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Triple Hierarchical Variational Inequalities, Systems of Variational Inequalities, and Fixed Point Problems

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Abstract: In this paper, we introduce a multiple hybrid implicit iteration method for finding a solution for a monotone variational inequality with a variational inequality constraint over the common solution set of a general system of variational inequalities, and a common fixed point problem of a countable family of uniformly Lipschitzian pseudocontractive mappings and an asymptotically nonexpansive mapping in Hilbert spaces. Strong convergence of the proposed method to the unique solution of the problem is established under some suitable assumptions.

Keywords: multiple hybrid implicit iteration method; triple hierarchical constrained variational inequality; general system of variational inequalities; fixed point; asymptotically nonexpansive mapping; pseudocontractive mapping; strong convergence; Hilbert spaces

1. Introduction

We suppose that H is a real Hilbert space. We use $\langle \cdot, \cdot \rangle$ to stand for the inner product and $\| \cdot \|$ the norm. We suppose that C is a convex closed nonempty set in the Hilbert space H, and P_C is the well-known metric projection from the space H onto the set C. Here, we also suppose that T is a nonlinear self mapping defined in C. Let $\mathrm{Fix}(T)$ be the set of all fixed points of T, that is, $\mathrm{Fix}(T) = \{x \in C : x = Tx\}$. We use the notations $\to$ and $\rightharpoonup$ to indicate the norm convergence and the weak convergence, respectively. Now, we suppose that $A : C \to H$ is a nonlinear nonself mapping in C to H. The well-known classical variational inequality (VI), whose set of all solutions denoted by $\mathrm{VI}(C, A)$, is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1}$$

A mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{\theta_n\} \subset [0, +\infty)$ with $\lim_{n\to\infty} \theta_n = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \theta_n \|x - y\|, \quad \forall n \geq 0,\ x, y \in C. \tag{2}$$

This mapping is Lipschitz continuous with the Lipschitz constant $L > 1$. Fixed points of Lipschitz continuous mappings are a hot topic and have a lot of applications both in theoretical research, such as in differential equations, control theory, equilibrium problems, and in engineering applications; see References [1–6] and the references therein. In particular, T is said to be nonexpansive if $|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C$, that is, $\theta \equiv 1$ for all n. Recently, the variational inequality problem (1) has been extensively studied via the iterative methods of Lipschitz continuous mappings, in particular, (asymptotically) nonexpansve mappings; see References [7–12] and the references therein.

We suppose that $B_1, B_2 : C \to H$ are two nonlinear monotone mappings. We also suppose that μ_1 and μ_2 are two positive real constants. We consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C. \end{cases} \tag{3}$$

Problem (3) is called a general system of variational inequalities (GSVI). From Reference [8], the GSVI (3) can be translated into a fixed point problem of a Lipschitz continuous nonlinear operator in the following way.

Lemma 1 ([8])**.** *We suppose that C is a convex subset in a Hilbert space H. Fix two elements x^* and y^* in C, (x^*, y^*) is a solution of GSVI (3) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, and $y^* = P_C(I - \mu_2 B_2)x^*$.*

The GSVI (3), which includes the variational inequality (1) as a special case, has been investigated via fixed-point algorithms recently in real or complex Hilbert spaces; see References [13–18] and the references therein.

A self mapping $f : C \to C$ is said to be a strict contraction on C if there is a number $\delta \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \delta\|x - y\|$ for all $x, y \in C$. A nonself mapping $F : C \to H$ is called monotone if $\langle Fx - Fy, x - y \rangle \geq 0\ \forall x, y \in C$. It is called η-strongly monotone if there is $\eta > 0$ such that

$$\eta\|x - y\|^2 \leq \langle Fx - Fy, x - y \rangle, \quad \forall x, y \in C.$$

Moreover, it is called α-inverse-strongly monotone (or α-cocoercive) if there is a constant $\alpha > 0$ such that

$$\alpha\|Fx - Fy\|^2 \leq \langle Fx - Fy, x - y \rangle, \quad \forall x, y \in C.$$

The class of inverse-strongly monotone operators or α-cocoercive operators has been in the spotlight of theoretical research and studied from the viewpoint of numerical computation and many results were obtained in Hilbert (and more generally, in Banach) spaces; see References [19–24] and the references therein.

Let X be a real Banach space whose dual space is denoted by X^*. The well-known normalized duality operator $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{\psi \in X^* : \langle x, \psi \rangle = \|x\|^2 = \|\psi\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between E and E^*. A mapping T with domain $D(T)$ and range $R(T)$ in X is called pseudocontractive if the inequality holds

$$\|x - y + r((I - T)x - (I - T)y)\| \geq \|x - y\|, \quad \forall x, y \in D(T), \forall r > 0.$$

Kato's results [25] told us that the notion of pseudocontraction is equivalent to the one that for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2.$$

The purpose of this paper is act as a continuation of Reference [26], that is to introduce and analyze a multiple hybrid implicit iteration method for solving a monotone variational inequality with a variational inequality constraint for two inverse-strongly monotone mappings and a common fixed point problem (CFPP) of a countable family of uniformly Lipschitzian pseudocontractive mappings and an asymptotically nonexpansive mapping in Hilbert spaces, which is called the triple hierarchical constrained variational inequality (THCVI). Here, the multiple hybrid implicit iteration method is based on the Moudafi's viscosity approximation method, Korpelevich's extragradient method, Mann's mean method, and the hybrid steepest-descent method. Under some suitable assumptions, strong convergence of the proposed method to the unique solution of the THCVI is derived.

2. Preliminaries

Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of continuous pseudocontractive self-mappings on C. Then, $\{T_n\}_{n=0}^{\infty}$ is said to be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C if there exists a constant $\ell > 0$ such that each T_n is ℓ-Lipschitz continuous. We fix an element x in H to see that there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called a metric projection of H onto C. It may be a set-valued operator. Further, C is assumed to be convex and closed, and X is assumed to be Hilbert, P_C is, in such a situation, a single-valued operator.

We need the following propositions and lemmas to prove our main results.

Proposition 1 ([27]). *We suppose C is a convex closed subset of a Banach space X. Let $S_0, S_1, \dots$ be a self-mapping sequence on C. Let $\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1} x\| : x \in C\} < \infty$. We conclude $\{S_n y\}$, where $y \in C$, converges strongly to some point in C. Moreover, we assume S is a self mapping on C generated by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Therefore, $\lim_{n\to\infty} \sup\{\|Sx - S_n x\| : x \in C\} = 0$.*

Proposition 2 ([28]). *We suppose C is a convex closed subset of a Banach space X and T is a continuous strong pseudocontraction self-mapping. Therefore, T enjoys fixed points. Indeed, it has a unique fixed point.*

The following lemma is trivial.

