D_x \subset C$ and $y_x \in C$ with

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0$$

$$\forall z \in C \setminus D_x.$$

(B2) C acts as a bounded set.

Lemma 10 ([38]). We suppose that $\Theta : C \times C \rightarrow \mathbf{R}$ has conditions (A1)–(A4) and $\varphi : C \rightarrow \mathbf{R}$ has the properties proper lower semicontinuous and convex, if either condition (B1) or condition (B2) is true. For $r > 0$ and $x \in H$, generate an operator $T_r^{(\Theta, \varphi)} : H \rightarrow C$ through

$$T_r^{(\Theta, \varphi)}(x) := \{z \in C : \varphi(y) + \Theta(z, y) - \varphi(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Then,

(i) Set $T_r^{(\Theta, \varphi)}(x)$ is a singleton set.

(ii) $\forall x, y \in H$,

$$\|T_r^{(\Theta, \varphi)}x - T_r^{(\Theta, \varphi)}y\|^2 \leq \langle T_r^{(\Theta, \varphi)}x - T_r^{(\Theta, \varphi)}y, x - y \rangle.$$

(iii) $\text{MEP}(\Theta, \varphi) = \text{Fix}(T_r^{(\Theta, \varphi)})$.

(iv) $\text{MEP}(\Theta, \varphi)$ is convex closed.

(v) $\|T_s^{(\Theta, \varphi)}x - T_t^{(\Theta, \varphi)}x\|^2 \leq \frac{s-t}{s} \langle T_s^{(\Theta, \varphi)}x - T_t^{(\Theta, \varphi)}x, T_s^{(\Theta, \varphi)}x - x \rangle, \forall s, t > 0$ and $\forall x \in H$.

Next, under some mild control conditions, we establish the strong convergence of the proposed algorithm to the unique element $\{x^*\} = \text{VI}(\text{VI}(\Omega, \mathcal{A}_1), \mathcal{A}_2)$ (i.e., the unique solution of a THCVI), where $\Omega := \bigcap_{i=1}^N \text{GMEP}(\Theta_i, \varphi_i, A_i) \cap \text{GSVI}(C, B_1, B_2) \cap \text{Fix}(S) \cap \text{Fix}(T)$.

Theorem 2. We suppose C is a convex nonempty closed set. Assume that, $\forall i = 1, 2, \dots, N, \Theta_i : C \times C \rightarrow \mathbf{R}$ a bifunction has Conditions (A1)–(A4), $\varphi_i : C \rightarrow \mathbf{R} \cup \{+\infty\}$ is a lower semicontinuous, convex proper function with Condition (B1) or Condition (B2), and $A_i : C \rightarrow H$ is an η_i -inverse-strongly monotone nonself mapping. Let $f : C \rightarrow C$ be a contraction with the parameter $\delta \in [0, 1)$, $\mathcal{A}_1 : C \rightarrow H$ be a ζ -inverse-strongly monotone nonself mapping and $\mathcal{A}_2 : C \rightarrow H$ be κ -Lipschitzian and η -strongly monotone with parameters $\kappa, \eta > 0$, where $\delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1], 0 < \mu < \frac{2\eta}{\kappa^2}$. Let the nonself mappings $B_1, B_2 : C \rightarrow H$ be α -inverse-strongly monotone and β -inverse-strongly monotone, respectively. Let self mapping T , defined on C , be a nonexpansive mapping and self mapping S , also defined on C , be an ℓ -Lipschitzian pseudocontractive mapping such that $\Omega := \bigcap_{i=1}^N \text{GMEP}(\Theta_i, \varphi_i, A_i) \cap \text{GSVI}(C, B_1, B_2) \cap \text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$ and $\text{VI}(\Omega, \mathcal{A}_1) \neq \emptyset$, where $\text{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ with $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} u_n = \gamma_n x_n + (1 - \gamma_n) S u_n, \\ v_n = P_C(u_n - \mu_2 B_2 u_n), \\ z_n = P_C(v_n - \mu_1 B_1 v_n), \\ y_n = \sigma_n x_n + (1 - \sigma_n) P_C(I - \delta_n \mathcal{A}_1) z_n, \\ x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) P_C(I - \alpha_n \mu \mathcal{A}_2) T \Delta^N y_n, \quad \forall n \geq 0, \end{cases} \tag{54}$$

where $\Delta^N = T_{r_1}^{(\Theta_1, \varphi_1)}(I - r_1 A_1) \cdots T_{r_N}^{(\Theta_N, \varphi_N)}(I - r_N A_N)$ with $r_i \in (0, 2\eta_i)$ for each $i = 1, 2, \dots, N$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\sigma_n\} \subset (0, 1]$ and $\{\delta_n\} \subset (0, 2\zeta]$ are the sequences such that

