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Optimal Derivative-Free Root Finding Methods Based on Inverse Interpolation

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Abstract: Finding a simple root for a nonlinear equation $f(x) = 0$, $f : I \subseteq \mathbb{R} \to \mathbb{R}$ has always been of much interest due to its wide applications in many fields of science and engineering. Newton's method is usually applied to solve this kind of problems. In this paper, for such problems, we present a family of optimal derivative-free root finding methods of arbitrary high order based on inverse interpolation and modify it by using a transformation of first order derivative. Convergence analysis of the modified methods confirms that the optimal order of convergence is preserved according to the Kung-Traub conjecture. To examine the effectiveness and significance of the newly developed methods numerically, several nonlinear equations including the van der Waals equation are tested.

Keywords: nonlinear equations; simple roots; inverse interpolation; optimal iterative methods; higher order of convergence

1. Introduction

In this paper, we present optimal derivative-free methods to solve a nonlinear equation of the form $f(x) = 0$ [1–3]. Multipoint iterative methods for this problem have been extensively studied in the last decade as they are computationally efficient than the one-point methods such as the methods of Newton, Halley and Laguerre. According to the conjecture of Kung and Traub [4], the order of convergence of any multipoint method requiring $n+1$ evaluations cannot exceed the bound 2^n. The methods that satisfy this bound are called optimal methods. There is a vast literature on optimal multipoint methods, which are developed by using the famous one-step Newton method or the Steffensen method at the first step. The following is the iteration of Newton's scheme to find a simple root α of a nonlinear equation $f(x) = 0$, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval I [1]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0. \tag{1}$$

For a background study of multi-point optimal methods, one may consult [2,3,5–9]. Steffensen's iterative scheme is a well-known modification of Newton's method obtained by using the approximation

$$f'(x_n) \approx \frac{f(x_n) - f(z_n)}{x_n - z_n} = f[x_n, z_n], \tag{2}$$

in the Newton's scheme and is given as follows [10]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f[z_n, x_n]}, n \geq 0, \tag{3}$$

where $z_n = x_n + f(x_n)$. Both methods are quadratic in some neighborhood of α but Steffensen's method has an advantage that it does not need the evaluation of the function's derivative, which may be problematic and expensive to calculate for certain functions. To determine the computational efficiency of an iterative method, Ostrowski [11] defined the efficiency index as $q^{1/n}$, where q is the convergence order and n is the number of functional evaluations per iterative step. In a recent paper [6], Cordero and Torregrosa conjectured an approximation of the first derivative:

$$f'(x_n) \approx f[x_n, z_n], z_n = x_n + \gamma f(x_n)^m, m \geq q, \gamma \in \mathbb{R} - \{0\} \tag{4}$$

to transform a multipoint with derivative iterative method of order 2^q to a derivative-free method possessing the same order. Here, we use this conjecture by omitting the parameter γ to develop a family of n-point optimal derivative-free methods based on inverse interpolation. In Section 2, we present optimal derivative-free iterative methods based on inverse interpolation. The derivative-free forms of the methods are obtained using the conjecture of Cordero and Torregrosa [6] such that the convergence order is preserved. In Section 3, we consider some nonlinear equations and van der Waals equation for the numerical comparisons of presented methods with the existing methods of same kind.

2. Optimal Iterative Methods Based on Inverse Interpolation

In this section, we present optimal iterative methods based on inverse interpolation involving first derivative. The first derivative is further approximated by using the conjecture of Cordero and Torregrosa [6] to transform derivative methods into derivative-free methods such that the convergence order is preserved.