Lemma 2. *In a real Hilbert space H, there holds the inequality*

$$2\langle y + x, y\rangle \geq \|x + y\|^2 - \|x\|^2, \quad \forall x, y \in H.$$

Lemma 3 ([29]). *We suppose that $\{a_n\}$ is a nonnegative number sequence satisfying the restrictions*

$$a_{n+1} \leq a_n + \lambda_n \gamma_n - \lambda_n a_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real sequences such that

(i) $\limsup_{n\to\infty} \gamma_n \leq 0$ or $\sum_{n=0}^{\infty} |\lambda_n \gamma_n| < \infty$;
(ii) $\{\lambda_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$, or equivalently,

$$\lim_{n\to\infty} \prod_{k=0}^{n} (1 - \lambda_k) = 0.$$

Hence, $a_n \to 0$ as $n \to \infty$.

The following lemma is a direct consequence of Yamada [30].

Lemma 4. *Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone. We suppose λ is a positive real number in $(0,1]$ and $T : C \to H$ is a nonexpansive nonself mapping, and we define the mapping $T^{\lambda} : C \to H$ by*

$$T^{\lambda} x := Tx - \lambda\mu F(Tx), \quad \forall x \in C.$$

If $0 < \mu < \frac{2\eta}{\kappa^2}$, *then* T^λ *is a contraction operator, that is,*

$$\|T^\lambda x - T^\lambda y\| \le (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 5 ([31])**.** *We suppose that the nonself mapping* $A : C \to H$ *is* α*-inverse-strongly monotone. Then, for a given* $\lambda \ge 0$,

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \le \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2.$$

In particular, if $0 \le \lambda \le 2\alpha$, *then* $I - \lambda A$ *is nonexpansive. Further, we suppose* $A : C \to H$ *is a monotone and hemicontinuous mapping. Then, the following hold:*

(i) $\mathrm{VI}(C, A) = \{x^* \in C : \langle Ay, y - x^* \rangle \ge 0, \ \forall y \in C\}$;
(ii) $\mathrm{VI}(C, A) = \mathrm{Fix}(P_C(I - \lambda A))$ *for all* $\lambda > 0$;
(iii) $\mathrm{VI}(C, A)$ *consists of one point if* A *is strongly monotone and Lipschitz continuous.*

Lemma 6 ([8])**.** *We suppose the nonself operators* $B_1, B_2 : C \to H$ *are* α*-inverse-strongly monotone and* β*-inverse-strongly monotone, respectively. Let the self operator* $G : C \to C$ *be defined in* $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$. $G : C \to C$ *is nonexpansive if* $0 \le \mu_1 \le 2\alpha$ *and* $0 \le \mu_2 \le 2\beta$.

Lemma 7 ([32])**.** *We suppose the Banach space* X *enjoys a weakly continuous duality mapping, and* C *is a convex closed set in* X. *Let* $T : C \to C$ *be an asymptotically nonexpansive self mapping on* C *with a nonempty fixed point set. Then,* $I - T$ *is demiclosed at zero, i.e., if* $\{x_n\}$ *is a sequence in* C *converging weakly to some* $x \in C$ *and the sequence* $\{(I - T)x_n\}$ *converges strongly to zero, then* $(I - T)x = 0$, *where* I *is the identity mapping of* X.

Lemma 8 ([33])**.** *Let both* $\{x_n\}$ *and* $\{h_n\}$ *be a bounded sequence in a Banach space* X. *Let* $\{\beta_n\} \subset (0, 1)$ *be a number sequence such that*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = \beta_n x_n + (1 - \beta_n)h_n \ \forall n \ge 0$ *and* $\limsup_{n\to\infty}(\|h_{n+1} - h_n\| - \|x_{n+1} - x_n\|) \le 0$. *So,* $\lim_{n\to\infty} \|h_n - x_n\| = 0$.

3. Main Results

Let C be a convex closed subset of a real Hilbert space H. Let $B_1, B_2 : C \to H$ be monotone mappings, $A - g : C \to H$ be a monotone mapping with $A, g : C \to H$, $T : C \to C$ be an asymptotically nonexpansive mapping, and $\{S_n\}_{n=0}^\infty$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings defined on C. We suppose $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T) \ne \emptyset$ and studied the variational inequality for monotone mapping $A - g$ over the common solution set Ω of the GSVI (3) and the CFPP of $\{S_n\}_{n=0}^\infty$ and T:

$$\text{Find } \bar{x} \in \mathrm{VI}(\Omega, A - g) := \{\bar{x} \in \Omega : \langle (A - g)\bar{x}, y - \bar{x} \rangle \ge 0 \ \forall y \in \Omega\}.$$

This section introduces the following monotone variational inequality problem with the inequality constraint over the common solution set of the GSVI (2) and the CFPP of T and $\{S_n\}_{n=0}^\infty$, which is named the triple hierarchical constrained variational inequality:

Assume that

(C1) $T : C \to C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$;

(C2) $\{S_n\}_{n=0}^\infty$ is a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C;

(C3) $B_1, B_2 : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively;

(C4) $\mathrm{GSVI}(C, B_1, B_2) := \mathrm{Fix}(G)$ where $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ for $\mu_1, \mu_2 > 0$;

(C5) $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T) \neq \emptyset$;

(C6) $\sum_{n=1}^{\infty} \sup_{x \in D} \|S_n x - S_{n-1} x\| < \infty$ for any bounded subset D of C;

(C7) $S : C \to C$ is the mapping defined by $Sx = \lim_{n\to\infty} S_n x \ \forall x \in C$, such that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$;

(C8) $g : C \to H$ is l-Lipschitzian and $A : C \to H$ is ζ-inverse-strongly monotone such that $A - g$ is monotone;

(C9) $f : C \to C$ is a contraction mapping with coefficient $\delta \in [0, 1)$ and $F : C \to H$ is κ-Lipschitzian and η-strongly monotone;

(C10) $\mathrm{VI}(\Omega, A - g) \neq \emptyset$.

Problem 1. *The objective is to*

$$\begin{aligned} &\text{find } x^* \in \mathrm{VI}(\mathrm{VI}(\Omega, A - g), I - f) \\ &\quad := \{x^* \in \mathrm{VI}(\Omega, A - g) : \langle (I - f)x^*, v - x^* \rangle \geq 0,\ \forall v \in \mathrm{VI}(\Omega, A - g)\}. \end{aligned}$$

Since the original problem is a variational inequality, in this paper, we call it a triple hierarchical constrained variational inequality. Since the mapping f is a contractive, we easily get that the solution of the problem is unique. Inspired by the results announced recently, we introduce the following multiple hybrid implicit iterative algorithm to find the solution of such a problem.

Algorithm 1: Multiple hybrid implicit iterative algorithm.

Step 0. Take $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$, and $\mu > 0$, choose $x_0 \in C$ arbitrarily, and let $n := 0$.