- (i) $\alpha_n \rightarrow 0$ as $n \rightarrow \infty, \sum_{n=0}^\infty \alpha_n = \infty$ and $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$.
- (ii) $\frac{\beta_n}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$ and $\sum_{n=0}^\infty |\beta_{n+1} - \beta_n| < \infty$.
- (iii) $0 < \liminf_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} \sigma_n < 1$ and $\sum_{n=0}^\infty |\sigma_{n+1} - \sigma_n| < \infty$.
- (iv) $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ and $\sum_{n=0}^\infty |\gamma_{n+1} - \gamma_n| < \infty$.
- (v) $\delta_n \leq \alpha_n \forall n \geq 0$ and $\sum_{n=0}^\infty |\delta_{n+1} - \delta_n| < \infty$.

Then, the sequence $\{x_n\}_{n=0}^\infty$ generated by (54) satisfies the following properties:

- (a) $\{x_n\}_{n=0}^\infty$ is bounded.
- (b) $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0, \lim_{n \rightarrow \infty} \|x_n - Gx_n\| = 0, \lim_{n \rightarrow \infty} \|x_n - T \Delta^N x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$.

(c) $\{x_n\}_{n=0}^\infty$ converges strongly to the unique element $\{x^*\} = \text{VI}(\text{VI}(\Omega, \mathcal{A}_1), \mathcal{A}_2)$ (i.e., the unique solution of a THCVI), provided $\|x_n - y_n\| = o(\delta_n)$.

Proof. First, let us show that for each $i = 1, 2, \dots, N$, the composite mapping $T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)$ with $r_i \in (0, 2\eta_i)$ is nonexpansive. Indeed, from Lemma 10 (iii), it is not difficult to obtain

$$\text{GMEP}(\Theta_i, \varphi_i, A_i) = \text{Fix}(T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)).$$

Utilizing Lemma 5 and Lemma 10 (ii), we have

$$\begin{aligned} & \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)y\|^2 \\ & \leq \|(I - r_i A_i)x - (I - r_i A_i)y\|^2 \\ & \leq \|x - y\|^2 + r_i(r_i - 2\eta_i)\|A_i - A_i y\|^2 \\ & \leq \|x - y\|^2. \end{aligned}$$

Thus, each composite mapping $T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)$ is nonexpansive. Moreover, we claim that $T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)$ is also attracting nonexpansive for each $i = 1, 2, \dots, N$. In fact, for all $x \notin \text{GMEP}(\Theta_i, \varphi_i, A_i)$ and $p \in \text{GMEP}(\Theta_i, \varphi_i, A_i)$, by the firm nonexpansivity of $T_{r_i}^{(\Theta_i, \varphi_i)}$ (due to Lemma 10 (ii)), we obtain

$$\begin{aligned} & \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - p\|^2 = \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)p\|^2 \\ & \leq \langle T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)p, (I - r_i A_i)x - (I - r_i A_i)p \rangle \\ & = \frac{1}{2} \{ \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)p\|^2 + \|(I - r_i A_i)x - (I - r_i A_i)p\|^2 \\ & \quad - \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)p - (I - r_i A_i)x + (I - r_i A_i)p\|^2 \} \\ & = \frac{1}{2} \{ \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - p\|^2 + \|(I - r_i A_i)x - (I - r_i A_i)p\|^2 \\ & \quad - \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - p - (I - r_i A_i)x + (I - r_i A_i)p\|^2 \}, \end{aligned}$$

which immediately implies that

$$\begin{aligned} & \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - p\|^2 \\ & \leq \|(I - r_i A_i)x - (I - r_i A_i)p\|^2 - \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - p - (I - r_i A_i)x + (I - r_i A_i)p\|^2. \end{aligned} \tag{55}$$

Next, we discuss two cases.