2.1. *Optimal Two-Point Fourth Order Method*

To construct an optimal two-point method, we use the following quadratic polynomial,

$$x = R(f(x)) = a_1 + b_1(f(x) - f(x_n)) + g_1(f(x) - f(x_n))^2. \tag{5}$$

By substituting $x = x_n$ into Equation (5), we get:

$$a_1 = x_n = R(f(x_n)). \tag{6}$$

Now by differentiating Equation (5) with respect to x, we get:

$$1 = R'(f(x))f'(x) = (b_1 + 2g_1(f(x) - f(x_n))^2)f'(x).$$

Therefore,

$$b_1 = \frac{1}{f'(x_n)}. \tag{7}$$

Now, by substituting $x = y_n$ and using Equations (6) and (7) in Equation (5), we get:

$$g_1 = \frac{1}{[f(y_n) - f(x_n)]f[y_n, x_n]} - \frac{1}{[f(y_n) - f(x_n)]f'(x_n)}, \tag{8}$$

where y_n is the Newton's iterate and $f[y, x] = \frac{f(y)-f(x)}{(y-x)}$.

Hence, by using Equations (6)–(8) in Equation (5), we obtain the following two-point optimal fourth order method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0, \\ x_{n+1} &= R(0) = y_n + g_1 f(x_n)^2, \end{aligned} \tag{9}$$

where g_1 is given by Equation (8).

Theorem 1. *Let α be a simple root of f, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a sufficiently differentiable function in an open interval I. If x_0 is sufficiently close to α, then the iterative method in Equation (9) is fourth order convergent and possesses the following error relation:*

$$e_{n+1} = c_2(-c_3 + 2c_2^2)e_n^4 + O(e_n^5), \tag{10}$$

where $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$, $j \geq 2$ and $e_n = x_n - \alpha$.

Proof. By using Taylor's expansions about α, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4] + O(e_n^5)$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3] + O(e_n^4)$$

By using above expressions in the first step of Equation (9), we get

$$y_n - \alpha = c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_3c_2 + 4c_2^3)e_n^4 + O(e_n^5).$$

Again, by using Taylor's expansion, we have

$$f(y_n) = c_2 f'(\alpha)e_n^2 + (2c_3 - 2c_2^2)f'(\alpha)e_n^3 + (3c_4 - 7c_3c_2 + 4c_2^3)f'(\alpha)e_n^4 + O(e_n^5).$$

Hence, we get the following error equation of the method in Equation (9) by using the above expressions in its second step:

$$e_{n+1} = c_2(-c_3 + 2c_2^2)e_n^4 + O(e_n^5).$$

Thus, the proof is complete. □

Now, we modify the new two-point optimal scheme in Equation (9) to obtain a derivative-free method by using the conjecture in Equation (4) such that the optimal order is preserved.

Therefore, with the help of the approximation given in Equation (4), the iterative method in Equation (9) is modified as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[z_n, x_n]}, z_n = x_n + f(x_n)^2, n \geq 0, \\ x_{n+1} &= y_n + h_1 f(x_n)^2, \end{aligned} \tag{11}$$

where

$$h_1 = \frac{1}{[f(y_n) - f(x_n)]f[y_n, x_n]} - \frac{1}{[f(y_n) - f(x_n)]f[z_n, x_n]}. \tag{12}$$

Similar to Theorem 1, we can prove that the iterative method in Equations (11) and (12) has convergence order four with the following error equation:

$$e_{n+1} = -c_2(c_3 + f'(\alpha)^2 c_2 - 2c_2^2)e_n^4 + O(e_n^5).$$

2.2. Optimal Three-Point Eighth Order Method

In [12], Neta and Petkovic proposed the following three-point optimal method based on inverse interpolation:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0, \\ w_n &= \psi_4(x_n, y_n), \\ x_{n+1} &= y_n + b_2 f(x_n)^2 - g_2 f(x_n)^3, \end{aligned} \tag{13}$$

where ψ_4 is a real function that should be chosen such that it provides the fourth order convergence of the sequence $\{x_n\}$ and it requires already computed values $f(x_n), f'(x_n)$ and $f(y_n)$. The iteration in Equation (9) is an example of such function. The values of g_2 and b_2 are given as:

$$\begin{aligned} g_2 = & \frac{1}{[f(y_n)-f(x_n)][f(y_n)-f(w_n)]f[y_n,x_n]} \\ & -\frac{1}{[f(w_n)-f(x_n)][f(y_n)-f(w_n)]f[w_n,x_n]} \\ & +\frac{1}{[f(w_n)-f(x_n)][f(y_n)-f(w_n)]f'(x_n)} \\ & -\frac{1}{[f(y_n)-f(x_n)][f(y_n)-f(w_n)]f'(x_n)}, \end{aligned} \tag{14}$$

$$b_2 = \frac{1}{[f(y_n)-f(x_n)]f[y_n,x_n]} - \frac{1}{f'(x_n)[f(y_n)-f(x_n)]} - g_2[f(y_n) - f(x_n)]. \tag{15}$$