Step 1. Given $x_n \in C$, compute $x_{n+1} \in C$ as

$$\left\{ \begin{array}{l} u_n = \gamma_n x_n + (1 - \gamma_n) S_n u_n, \\ v_n = P_C(u_n - \mu_2 B_2 u_n), \\ z_n = P_C(v_n - \mu_1 B_1 v_n), \\ y_n = P_C[\alpha_n g(x_n) + (I - \alpha_n A) z_n], \\ w_n = P_C[\alpha_n x_n + (I - \alpha_n \mu F) T^n y_n], \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) w_n. \end{array} \right. \tag{4}$$

Update $n := n + 1$ and go to Step 1.

We remark here that our algorithm is quite general. It includes mean-valued techniques, gradient techniques, and implicit iteration techniques. Our algorithm can also generate a strong convergence without any compact assumptions in infinite dimensional spaces.

We now state and prove the main result of this paper, that is, the following convergence analysis is presented for our Algorithm 1.

Theorem 1. *We suppose $\mu_1 \in (0, 2\alpha)$, $\mu_2 \in (0, 2\beta)$, and $l + \delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$ for $\mu \in (0, \frac{2\eta}{\kappa^2})$. Let number sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ lie in $(0, 1]$ such that*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\alpha_n + \beta_n \leq 1\ \forall n \geq 0$;
(iv) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;
(v) $\lim_{n\to\infty} \|T^{n+1} y_n - T^n y_n\| = 0$.*Then, we have the following conclusions:*

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*
(b) $\lim_{n\to\infty}\|x_n - y_n\| = 0$ *and* $\lim_{n\to\infty}\|x_n - w_n\| = 0$*;*
(c) $\lim_{n\to\infty}\|x_n - Gx_n\| = 0,\ \lim_{n\to\infty}\|x_n - Tx_n\| = 0$ *and* $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$*;*
(d) If $(\|x_n - y_n\| + \|x_n - w_n\|) = o(\alpha_n)$*, then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique solution of the Problem 1.*

Proof. Observe that the metric projection $P_{\mathrm{VI}(\Omega,A-g)}$ is nonexpansive. Indeed, it is firmly nonexpansive. The mapping f is contractive. Thus, the composition mapping $P_{\mathrm{VI}(\Omega,A-g)}f$ is a contraction mapping and hence $P_{\mathrm{VI}(\Omega,A-g)}f$ has a unique fixed point. Say $x^* \in C$, that is, $x^* = P_{\mathrm{VI}(\Omega,A-g)}f(x^*)$. By Lemma 5,

$$\{x^*\} = \mathrm{Fix}(P_{\mathrm{VI}(\Omega,A-g)}f) = \mathrm{VI}(\mathrm{VI}(\Omega, A-g), I-f).$$

Therefore, Problem 1 has a unique solution. Without loss of the generality, we can assume that $\{\alpha_n\} \subset (0, 2\zeta]$ and $\{\gamma_n\} \subset [a,b] \subset (0,1)$ for some $a, b \in (0,1)$. By Lemma 6, we know that G is nonexpansive. It is easy to see that for each $n \geq 0$ there exists a unique element $u_n \in C$ such that

$$u_n = \gamma_n x_n + (1-\gamma_n)S_n u_n. \tag{5}$$

Therefore, it can be seen that the multiple hybrid implicit iterative scheme (4) can be rewritten as

$$\begin{cases} u_n = \gamma_n x_n + (1-\gamma_n)S_n u_n, \\ z_n = Gu_n, \\ y_n = P_C[\alpha_n g(x_n) + (I-\alpha_n A)z_n], \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)P_C[\alpha_n x_n + (I-\alpha_n\mu F)T^n y_n], \quad \forall n \geq 0. \end{cases} \tag{6}$$

Next, we divide the rest of the proof into several steps.

Step 1. We prove $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{v_n\}, \{T^n y_n\}$, and $\{F(T^n y_n)\}$ are bounded. Indeed, We can take an element $p \in \Omega = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T)$ arbitrarily. Then, we have $S_n p = p$, $Gp = p$ and $Tp = p$. Since $S_n : C \to C$ is a pseudocontraction self mapping, one can show that

$$\|u_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 0. \tag{7}$$

Hence, we get

$$\|z_n - p\| = \|Gu_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{8}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $1 > \limsup_{n\to\infty}\beta_n \geq \liminf_{n\to\infty}\beta_n > 0$, we may assume that $\{\alpha_n + \beta_n\}$ is a set in $[c,d]$. Here, $c, d \in (0,1)$. In addition, since $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$, we may further assume that

$$\theta_n(1+\alpha_n l) \leq \frac{\alpha_n(\tau - l - \delta)}{2}\ (\leq \alpha_n(\tau - l - \delta)).$$

From Lemma 5 and (8), we can prove that

$$\begin{aligned} \|y_n - p\| &\leq \|(I-\alpha_n A)z_n - (I-\alpha_n A)p + \alpha_n(g(x_n) - Ap)\| \\ &\leq \|z_n - p\| + \alpha_n\|g(x_n) - Ap\| \\ &\leq (1+\alpha_n l)\|x_n - p\| + \alpha_n\|g(p) - Ap\|. \end{aligned} \tag{9}$$