Case 1. If $A_i x = A_i p$, then from (55) we have

$$\begin{aligned} & \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - p\|^2 \\ & \leq \|(I - r_i A_i)x - (I - r_i A_i)p\|^2 - \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - p - (I - r_i A_i)x + (I - r_i A_i)p\|^2 \\ & = \|x - p\|^2 - \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - x\|^2 \\ & < \|x - p\|^2. \end{aligned}$$

Case 2. If $A_i x \neq A_i p$, then from (55) we get

$$\begin{aligned} & \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - p\|^2 \\ & \leq \|(x - r_i A_i x) - (p - r_i A_i p)\|^2 - \|T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)x - p - (I - r_i A_i)x + (I - r_i A_i)p\|^2 \\ & \leq \|(x - r_i A_i x) - (p - r_i A_i p)\|^2 \\ & \leq \|x - p\|^2 + r_i(r_i - 2\eta_i)\|A_i x - A_i p\|^2 \\ & < \|x - p\|^2 \end{aligned}$$

(due to $r_i \in (0, 2\eta_i)$). Summing up the above two cases, we know that each composite mapping $T_{r_i}^{(\Theta_i, \varphi_i)}(I - r_i A_i)$ is also attracting nonexpansive. Therefore, by Lemma 9, we conclude that $\text{Fix}(T\Delta^N) = \bigcap_{i=1}^N \text{GMEP}(\Theta_i, \varphi_i, A_i) \cap \text{Fix}(T)$. Then, we get the desired result by Theorem 1 easily. \square

Author Contributions: These authors contributed equally to this work.

Funding: This research was funded by the Natural Science Foundation of Shandong Province of China (ZR2017LA001) and Youth Foundation of Linyi University (LYDX2016BS023). The first author was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002) and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).

Acknowledgments: The authors are grateful to the editor and the referees for useful suggestions which improved the contents of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ceng, L.C.; Wang, C.Y.; Yao, J.C. Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. *Math. Methods Oper. Res.* **2008**, *67*, 375–390. [[CrossRef](#)]
2. Ceng, L.C.; Kong, Z.R.; Wen, C.F. On general systems of variational inequalities. *Comput. Math. Appl.* **2013**, *66*, 1514–1532. [[CrossRef](#)]
3. Elster, R.; Hebestreit, N.; Khan, A.A.; Tammer, C. Inverse generalized vector variational inequalities with respect to variable domination structures and applications to vector approximation problems. *Appl. Anal. Optim.* **2018**, *2*, 341–372.
4. Qin, X.; Cho, S.Y. Convergence analysis of a monotone projection algorithm in reflexive Banach spaces. *Acta Math. Sci.* **2017**, *37*, 488–502. [[CrossRef](#)]
5. Ceng, L.C.; Yao, J.C. A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem. *Nonlinear Anal.* **2010**, *72*, 1922–1937. [[CrossRef](#)]
6. Zhao, X.; Ng, K.F.; Li, C.; Yao, J.-C. Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems. *Appl. Math. Optim.* **2018**, *78*, 613–641. [[CrossRef](#)]
7. Qin, X.; Petrusel, A.; Yao, J.C. CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces. *J. Nonlinear Convex Anal.* **2018**, *19*, 157–165.
8. Takahashi, W.; Wen, C.F.; Yao, J.C. Split common fixed point problems and hierarchical variational inequality problems in Hilbert spaces. *J. Nonlinear Convex Anal.* **2017**, *18*, 777–797.
9. Chang, S.S.; Wen, C.F.; Yao, J.C. Generalized viscosity implicit rules for solving quasi-inclusion problems of accretive operators in Banach spaces. *Optimization* **2017**, *66*, 1105–1117. [[CrossRef](#)]
10. Cho, S.Y. Weak convergence of a splitting algorithm in Hilbert spaces. *J. Appl. Anal. Comput.* **2017**, *7*, 427–438.
11. Takahashi, W.; Wen, C.F.; Yao, J.C. The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space. *Fixed Point Theory* **2018**, *19*, 407–419. [[CrossRef](#)]
12. Kato, T. Nonlinear semigroups and evolution equations. *J. Math. Soc. Jpn.* **1967**, *19*, 508–520. [[CrossRef](#)]
13. Qin, X.; Yao, J.C. Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators. *J. Inequal. Appl.* **2016**, *2016*, 232. [[CrossRef](#)]
14. Qin, X.; Yao, J.C. Projection splitting algorithms for nonself operators. *J. Nonlinear Convex Anal.* **2017**, *18*, 925–935.
15. Qin, X.; Cho, S.Y.; Wang, L. A regularization method for treating zero points of the sum of two monotone operators. *Fixed Point Theory Appl.* **2014**, *2014*, 75. [[CrossRef](#)]
16. Djedidi, M.; Mansour, A.; Nachi, K. Fixed point theorem for generalized set-valued contractions on partial metric space. *Tamkang J. Math.* **2017**, *48*, 331–344.
17. Chidume, C.E.; Nnakwe, M.O. Convergence theorems of subgradient extragradient algorithm for solving variational inequalities and a convex feasibility problem. *Fixed Point Theory Appl.* **2018**, *2018*, 16. [[CrossRef](#)]
18. Chang, S.S.; Wen, C.F.; Yao, J.C. Zero point problem of accretive operators in Banach spaces. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 105–118. [[CrossRef](#)]