By using the approximation given by Equation (4) and the iterative scheme in Equations (11) and (12) at the second step of the three-point method in Equation (13), we obtain a new optimal derivative-free method as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[z_n, x_n]}, z_n = x_n + f(x_n)^3, n \geq 0, \\ w_n &= y_n + h_2 f(x_n)^2, \\ x_{n+1} &= y_n + b_3 f(x_n)^2 - g_3 f(x_n)^3, \end{aligned} \tag{16}$$

where

$$\begin{aligned} h_2 &= \frac{1}{[f(y_n) - f(x_n)]f[y_n, x_n]} - \frac{1}{[f(y_n) - f(x_n)]f[z_n, x_n]}, \\ g_3 &= \frac{1}{[f(y_n) - f(x_n)][f(y_n) - f(w_n)]f[y_n, x_n]} \\ & \quad -\frac{1}{[f(w_n) - f(x_n)][f(y_n) - f(w_n)]f[w_n, x_n]} \\ & \quad +\frac{1}{[f(w_n) - f(x_n)][f(y_n) - f(w_n)]f[z_n, x_n]} \\ & \quad -\frac{1}{[f(y_n) - f(x_n)][f(y_n) - f(w_n)]f[z_n, x_n]}, \\ b_3 &= \frac{1}{[f(y_n) - f(x_n)]f[y_n, x_n]} - \frac{1}{f[z_n, x_n][f(y_n) - f(x_n)]} - g_3[f(y_n) - f(x_n)]. \end{aligned} \tag{17}$$

Theorem 2. *Let α be a simple root of f, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a sufficiently differentiable function in an open interval I. For an initial approximation x_0 sufficiently close to α, the iterative methods given by Equations (16) and (17) is optimal eighth order convergent with the following error relation:*

$$\begin{aligned} e_{n+1} &= -c_2^2(-10c_2^5 + 15c_2^3 c_3 - 5c_2 c_3^2 - 2c_4 c_2^2 + 2c_2^3 f'(\alpha)^3 \\ & \quad + c_3 c_4 - c_3 f'(\alpha)^3 c_2) e_n^8 + O(e_n^9), \end{aligned}$$

where $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$, $j \geq 2$ and $e_n = x_n - \alpha$.

Proof. With the help of Taylor's expansions, the proof is similar to the proof for Theorem 1 and those already taken in [6,8]. Hence, it is omitted. □

2.3. Optimal Four-Point Sixteenth Order Method

Neta and Petkovic [12] also presented an optimal four-point scheme with sixteenth order convergence using inverse interpolation requiring an evaluation of first derivative $f'(x_n)$ at each step and four evaluations of the function.

We transform their four-point optimal scheme and use the scheme in Equations (11) and (12) at the second step to develop a new derivative-free optimal four-point method as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[z_n, x_n]}, z_n = x_n + f(x_n)^4, n \geq 0, \\ w_n &= y_n + h_2 f(x_n)^2, \\ t_n &= y_n + b_3 f(x_n)^2 - g_3 f(x_n)^3, \\ x_{n+1} &= y_n + b_4 f(x_n)^2 - g_4 f(x_n)^3 + g_5 f(x_n)^4, \end{aligned} \tag{18}$$

where h_2, $g_3$and b_3 are given in Equation (17) and the values of g_5, g_4 and b_4 are given as:

$$\begin{aligned} g_5 &= \frac{\frac{\varphi_t - \varphi_w}{[f(t_n) - f(w_n)]} - \frac{\varphi_y - \varphi_w}{[f(y_n) - f(w_n)]}}{[f(t_n) - f(y_n)]}, \\ g_4 &= \frac{\varphi_t - \varphi_w}{[f(t_n) - f(w_n)]} - g_5([f(t_n) - f(x_n)] + [f(w_n) - f(x_n)]), \\ b_4 &= \varphi_t - g_4[f(t_n) - f(x_n)] - g_5[f(t_n) - f(x_n)]^2 \end{aligned} \tag{19}$$

where

$$\begin{aligned} \varphi_t &= \frac{1}{f[t_n, x_n][f(t_n) - f(x_n)]} - \frac{1}{f[z_n, x_n][f(t_n) - f(x_n)]}, \\ \varphi_w &= \frac{1}{f[w_n, x_n][f(w_n) - f(x_n)]} - \frac{1}{f[z_n, x_n][f(w_n) - f(x_n)]}, \\ \varphi_y &= \frac{1}{f[y_n, x_n][f(y_n) - f(x_n)]} - \frac{1}{f[z_n, x_n][f(y_n) - f(x_n)]}. \end{aligned}$$

Theorem 3. *Let α be a simple root of f, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a sufficiently differentiable function in an open interval I. For an initial approximation x_0 close enough to α, the iterative method given by Equations (18) and (19) is optimal sixteenth order convergent with the following error equation:*

$$\begin{aligned} e_{n+1} = & -c_2^4(c_4 c_2 f'(\alpha)^4 c_3^2 - 4c_4 c_3 f'(\alpha)^4 c_2^3 - 4c_5 c_2^2 c_4 c_3 - 3c_4 c_3^4 + 20 c_2^7 c_5 \\ & -24 c_2^5 c_4^2 + 15 c_3^5 c_2 - 180 c_3^4 c_2^3 + 715 c_3^3 c_2^5 + 20 f'(\alpha)^4 c_2^8 - 176 c_2^8 c_4 \\ & -1250 c_2^7 c_3^2 + 980 c_3 c_2^9 - 280 c_2^{11} - 5 c_2^2 f'(\alpha)^4 c_3^3 + 24 c_2^3 c_4^2 c_3 - 6 c_4^2 c_3^2 c_2 \\ & +4 c_2^4 c_5 c_4 - 40 c_3 f'(\alpha)^4 c_2^6 - 5 c_5 c_3^3 c_2 + 25 c_3^2 c_2^3 c_5 + c_3^2 c_5 c_4 \\ & +25 c_3^2 f'(\alpha)^4 c_2^4 - 260 c_4 c_3^2 c_2^4 - 40 c_3 c_2^5 c_5 + 63 c_4 c_3^3 c_2^2 + 380 c_2^6 c_4 c_3 \\ & +4 c_4 f'(\alpha)^4 c_2^5) e_n^{16} + O(e_n^{17}), \end{aligned} \tag{20}$$

where $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$, $j \geq 2$ and $e_n = x_n - \alpha$.

Proof. The proof is similar to those already taken in [6,8] by using the Taylor's expansions. □

2.4. *n*-Point Method of Optimal Order 2^n

In [2], Petkovic et al. proposed an n-point n-step iterative scheme of order 2^n that requires one evaluation of first derivative $f'(x_n)$ and n evaluations of function at each step. We develop the following n-point n-step derivative-free method by using the approximation given in Equation (4):

$$\begin{aligned}
\phi_1(x_k) &= x_k - \frac{f(x_k)}{f[z_k,x_k]}, z_k = x_k + f(x_k)^m, m \geq n \geq 0, \\
\phi_2(x_k) &= R_2(0), \\
&\vdots \\
x_{k+1} &= \phi_n(x_n) = R_n(0).
\end{aligned} \tag{21}$$

where R_n is an inverse interpolating polynomial of degree n given as:

$$x = R_n(f(x)) = d_0 + d_1[f(x) - f(x_k)] + d_2[f(x) - f(x_k)]^2 + \cdots + d_n[f(x) - f(x_k)]^n, \tag{22}$$

with the conditions:

$$\begin{aligned}
R_n(f(x_k)) &= x_k, R_n'(f(x_k)) = \frac{1}{f[z_n,x_n]}, \\
R_n(f(\phi_1)) &= f(\phi_1), ..., R_n(f(\phi_n)) = f(\phi_n).
\end{aligned} \tag{23}$$