We have from (6) and using Lemma 4 and (9) that

$$
\begin{aligned}
&\|x_{n+1}-p\| \\
&\le \alpha_n\|f(x_n)-p\| + \beta_n\|x_n-p\| + (1-\alpha_n-\beta_n)\|P_C[\alpha_n x_n + (I-\alpha_n\mu F)T^n y_n] - p\| \\
&\le \alpha_n\|f(x_n)-f(p)+f(p)-p\| + \beta_n\|x_n-p\| \\
&\quad + (1-\alpha_n-\beta_n)\|\alpha_n(x_n-p) + (I-\alpha_n\mu F)T^n y_n - (I-\alpha_n\mu F)p + \alpha_n(I-\mu F)p\| \\
&\le (\alpha_n\delta+\beta_n)\|x_n-p\| + \alpha_n\|f(p)-p\| \\
&\quad + (1-\alpha_n-\beta_n)[\alpha_n\|x_n-p\| + (1-\alpha_n\tau)(1+\theta_n)\|y_n-p\| + \alpha_n\|(I-\mu F)p\|] \\
&\le (\alpha_n\delta+\beta_n)\|x_n-p\| + \alpha_n\|f(p)-p\| \\
&\quad + (1-\alpha_n-\beta_n)[\alpha_n\|x_n-p\| + (1-\alpha_n\tau+\theta_n)\|y_n-p\| + \alpha_n\|(I-\mu F)p\|] \\
&\le (\alpha_n\delta+\beta_n)\|x_n-p\| + \alpha_n\|f(p)-p\| + (1-\alpha_n-\beta_n)\{\alpha_n\|x_n-p\| \\
&\quad + (1-\alpha_n\tau)[(1+\alpha_n l)\|x_n-p\| + \alpha_n\|g(p)-Ap\|] + \theta_n[(1+\alpha_n l)\|x_n-p\| \\
&\quad + \alpha_n\|g(p)-Ap\|] + \alpha_n\|(I-\mu F)p\|\} \\
&\le (\alpha_n\delta+\beta_n)\|x_n-p\| + \alpha_n\|f(p)-p\| + (1-\alpha_n-\beta_n)\{\alpha_n\|x_n-p\| \\
&\quad + [1-\alpha_n(\tau-l)]\|x_n-p\| + (1-\alpha_n\tau)\alpha_n\|g(p)-Ap\| + \theta_n(1+\alpha_n l)\|x_n-p\| \\
&\quad + \alpha_n^2\tau\|g(p)-Ap\|\} + \alpha_n\|(I-\mu F)p\| \\
&\le (\alpha_n\delta+\beta_n)\|x_n-p\| + \alpha_n\|f(p)-p\| + (1-\alpha_n-\beta_n)[\alpha_n + 1 - \alpha_n(\tau-l)]\|x_n-p\| \\
&\quad + \alpha_n\|g(p)-Ap\| + \theta_n(1+\alpha_n l)\|x_n-p\| + \alpha_n\|(I-\mu F)p\| \\
&= \{1-\alpha_n(\tau-l-\delta) - (\alpha_n+\beta_n)\alpha_n[1-(\tau-l)]\}\|x_n-p\| \\
&\quad + \theta_n(1+\alpha_n l)\|x_n-p\| + \alpha_n(\|f(p)-p\| + \|g(p)-Ap\| + \|(I-\mu F)p\|) \\
&\le [1-\alpha_n(\tau-l-\delta)]\|x_n-p\| + \frac{\alpha_n(\tau-l-\delta)}{2}\|x_n-p\| \\
&\quad + \alpha_n(\|f(p)-p\| + \|g(p)-Ap\| + \|(I-\mu F)p\|) \\
&= [1-\frac{\alpha_n(\tau-l-\delta)}{2}]\|x_n-p\| + \frac{\alpha_n(\tau-l-\delta)}{2}\cdot\frac{2(\|f(p)-p\| + \|g(p)-Ap\| + \|(I-\mu F)p\|)}{\tau-l-\delta} \\
&\le \max\{\|x_n-p\|, \frac{2(\|f(p)-p\| + \|g(p)-Ap\| + \|(I-\mu F)p\|)}{\tau-l-\delta}\}.
\end{aligned}
$$

By induction, we have

$$\|x_{n+1}-p\| \le \max\{\frac{2(\|p-f(p)\| + \|Ap-g(p)\| + \|(I-\mu F)p\|)}{\tau-l-\delta}, \|p-x_0\|\}, \quad \forall n \ge 0.$$

Thus, $\{x_n\}$ is a bounded sequence, and so are the sequences $\{y_n\}, \{z_n\}, \{u_n\}, \{T^n y_n\}$, and $\{F(T^n y_n)\}$. Since $\{S_n\}$ is ℓ-uniformly Lipschitzian on C, we know that

$$\|S_n u_n\| \le \|S_n u_n - p\| + \|p\| \le \ell\|u_n - p\| + \|p\|,$$

which implies that the set $\{S_n u_n\}$ is bounded. Additionally, from Lemma 1 and $p \in \Omega \subset \mathrm{GSVI}(C, B_1, B_2)$, it follows that (p, q) is a solution of the GSVI (3), where $q = P_C(I-\mu_2 B_2)p$. Noting that $v_n = P_C(I-\mu_2 B_2)u_n$ for all $n \ge 0$, by Lemma 5, we have

$$
\begin{aligned}
\|v_n\| &\le \|P_C(I-\mu_2 B_2)u_n - q\| + \|q\| \\
&= \|P_C(I-\mu_2 B_2)u_n - P_C(I-\mu_2 B_2)q\| + \|q\| \\
&\le \|u_n - q\| + \|q\|,
\end{aligned}
$$

which shows that $\{v_n\}$ also is bounded.

Step 2. We prove that $\|x_{n+1}-x_n\| \to 0$ and $\|y_{n+1}-y_n\| \to 0$ as $n \to \infty$. Indeed, we set

$$x_{n+1} = \beta_n x_n + (1-\beta_n)h_n$$

and notice

$$w_n = P_C[(I-\alpha_n\mu F)T^n y_n + \alpha_n x_n].$$

Then,

$$h_n = \frac{\alpha_n}{1-\beta_n} f(x_n) + (1 - \frac{\alpha_n}{1-\beta_n}) P_C[\alpha_n x_n + (I - \alpha_n \mu F) T^n y_n].$$

Simple calculations show that

$$\begin{aligned}
&h_{n+1} - h_n \\
&= \tfrac{\alpha_{n+1}}{1-\beta_{n+1}} (f(x_{n+1}) - f(x_n)) + (1 - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}) \{P_C[\alpha_{n+1} x_{n+1} + (I - \alpha_{n+1}\mu F) T^{n+1} y_{n+1}] \\
&\quad - P_C[\alpha_n x_n + (I - \alpha_n \mu F) T^n y_n)\} + (\tfrac{\alpha_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n}{1-\beta_n})(f(x_n) - w_n).
\end{aligned}$$

It follows from (6) that

$$\begin{aligned}
&\|h_{n+1} - h_n\| \\
&\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_n) - f(x_{n+1})\| + (1 - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}) \|P_C[\alpha_{n+1} x_{n+1} + (I - \alpha_{n+1}\mu F) T^{n+1} y_{n+1}] \\
&\quad - P_C[\alpha_n x_n + (I - \alpha_n \mu F) T^n y_n]\| + |\tfrac{\alpha_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n}{1-\beta_n}| \|w_n - f(x_n)\| \\
&\le \tfrac{\alpha_{n+1}\delta}{1-\beta_{n+1}} \|x_n - x_{n+1}\| + (1 - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}) \|T^{n+1} y_{n+1} - T^{n+1} y_n + T^{n+1} y_n - T^n y_n \\
&\quad + \alpha_{n+1}(x_{n+1} - \mu F(T^{n+1} y_{n+1})) - \alpha_n (x_n - \mu F(T^n y_n))\| \\
&\quad + |\tfrac{\alpha_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n}{1-\beta_n}| \|f(x_n) - T^n y_n - \alpha_n (x_n - \mu F(T^n y_n))\| \\
&\le \tfrac{\alpha_{n+1}\delta}{1-\beta_{n+1}} \|x_{n+1} - x_n\| + (1 - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}})(1 + \theta_{n+1}) \|y_{n+1} - y_n\| + \|T^n y_n - T^{n+1} y_n\| \\
&\quad + \alpha_{n+1} \|x_{n+1} - \mu F(T^{n+1} y_{n+1})\| + \alpha_n \|x_n - \mu F(T^n y_n)\| \\
&\quad + |\tfrac{\alpha_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n}{1-\beta_n}| \|f(x_n) - T^n y_n - \alpha_n (x_n - \mu F(T^n y_n))\| \\
&\le \|y_{n+1} - y_n\| + \theta_{n+1} \|y_n - y_{n+1}\| + \tfrac{\alpha_{n+1}\delta}{1-\beta_{n+1}} \|x_{n+1} - x_n\| + \|T^{n+1} y_n - T^n y_n\| \\
&\quad + \alpha_{n+1} \|x_{n+1} - \mu F(T^{n+1} y_{n+1})\| + \alpha_n \|x_n - \mu F(T^n y_n)\| \\
&\quad + |\tfrac{\alpha_{n+1}}{1-\beta_{n+1}} - \tfrac{\alpha_n}{1-\beta_n}| \|f(x_n) - T^n y_n - \alpha_n (x_n - \mu F(T^n y_n))\|.
\end{aligned} \tag{10}$$