19. Qin, X.; Cho, S.Y.; Wang, L. Strong convergence of an iterative algorithm involving nonlinear mappings of nonexpansive and accretive type. *Optimization* **2018**, *67*, 1377–1388. [[CrossRef](#)]
20. Ceng, L.C.; Petrusel, A.; Wong, M.M.; Yu, S.J. Strong convergence of implicit viscosity approximation methods for pseudocontractive mappings in Banach spaces. *Optimization* **2011**, *60*, 659–670. [[CrossRef](#)]
21. Reich, S. Weak convergence theorems for nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.* **1979**, *67*, 274–276. [[CrossRef](#)]
22. Deimling, K. Zeros of accretive operators. *Manuscr. Math.* **1974**, *13*, 365–374. [[CrossRef](#)]
23. Xue, Z.; Zhou, H.; Cho, Y.J. Iterative solutions of nonlinear equations for m-accretive operators in Banach spaces. *J. Nonlinear Convex Anal.* **2000**, *1*, 313–320.
24. Xu, H.K.; Kim, T.H. Convergence of hybrid steepest-descent methods for variational inequalities. *J. Optim. Theory Appl.* **2003**, *119*, 185–201. [[CrossRef](#)]
25. Lim, T.C.; Xu, H.K. Fixed point theorems for asymptotically nonexpansive mappings. *Nonlinear Anal.* **1994**, *22*, 1345–1355. [[CrossRef](#)]
26. Iiduka, H. Iterative algorithm for solving triple-hierarchical constrained optimization problem. *J. Optim. Theory Appl.* **2011**, *148*, 580–592. [[CrossRef](#)]
27. Chancelier, J.P. Iterative schemes for computing fixed points of nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.* **2009**, *353*, 141–153. [[CrossRef](#)]
28. Qin, X.; Cho, S.Y.; Wang, L. Iterative algorithms with errors for zero points of m-accretive operators. *Fixed Point Theory Appl.* **2013**, *2013*, 148. [[CrossRef](#)]
29. Yao, Y.; Cho, Y.J.; Liou, Y.C. Constructed nets with perturbations for equilibrium and fixed point problems. *J. Inequal. Appl.* **2014**, *2014*, 334. [[CrossRef](#)]
30. Blum, E.; Oettli, W. From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **1994**, *63*, 123–145.
31. Yao, Y.; Postolache, Y.; Qin, X.; Yao, J.-C. Iterative algorithms for the proximal split feasibility problem. *Univ. Politeh. Buchar. Sci. Bull. Appl. Math. Phys.* **2018**, *80*, 37–44.
32. Yao, Y.; Leng, L.; Liou, Y.C. Strong convergence of an iteration for the split common fixed points of demicontractive operators. *J. Nonlinear Convex Anal.* **2018**, *19*, 197–205.
33. Cho, S.Y.; Qin, X.; Yao, J.-C.; Yao, Y. Viscosity approximation splitting methods for monotone and nonexpansive operators in Hilbert spaces. *J. Nonlinear Convex Anal.* **2018**, *19*, 251–264.
34. Takahashi, S.; Takahashi, W. Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space. *Nonlinear Anal.* **2008**, *69*, 1025–1033. [[CrossRef](#)]
35. Yao, Y.; Qin, X.; Yao, J.C. Projection methods for firmly type nonexpansive operators. *J. Nonlinear Convex Anal.* **2018**, *19*, 407–415.
36. Ceng, L.C. Approximation of common solutions of a split inclusion problem and a fixed-point problem. *J. Appl. Numer. Optim.* **2019**, *1*, 1–12.
37. Argyros, I.K.; George, S. Extended optimality of the secant method on Banach spaces. *Commun. Optim. Theory* **2018**, *2018*, 3.
38. Ceng, L.C.; Yao, J.C. A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. *J. Comput. Appl. Math.* **2008**, *214*, 186–201. [[CrossRef](#)]