By the use of the conditions in Equation (23), the coefficients $d_0, d_1, ..., d_n$ can be determined easily and hence we obtain an n-point derivative-free family of following form:

$$\begin{aligned}
\phi_1(x_k) &= x_k - \frac{f(x_k)}{f[z_k,x_k]}, z_k = x_k + f(x_k)^m, m \geq n \geq 0, \\
\phi_2(x_k) &= R_2(0), \\
&\vdots \\
x_{k+1} &= \phi_n(x_k) = R_n(0) = \phi_1(x_k) + d_2 f(x_k)^2 + \cdots + (-1)^n d_n f(x_k)^n.
\end{aligned} \tag{24}$$

Theorem 4. *Let α be a simple root of f, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a sufficiently differentiable function in an open interval I. For an initial approximation x_0 sufficiently close to α, the n-point iterative method defined by Equation (24) is of optimal order 2^n.*

3. Review of Some Four-Point Optimal Methods and Numerical Results

Here, we use the transformation in Equation (4) to modify optimal four-point iterative scheme of Sharifi et al. [13] denoted by SL16 and optimal four-point method by Geum and Kim [14] denoted by GK16. Table 1 presents the original and modified forms of four-points methods where

$$\begin{aligned}
K_1(u_n) &= \frac{1 + \rho u_n + (-9 + 5\rho/2)u_n^2}{1 + (\rho - 2)u_n + (-4 + \rho/2)u_n^2}, \\
K_2(u_n, v_n, w_n) &= \frac{1 + 2u_n + (2 + \xi)w_n}{1 - v_n + \xi w_n}, \\
K_3(u_n, v_n, w_n, t_n) &= \frac{1 + 2u_n + (2 + \xi)v_n w_n}{1 - v_n - 2w_n - t_n + 2(1 + \xi)v_n w_n} - \frac{1}{2}u_n w_n[6 + 12u_n \\
&\quad + u_n^2(24 - 11\rho) + u_n^3(11\rho^2 - 66\rho + 136) + 4\xi] \\
&\quad + (2u_n(\xi^2 - 2\xi - 9) - 4\xi - 6)w_n^2
\end{aligned}$$

and

$$\begin{aligned} L_1(u_n) &= 1+2u_n+5u_n^2-6u_n^3, \\ L_2(u_n,v_n,w_n) &= 1+2u_n+4w_n+6u_n^2+v_n \\ L_3(u_n,v_n,w_n,p_n,q_n,t_n) &= 1+6u_n^2+2u_n-v_n^3+v_n+4w_n-4w_n^2+u_nw_n \\ &\quad +6u_n^2w_n+2u_n^3w_n-10u_nw_n^2+t_n+2q_n+8p_n \\ &\quad +2u_nt_n+2v_nw_n+6u_n^2t_n-4v_n^2w_n+24u_n^4w_n, \end{aligned}$$

are weight functions,

$$u_n=\frac{f(y_n)}{f(x_n)}, v_n=\frac{f(r_n)}{f(y_n)}, w_n=\frac{f(r_n)}{f(x_n)}, t_n=\frac{f(s_n)}{f(r_n)}, p_n=\frac{f(s_n)}{f(x_n)}, q_n=\frac{f(s_n)}{f(y_n)} \tag{25}$$

and ρ, ξ are free parameters. Here, we have chosen $\rho = 2$, $\xi = -2$.