Since $\{\alpha_n\} \subset (0, 2\zeta]$ and A is ζ-inverse-strongly monotone, by Lemma 5, we obtain

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&= \|P_C[\alpha_n g(x_n) + (I - \alpha_n A) z_n] - P_C[\alpha_{n+1} g(x_{n+1}) + (I - \alpha_{n+1} A) z_{n+1}]\| \\
&\le \|(I - \alpha_{n+1} A) z_{n+1} - (I - \alpha_n A) z_n + \alpha_{n+1} g(x_{n+1}) - \alpha_n g(x_n)\| \\
&= \|(I - \alpha_{n+1} A) z_{n+1} - (I - \alpha_{n+1} A) z_n + (\alpha_n - \alpha_{n+1}) A z_n + \alpha_{n+1} g(x_{n+1}) - \alpha_n g(x_n)\| \\
&\le \|(I - \alpha_{n+1} A) z_{n+1} - (I - \alpha_{n+1} A) z_n\| + |\alpha_n - \alpha_{n+1}| \|A z_n\| + \|\alpha_{n+1} g(x_{n+1}) - \alpha_n g(x_n)\| \\
&\le \|z_{n+1} - z_n\| + |\alpha_n - \alpha_{n+1}| \|A z_n\| + \alpha_{n+1} \|g(x_{n+1})\| + \alpha_n \|g(x_n)\| \\
&\le \|u_{n+1} - u_n\| + \|A z_n\| |\alpha_n - \alpha_{n+1}| + \alpha_{n+1} \|g(x_{n+1})\| + \alpha_n \|g(x_n)\|.
\end{aligned} \tag{11}$$

Furthermore, simple calculations show that

$$u_{n+1} - u_n = \gamma_{n+1}(x_{n+1} - x_n) + (1 - \gamma_{n+1})(S_{n+1} u_{n+1} - S_n u_n) + (\gamma_{n+1} - \gamma_n)(x_n - S_n u_n),$$

which hence yields

$$\begin{aligned}
&\|u_{n+1} - u_n\|^2 \\
&= \gamma_{n+1} \langle x_{n+1} - x_n, u_{n+1} - u_n \rangle + (1 - \gamma_{n+1}) \langle S_{n+1} u_{n+1} - S_n u_n, u_{n+1} - u_n \rangle \\
&\quad + (\gamma_{n+1} - \gamma_n) \langle x_n - S_n u_n, u_{n+1} - u_n \rangle \\
&= \gamma_{n+1} \langle x_{n+1} - x_n, u_{n+1} - u_n \rangle + (1 - \gamma_{n+1}) [\langle S_{n+1} u_{n+1} - S_n u_{n+1}, u_{n+1} - u_n \rangle \\
&\quad + \langle S_n u_{n+1} - S_n u_n, u_{n+1} - u_n \rangle] + (\gamma_{n+1} - \gamma_n) \langle x_n - S_n u_n, u_{n+1} - u_n \rangle \\
&\le \gamma_{n+1} \|x_{n+1} - x_n\| \|u_{n+1} - u_n\| + (1 - \gamma_{n+1}) [\|S_{n+1} u_{n+1} - S_n u_{n+1}\| \|u_{n+1} - u_n\| \\
&\quad + \|u_{n+1} - u_n\|^2] + |\gamma_{n+1} - \gamma_n| \|x_n - S_n u_n\| \|u_{n+1} - u_n\|.
\end{aligned}$$

So it follows that

$$\begin{aligned}
\|u_{n+1} - u_n\| \quad &\le \gamma_{n+1} \|x_{n+1} - x_n\| + (1 - \gamma_{n+1}) [\|S_{n+1} u_{n+1} - S_n u_{n+1}\| \\
&\quad + \|u_{n+1} - u_n\|] + |\gamma_{n+1} - \gamma_n| \|x_n - S_n u_n\|,
\end{aligned}$$

which immediately leads to

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|x_{n+1}-x_n\| + \tfrac{1-\gamma_{n+1}}{\gamma_{n+1}}\|S_{n+1}u_{n+1}-S_nu_{n+1}\| + |\gamma_{n+1}-\gamma_n|\tfrac{\|x_n-S_nu_n\|}{\gamma_{n+1}} \\ &\le \|x_{n+1}-x_n\| + \tfrac{1}{a}\|S_{n+1}u_{n+1}-S_nu_{n+1}\| + |\gamma_{n+1}-\gamma_n|\tfrac{\|x_n-S_nu_n\|}{a}.\end{aligned} \tag{12}$$

Put $D=\{u_n : n\ge 0\}$. Since $\{u_n\}$ is a bounded sequence, we know that D is a bounded set. Then, by the assumption of this theorem, we get

$$\sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x-S_nx\|<\infty.$$

Noticing that

$$\|S_{n+1}u_{n+1}-S_nu_{n+1}\|\le\sup_{x\in D}\|S_{n+1}x-S_nx\|,\quad \forall n\ge 0,$$