A family of optimal derivative-free iterative methods of arbitrary high order by using polynomial interpolation presented by Cordero et al. [7] is given as follows:

$$\begin{aligned} y_0 &= x_k, k \geq 0, \\ y_1 &= y_0 + f(y_0), \\ x_{k+1} = y_{k+1} &= y_j - \frac{f(y_j)}{p_j'(y_j)}, j = 1, 2, ..., n, \end{aligned} \tag{26}$$

where p_j is the polynomial that interpolates f in $y_0, y_1, ..., y_j$.

Table 1. Four-point methods and their modifications.

Original Iterative Method	**Modified Iterative Method**
GK16:	MGK16:
$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0,$	$y_n = x_n - \frac{f(x_n)}{f[z_n,x_n]}, z_n = x_n + f(x_n)^4, n \geq 0,$
$r_n = y_n - K_1(u_n)\frac{f(y_n)}{f'(x_n)},$	$r_n = y_n - K_1(u_n)\frac{f(y_n)}{f[z_n,x_n]},$
$s_n = r_n - K_2(u_n,v_n,w_n)\frac{f(r_n)}{f'(x_n)},$	$s_n = r_n - K_2(u_n,v_n,w_n)\frac{f(r_n)}{f[z_n,x_n]},$
$x_{n+1} = s_n - K_3(u_n,v_n,w_n,t_n)\frac{f(s_n)}{f'(x_n)}.$	$x_{n+1} = s_n - K_3(u_n,v_n,w_n,t_n)\frac{f(s_n)}{f[z_n,x_n]}.$
SL16:	MSL16:
$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0,$	$y_n = x_n - \frac{f(x_n)}{f[z_n,x_n]}, z_n = x_n + f(x_n)^4, n \geq 0,$
$r_n = y_n - L_1(u_n)\frac{f(y_n)}{f'(x_n)},$	$r_n = y_n - L_1(u_n)\frac{f(y_n)}{f[z_n,x_n]},$
$s_n = r_n - L_2(u_n,v_n,w_n)\frac{f(r_n)}{f'(x_n)},$	$s_n = r_n - L_2(u_n,v_n,w_n)\frac{f(r_n)}{f[z_n,x_n]},$
$x_{n+1} = s_n - L_3(u_n,v_n,w_n,p_n,q_n,t_n)\frac{f(s_n)}{f'(x_n)}.$	$x_{n+1} = s_n - L_3(u_n,v_n,w_n,p_n,q_n,t_n)\frac{f(s_n)}{f[z_n,x_n]}.$

Now, we test all the discussed optimal with- and without-derivative methods using different types of nonlinear equations. We employed multi-precision arithmetic with 4000 significant decimal digits in the programming package of Maple 16 (Waterloo Maple Inc., Waterloo, ON, Canada) to obtain a high accuracy and to avoid the loss of significant digits.

In chemistry, several nonlinear systems can be found, for example in the investigation of stability of chemical reactions. Here, our concern is to deal with the solution of Van der Waals equation for finding the volume of a gas. Van der Waals equation is given by

$$(P + \frac{an^2}{V^2})(V - nb) = nRT, \tag{27}$$

which relates the pressure (P), volume (V) and temperature (T) of a gas. Here, n represents the number of moles of gas present and R is the universal gas constant. The term involving the parameter a corrects the pressure for intermolecular attractive forces, while the term involving b is a correction for that portion of the volume of gas that is not compressible due to the intrinsic volume of the gas molecules. We have to find the solution of a nonlinear equation in V to determine the volume V of the gas in terms of the remaining parameters as follows:

$$PV^3 - (nbP + nRT)V^2 + an^2V - an^2b = 0. \quad (28)$$

Suppose that one mole of chlorine gas has a pressure of 2 atmospheres and a temperature of 313 K. For chlorine gas, $a = 6.29$ atm·L^2/mol^2 and $b = 0.0562$ L/mol. The universal gas constant has the value $R = 0.08206$ atm·L/mol·K. Hence, we obtain following equation, which is cubic in V:

$$f_1(V) = 2V^3 - 25.79718V^2 + 6.29V - 0.353498, \quad (29)$$

with a root $\alpha = 0.087113811$.

We compare the convergence behavior of the modified methods MGK16, MSL16 and Equations (18) and (19) denoted by MNP16 with their with derivative versions, i.e., GK16, SL16 and (NP16) [12], respectively, and the family (Equation (26)) denoted by (CT16) for $n = 4$, using $f_1(V)$ and the nonlinear functions given in Table 2. The error between approximated and exact root ($|x_n - \alpha|$) and the computational order of convergence (coc) for first three iterations of various methods is displayed in Tables 3–9, where $E(-i)$ denotes $E \times 10^{-i}$, which means that the absolute error for the corespondent method is zero up to $i - 1$ decimal places. It is observed that the computational order of convergence (coc) supports the theoretical order of convergence. The formula to compute the computational order of convergence (coc) is given by [15]:

$$\text{coc} \approx \frac{\log|(x_{n+1} - \alpha)/(x_n - \alpha)|}{\log|(x_n - \alpha)/(x_{n-1} - \alpha)|}.$$

It was observed from the numerical experiments that the modified four-point methods MGK16, MSL16 and MNP16 are comparable and competitive to the methods GK16, SL16, NP16 and CT16. Especially, for the case of f_7, all the with-derivative four-point methods failed to converge when the initial guess was taken far from the required root, while the modified derivative-free four-points methods provided remarkably fast convergence. We see that schemes GK16 and CT16 failed to converge to the required root for the case of f_4.

Table 2. Test functions.

Test Functions	Exact Root α	x_0
$f_2(x) = (2 + x^3)\cos(\frac{\pi x}{2}) + \log(x^2 + 2x + 2)$	-1	-0.93
$f_3(x) = x^2e^x + x\cos\frac{1}{x^3} + 1$	$-1.5650602...$	-1.25
$f_4(x) = xe^x + \log(1 + x + x^4)$	0	-0.5
$f_5(x) = (x - 1)(x + 1 + \log(2 + x + x^2))$	1	1.05
$f_6(x) = -20x^5 - \frac{x}{2} + \frac{1}{2}$	0.42767729...	0.38
$f_7(x) = e^{\sin(8x)} - 4x$	0.34985721...	7

Table 3. Numerical results for f_1, $x_0 = 0$.

Error	GK16	SL16	NP16	CT16	MGK16	MSL16	MNP16
$\lvert x_1 - \alpha\rvert$	2.81(−5)	3.02(−4)	5.28(−5)	1.73(−4)	9.73(−6)	2.87(−4)	4.36(−5)
$\lvert x_2 - \alpha\rvert$	6.12(−54)	4.70(−38)	1.17(−49)	2.96(−40)	2.44(−61)	2.05(−38)	5.52(−51)
$\lvert x_3 - \alpha\rvert$	1.42(−832)	6.75(−580)	4.03(−764)	1.68(−612)	5.89(−951)	1.19(−585)	2.36(−785)
coc	16.00	16.00	16.00	16.00	16.00	16.00	16.00

Table 4. Numerical results for f_2, $x_0 = -0.93$.

Error	GK16	SL16	NP16	CT16	MGK16	MSL16	MNP16
$\|x_1 - \alpha\|$	1.44(−10)	2.63(−9)	1.83(−10)	6.42(−10)	1.65(−10)	2.59(−9)	1.83(−10)
$\|x_2 - \alpha\|$	8.08(−147)	1.61(−127)	2.77(−145)	9.99(−136)	6.82(−147)	1.23(−127)	2.58(−145)
$\|x_3 - \alpha\|$	7.29(−2327)	6.48(−2019)	1.98(−2302)	1.18(−2148)	4.69(−2327)	8.22(−2021)	6.18(−2303)
coc	16.00	16.00	16.00	16.00	16.00	16.00	16.00

Table 5. Numerical results for f_3, $x_0 = -1.25$.