we have

$$\sum_{n=0}^{\infty}\|S_{n+1}u_{n+1}-S_nu_{n+1}\|<\infty. \tag{13}$$

Therefore, from (10)–(12) we deduce that

$$\begin{aligned}&\|h_{n+1}-h_n\|\\ &\le \theta_{n+1}\|y_{n+1}-y_n\| + \frac{\alpha_{n+1}\delta}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \|T^{n+1}y_n-T^ny_n\| + \|y_{n+1}-y_n\|\\ &\quad + \alpha_{n+1}\|x_{n+1}-\mu F(T^{n+1}y_{n+1})\| + \alpha_n\|x_n-\mu F(T^ny_n)\|\\ &\quad + |\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)-T^ny_n-\alpha_n(x_n-\mu F(T^ny_n))\|\\ &\le \|u_{n+1}-u_n\| + |\alpha_n-\alpha_{n+1}|\|Az_n\| + \alpha_{n+1}\|g(x_{n+1})\| + \alpha_n\|g(x_n)\|\\ &\quad + \theta_{n+1}\|y_{n+1}-y_n\| + \frac{\alpha_{n+1}\delta}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \|T^{n+1}y_n-T^ny_n\|\\ &\quad + \alpha_{n+1}\|x_{n+1}-\mu F(T^{n+1}y_{n+1})\| + \alpha_n\|x_n-\mu F(T^ny_n)\|\\ &\quad + |\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)-T^ny_n-\alpha_n(x_n-\mu F(T^ny_n))\|\\ &\le \|x_{n+1}-x_n\| + \frac{1}{a}\|S_{n+1}u_{n+1}-S_nu_{n+1}\| + |\gamma_{n+1}-\gamma_n|\frac{\|x_n-S_nu_n\|}{a}\\ &\quad + |\alpha_n-\alpha_{n+1}|\|Az_n\| + \alpha_{n+1}\|g(x_{n+1})\| + \alpha_n\|g(x_n)\| + \theta_{n+1}\|y_{n+1}-y_n\|\\ &\quad + \frac{\alpha_{n+1}\delta}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \|T^{n+1}y_n-T^ny_n\| + \alpha_{n+1}\|x_{n+1}-\mu F(T^{n+1}y_{n+1})\|\\ &\quad + \alpha_n\|x_n-\mu F(T^ny_n)\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}|\|f(x_n)-T^ny_n-\alpha_n(x_n-\mu F(T^ny_n))\|,\end{aligned}$$

which immediately attains

$$\begin{aligned}\|h_{n+1}-h_n\|-\|x_{n+1}-x_n\| &\le \tfrac{1}{a}\|S_{n+1}u_{n+1}-S_nu_{n+1}\| + |\gamma_{n+1}-\gamma_n|\tfrac{\|x_n-S_nu_n\|}{a}\\ &\quad + |\alpha_n-\alpha_{n+1}|\|Az_n\| + \alpha_{n+1}\|g(x_{n+1})\| + \alpha_n\|g(x_n)\| + \theta_{n+1}\|y_{n+1}-y_n\|\\ &\quad + \tfrac{\alpha_{n+1}\delta}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \|T^{n+1}y_n-T^ny_n\| + \alpha_{n+1}\|x_{n+1}-\mu F(T^{n+1}y_{n+1})\|\\ &\quad + \alpha_n\|x_n-\mu F(T^ny_n)\| + |\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}-\tfrac{\alpha_n}{1-\beta_n}|\|f(x_n)-T^ny_n-\alpha_n(x_n-\mu F(T^ny_n))\|.\end{aligned} \tag{14}$$

Since $\lim_{n\to\infty}\theta_n=0$ and $\lim_{n\to\infty}\|T^{n+1}y_n-T^ny_n\|=0$ (due to condition (v)), from (13) and conditions (i), (iii), (iv), it follows that

$$\limsup_{n\to\infty}(\|h_{n+1}-h_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence, by condition (iii) and Lemma 8, we get $\lim_{n\to\infty}\|h_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|h_n - x_n\| = 0. \tag{15}$$

Again from (11) and (12), we conclude that

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\le \|u_{n+1}-u_n\| + |\alpha_n - \alpha_{n+1}|\|Az_n\| + \alpha_{n+1}\|g(x_{n+1})\| + \alpha_n\|g(x_n)\| \\
&\le \|x_{n+1}-x_n\| + \tfrac{1}{a}\|S_{n+1}u_{n+1} - S_n u_{n+1}\| + |\gamma_{n+1}-\gamma_n|\tfrac{\|x_n - S_n u_n\|}{a} + |\alpha_n - \alpha_{n+1}|\|Az_n\| \\
&\quad + \alpha_{n+1}\|g(x_{n+1})\| + \alpha_n\|g(x_n)\| \to 0 \quad (n\to\infty),
\end{aligned}$$

and

$$\|z_n - z_{n+1}\| = \|Gu_n - Gu_{n+1}\| \le \|u_n - u_{n+1}\| \to 0 \quad (n\to\infty).$$

Thus,

$$\lim_{n\to\infty}\|y_n - y_{n+1}\| = 0, \quad \lim_{n\to\infty}\|u_n - u_{n+1}\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|z_n - z_{n+1}\| = 0.$$

Step 3. We prove $\|x_n - Gx_n\| \to 0$ as $n\to\infty$. Indeed, noticing $w_n = P_C[(I-\alpha_n\mu F)T^n y_n + \alpha_n x_n]$ for all $n \ge 0$, we have

$$\langle (I-\alpha_n\mu F)T^n y_n + \alpha_n x_n - P_C[\alpha_n x_n + (I-\alpha_n\mu F)T^n y_n], p - w_n\rangle \le 0. \tag{16}$$

From (16), we have

$$\begin{aligned}
\|w_n - p\|^2 &= \langle P_C[(I-\alpha_n\mu F)T^n y_n + \alpha_n x_n] - p, w_n - p\rangle \\
&= \langle P_C[(I-\alpha_n\mu F)T^n y_n + \alpha_n x_n] - \alpha_n x_n - (I-\alpha_n\mu F)T^n y_n, w_n - p\rangle \\
&\quad + \langle \alpha_n x_n + (I-\alpha_n\mu F)T^n y_n - p, w_n - p\rangle \\
&\le \langle \alpha_n x_n + (I-\alpha_n\mu F)T^n y_n - p, w_n - p\rangle \\
&= \langle (I-\alpha_n\mu F)T^n y_n - (I-\alpha_n\mu F)p, w_n - p\rangle + \alpha_n\langle x_n - \mu Fp, w_n - p\rangle \\
&\le (1-\alpha_n\tau)\|T^n y_n - p\|\|w_n - p\| + \alpha_n\langle x_n - \mu Fp, w_n - p\rangle \\
&\le \frac{1}{2}(1-\alpha_n\tau)^2\|T^n y_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 + \alpha_n\langle x_n - \mu Fp, w_n - p\rangle.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|w_n - p\|^2 &\le (1-\alpha_n\tau)^2\|T^n y_n - p\|^2 + 2\alpha_n\langle x_n - \mu Fp, w_n - p\rangle \\
&\le (1-\alpha_n\tau)(1+\theta_n)^2\|y_n - p\|^2 + 2\alpha_n\langle x_n - \mu Fp, w_n - p\rangle \\
&= (1-\alpha_n\tau)[\|y_n - p\|^2 + \theta_n(2+\theta_n)\|y_n - p\|^2] + 2\alpha_n\langle x_n - \mu Fp, w_n - p\rangle \\
&\le (1-\alpha_n\tau)\|y_n - p\|^2 + \theta_n(2+\theta_n)\|y_n - p\|^2 + 2\alpha_n\langle x_n - \mu Fp, w_n - p\rangle.
\end{aligned} \tag{17}$$