Error	GK16	SL16	NP16	CT16	MGK16	MSL16	MNP16
$\|x_1 - \alpha\|$	5.84(−13)	7.09(−9)	2.49(−11)	4.10(−6)	1.37(−11)	1.57(−8)	1.46(−11)
$\|x_2 - \alpha\|$	2.58(−201)	7.05(−135)	5.79(−176)	2.16(−89)	2.07(−179)	2.53(−129)	4.07(−180)
$\|x_3 - \alpha\|$	5.49(−3215)	6.51(−2151)	4.04(−2810)	7.88(−1422)	1.46(−2864)	5.25(−2062)	5.01(−2877)
coc	16.00	16.00	16.00	16.00	16.00	16.00	16.00

Table 6. Numerical results for f_4, $x_0 = -0.5$.

Error	GK16	SL16	NP16	CT16	MGK16	MSL16	MNP16
$\|x_1 - \alpha\|$	Diverges	2.9732...	1.42(−4)	Diverges	1.20(−9)	1.51(−7)	6.69(−10)
$\|x_2 - \alpha\|$	Diverges	7.74(−1)	1.65(−67)	Diverges	3.50(−149)	1.24(−112)	2.43(−152)
$\|x_3 - \alpha\|$	Diverges	4.59(−6)	1.85(−1074)	Diverges	9.05(−2382)	4.90(−1794)	2.35(−2431)
coc	Diverges	8.94	16.00	Diverges	16.00	16.00	16.00

Table 7. Numerical results for f_5, $x_0 = 1.05$.

Error	GK16	SL16	NP16	CT16	MGK16	MSL16	MNP16
$\|x_1 - \alpha\|$	1.02(−23)	6.99(−25)	1.29(−23)	3.40(−21)	4.19(−20)	8.96(−23)	2.69(−21)
$\|x_2 - \alpha\|$	2.44(−370)	8.60(−390)	8.99(−369)	1.83(−327)	4.78(−310)	3.43(−356)	7.83(−330)
$\|x_3 - \alpha\|$	0	0	1(−3999)	0	0	0	0
coc	16.00	16.00	16.00	16.00	16.00	16.00	16.00

Table 8. Numerical results for f_6, $x_0 = 0.38$.

Error	GK16	SL16	NP16	CT16	MGK16	MSL16	MNP16
$\|x_1 - \alpha\|$	1.65(−5)	1.28(−3)	8.41(−11)	1.70(−11)	1.10(−5)	1.04(−3)	2.78(−11)
$\|x_2 - \alpha\|$	2.04(−66)	2.68(−34)	1.59(−151)	1.86(−161)	8.28(−69)	1.04(−35)	5.53(−160)
$\|x_3 - \alpha\|$	5.69(−1041)	5.39(−525)	4.62(−2403)	7.53(−2561)	8.66(−1079)	1.60(−547)	3.22(−2539)
coc	16.00	16.00	16.00	16.00	16.00	16.00	16.00

Table 9. Numerical results for f_7, $x_0 = 7$.

Error	GK16	SL16	NP16	CT16	MGK16	MSL16	MNP16
$\|x_1 - \alpha\|$	Diverges	Diverges	Diverges	1.98(−2)	4.92(−3)	3.00(−2)	1.50(−2)
$\|x_2 - \alpha\|$	Diverges	Diverges	Diverges	3.89(−12)	3.72(−26)	5.71(−11)	3.31(−17)
$\|x_3 - \alpha\|$	Diverges	Diverges	Diverges	1.20(−168)	5.72(−398)	8.21(−155)	9.46(−225)
coc	Diverges	Diverges	Diverges	16.12	16.08	16.49	16.20

4. Conclusions

In this paper, we have developed a family of optimal derivative-free root finding methods of arbitrary high order based on inverse interpolation by applying the conjecture of Cordero and Torregrosa. Some existing derivative based methods are modified using this conjecture. Convergence analysis is studied for the proposed optimal methods. Finally, numerical tests are provided that support the theoretical results. It was observed that the modified derivative free methods can compete and work better than their with-derivative versions. Especially, the proposed derivative-free methods provided remarkably fast convergence for the case of f_7, even when the initial guess was taken far from the required root, while the derivative based methods failed to converge.

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