From (9) and (17), we get

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&= \|\beta_n(x_n-p)+\alpha_n(f(x_n)-f(p))+(1-\alpha_n-\beta_n)(w_n-p)+\alpha_n(f(p)-p)\|^2\\
&\le \|\beta_n(x_n-p)+\alpha_n(f(x_n)-f(p))+(1-\alpha_n-\beta_n)(w_n-p)\|^2+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \alpha_n\|f(x_n)-f(p)\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)\|w_n-p\|^2+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)[(1-\alpha_n\tau)\|y_n-p\|^2\\
&\quad+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\langle x_n-\mu Fp,w_n-p\rangle]+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)\{(1-\alpha_n\tau)(\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)^2\\
&\quad+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\langle x_n-\mu Fp,w_n-p\rangle\}+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)\{(1-\alpha_n\tau)\|z_n-p\|^2\\
&\quad+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2\\
&\quad+2\alpha_n\langle x_n-\mu Fp,w_n-p\rangle\}+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)(1-\alpha_n\tau)\|z_n-p\|^2\\
&\quad+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2\\
&\quad+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|.
\end{aligned}
\tag{18}
$$

We now note that $q=P_C(p-\mu_2B_2p)$, $v_n=P_C(u_n-\mu_2B_2u_n)$, and $z_n=P_C(v_n-\mu_1B_1v_n)$. Then, $z_n=Gu_n$. By Lemma 5, we have

$$
\begin{aligned}
\|v_n-q\|^2 &= \|P_C(u_n-\mu_2B_2u_n)-P_C(p-\mu_2B_2p)\|^2\\
&\le \|u_n-p-\mu_2(B_2u_n-B_2p)\|^2\\
&\le \|u_n-p\|^2-\mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
\|z_n-p\|^2 &= \|P_C(v_n-\mu_1B_1v_n)-P_C(q-\mu_1B_1q)\|^2\\
&\le \|v_n-q-\mu_1(B_1v_n-B_1q)\|^2\\
&\le \|v_n-q\|^2-\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2.
\end{aligned}
\tag{20}
$$

Substituting (19) for (20), we obtain from (7) that

$$
\begin{aligned}
\|z_n-p\|^2 &\le \|u_n-p\|^2-\mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2-\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2\\
&\le \|x_n-p\|^2-\mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2-\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2.
\end{aligned}
\tag{21}
$$

Combining (18) and (21), we get

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)(1-\alpha_n\tau)\|z_n-p\|^2\\
&\quad+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2\\
&\quad+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\
&\le \alpha_n\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\|x_n-p\|^2\\
&\quad-\mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2-\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2]\\
&\quad+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2\\
&\quad+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\
&= [1-\alpha_n(1-\alpha_n-\beta_n)\tau]\|x_n-p\|^2-(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2\\
&\quad+\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2]+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)\\
&\quad+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\
&\le \|x_n-p\|^2-(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2\\
&\quad+\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2]+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)\\
&\quad+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|,
\end{aligned}
$$

which immediately yields

$$\begin{aligned}
&(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2+\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2]\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)\\
&\quad+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)\\
&\quad+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|.
\end{aligned}$$

$(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2+\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2]\to 0$ as $n\to\infty$. Since $\liminf_{n\to\infty}(1-\alpha_n-\beta_n)>0$ (due to condition (iii)), $\mu_1\in(0,2\alpha)$, $\mu_2\in(0,2\beta)$, $\lim_{n\to\infty}\theta_n=0$ and $\lim_{n\to\infty}\alpha_n=0$, we obtain from (15) that

$$\lim_{n\to\infty}\|B_2u_n-B_2p\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|B_1v_n-B_1q\|=0. \tag{22}$$

On the other hand, we have

$$\begin{aligned}
\|v_n-q\|^2 &= \|P_C(u_n-\mu_2B_2u_n)-P_C(p-\mu_2B_2p)\|^2\\
&\le \langle u_n-\mu_2B_2u_n-(p-\mu_2B_2p),v_n-q\rangle\\
&= \langle u_n-p,v_n-q\rangle+\mu_2\langle B_2p-B_2u_n,v_n-q\rangle\\
&\le \tfrac{1}{2}[\|u_n-p\|^2+\|v_n-q\|^2-\|u_n-v_n-(p-q)\|^2]+\mu_2\|B_2p-B_2u_n\|\|v_n-q\|,
\end{aligned}$$

which implies that

$$\|v_n-q\|^2\le\|u_n-p\|^2-\|u_n-v_n-(p-q)\|^2+2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|. \tag{23}$$

In the same way, we derive

$$\begin{aligned}
\|z_n-p\|^2 &= \|P_C(v_n-\mu_1B_1v_n)-P_C(q-\mu_1B_1q)\|^2\\
&\le \langle v_n-\mu_1B_1v_n-(q-\mu_1B_1q),z_n-p\rangle\\
&= \langle v_n-q,z_n-p\rangle+\mu_1\langle B_1q-B_1v_n,z_n-p\rangle\\
&\le \tfrac{1}{2}[\|v_n-q\|^2+\|z_n-p\|^2-\|v_n-z_n+(p-q)\|^2]+\mu_1\|B_1q-B_1v_n\|\|z_n-p\|,
\end{aligned}$$

which implies that

$$\|z_n-p\|^2\le\|v_n-q\|^2-\|v_n-z_n+(p-q)\|^2+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\|. \tag{24}$$

Substituting (23) for (24), we deduce from (7) that

$$\begin{aligned}
\|z_n-p\|^2 &\le \|u_n-p\|^2-\|u_n-v_n-(p-q)\|^2-\|v_n-z_n+(p-q)\|^2\\
&\quad+2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\|\\
&\le \|x_n-p\|^2-\|u_n-v_n-(p-q)\|^2-\|v_n-z_n+(p-q)\|^2\\
&\quad+2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\|.
\end{aligned} \tag{25}$$

Combining (18) and (25), we have

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)(1-\alpha_n\tau)\|z_n-p\|^2 \\
&\quad +\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2 \\
&\quad +2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\| \\
&\le \alpha_n\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\|x_n-p\|^2 \\
&\quad -\|u_n-v_n-(p-q)\|^2-\|v_n-z_n+(p-q)\|^2+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\| \\
&\quad +2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|]+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|) \\
&\quad +\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\| \\
&= [1-\alpha_n(1-\alpha_n-\beta_n)\tau]\|x_n-p\|^2-(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\|u_n-v_n-(p-q)\|^2 \\
&\quad +\|v_n-z_n+(p-q)\|^2]+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\|+2\mu_2\|B_2p-B_2u_n\|\|v_n-q\| \\
&\quad +\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2 \\
&\quad +2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\| \\
&\le \|x_n-p\|^2-(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\|u_n-v_n-(p-q)\|^2+\|v_n-z_n+(p-q)\|^2] \\
&\quad +2\mu_1\|B_1q-B_1v_n\|\|z_n-p\|+2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|+\alpha_n\|g(x_n)-Ap\| \\
&\quad \times(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2 \\
&\quad +2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|,
\end{aligned}
$$

which hence yields

$$
\begin{aligned}
&(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\|u_n-v_n-(p-q)\|^2+\|v_n-z_n+(p-q)\|^2] \\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\| \\
&\quad +2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|) \\
&\quad +\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\| \\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\| \\
&\quad +2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|) \\
&\quad +\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|.
\end{aligned}
$$

Since $\liminf_{n\to\infty}(1-\alpha_n-\beta_n)>0$ (due to condition (iii)), $\lim_{n\to\infty}\theta_n=0$ and $\lim_{n\to\infty}\alpha_n=0$, we conclude from (15) and (22) that

$$
\lim_{n\to\infty}\|u_n-v_n-(p-q)\|=0 \quad \text{and} \quad \lim_{n\to\infty}\|v_n-z_n+(p-q)\|=0. \tag{26}
$$

It follows that

$$
\|u_n-z_n\|\le\|u_n-v_n-(p-q)\|+\|v_n-z_n+(p-q)\|\to 0 \quad (n\to\infty).
$$

That is,

$$
\lim_{n\to\infty}\|u_n-Gu_n\|=\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{27}
$$

Additionally, according to (6), we have

$$
\begin{aligned}
\|u_n-p\|^2 &= \gamma_n\langle x_n-p,u_n-p\rangle+(1-\gamma_n)\langle S_nu_n-p,u_n-p\rangle \\
&\le \gamma_n\langle x_n-p,u_n-p\rangle+(1-\gamma_n)\|u_n-p\|^2,
\end{aligned}
$$

which implicitly yields that

$$
\begin{aligned}
2\|u_n-p\|^2 &\le 2\langle x_n-p,u_n-p\rangle \\
&= \|x_n-p\|^2-\|x_n-u_n\|^2+\|u_n-p\|^2.
\end{aligned}
$$

This immediately implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2,$$

which together with (3.16), yields

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)(1-\alpha_n\tau)\|z_n-p\|^2\\ &\quad+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2\\ &\quad+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\ &\le \alpha_n\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)(1-\alpha_n\tau)[\|x_n-p\|^2-\|x_n-u_n\|^2]\\ &\quad+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2\\ &\quad+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\ &= [1-\alpha_n(1-\alpha_n-\beta_n)\tau]\|x_n-p\|^2-(1-\alpha_n-\beta_n)(1-\alpha_n\tau)\|x_n-u_n\|^2\\ &\quad+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)+\theta_n(2+\theta_n)\|y_n-p\|^2\\ &\quad+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\ &\le \|x_n-p\|^2+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)\\ &\quad+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|\\ &\quad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|-(1-\alpha_n-\beta_n)(1-\alpha_n\tau)\|x_n-u_n\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}&(1-\alpha_n-\beta_n)(1-\alpha_n\tau)\|x_n-u_n\|^2\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)\\ &\quad+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\ &\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_n\|g(x_n)-Ap\|(2\|z_n-p\|+\alpha_n\|g(x_n)-Ap\|)\\ &\quad+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|x_n-\mu Fp\|\|w_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|.\end{aligned}$$

Since $\liminf_{n\to\infty}(1-\alpha_n-\beta_n)>0$, $\lim_{n\to\infty}\theta_n=0$ and $\lim_{n\to\infty}\alpha_n=0$, we obtain from (15) that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{28}$$

Moreover, observe that

$$\|x_n-z_n\|\le\|x_n-u_n\|+\|Gu_n-u_n\|,$$

$$\|x_n-Gx_n\|\le\|x_n-z_n\|+\|Gu_n-Gx_n\|\le\|x_n-z_n\|+\|u_n-x_n\|,$$

and

$$\|x_n-y_n\|\le\|x_n-\alpha_n g(x_n)-(I-\alpha_n A)z_n\|\le\|x_n-z_n\|+\alpha_n\|g(x_n)-Az_n\|.$$

Then, from (27) and (28), it follows that

$$\lim_{n\to\infty}\|x_n-z_n\|=0,\quad \lim_{n\to\infty}\|x_n-Gx_n\|=0\quad\text{and}\quad \lim_{n\to\infty}\|x_n-y_n\|=0. \tag{29}$$

Step 4. Let us prove $\|x_n-S_nx_n\|\to0$, $\|x_n-w_n\|\to0$ and $\|x_n-Tx_n\|\to0$ as $n\to\infty$. Indeed, combining (5) and (8), we obtain that

$$\|S_nu_n-u_n\|=\frac{\gamma_n}{1-\gamma_n}\|x_n-u_n\|\le\frac{b}{1-b}\|x_n-u_n\|\to0\quad(n\to\infty).$$

That is,

$$\lim_{n\to\infty}\|S_nu_n-u_n\|=0. \tag{30}$$

Observe $\{S_n\}_{n=0}^{\infty}$ is ℓ-uniformly Lipschitzian. We further get from (28) and (30) that

$$\begin{aligned}\|S_n x_n - x_n\| &\le \|S_n x_n - S_n u_n\| + \|S_n u_n - u_n\| + \|u_n - x_n\| \\ &\le \ell\|x_n - u_n\| + \|S_n u_n - u_n\| + \|u_n - x_n\| \\ &= (\ell+1)\|x_n - u_n\| + \|S_n u_n - u_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

That is,

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{31}$$

We note that $\{\alpha_n + \beta_n\} \subset [c,d] \subset (0,1)$ for some $c, d \in (0,1)$, and observe that

$$\begin{aligned}&\|x_n - T^n y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n y_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - T^n y_n\| + \beta_n\|x_n - T^n y_n\| \\ &\quad + (1-\alpha_n-\beta_n)\|P_C[\alpha_n x_n + (I-\alpha_n\mu F)T^n y_n] - T^n y_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - T^n y_n\| + \beta_n\|x_n - T^n y_n\| + \alpha_n\|x_n - \mu F(T^n y_n)\|.\end{aligned}$$

Then,

$$\begin{aligned}\|x_n - T^n y_n\| &\le \tfrac{1}{1-\beta_n}\{\|x_n - x_{n+1}\| + \alpha_n(\|f(x_n) - T^n y_n\| + \|x_n - \mu F(T^n y_n)\|)\} \\ &\le \tfrac{1}{1-d}\{\|x_n - x_{n+1}\| + \alpha_n(\|f(x_n) - T^n y_n\| + \|x_n - \mu F(T^n y_n)\|)\}.\end{aligned}$$

Hence, we get

$$\begin{aligned}\|y_n - T^n y_n\| &\le \|y_n - x_n\| + \|x_n - T^n y_n\| \\ &\le \|y_n - x_n\| + \tfrac{1}{1-d}\{\|x_n - x_{n+1}\| + \alpha_n(\|f(x_n) - T^n y_n\| + \|x_n - \mu F(T^n y_n)\|)\}.\end{aligned}$$

Consequently, from (15), (29) and $\lim_{n\to\infty} \alpha_n = 0$, we obtain that

$$\lim_{n\to\infty} \|x_n - T^n y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - T^n y_n\| = 0. \tag{32}$$

So it follows that

$$\begin{aligned}\|x_n - w_n\| &\le \|x_n - \alpha_n x_n - (I-\alpha_n\mu F)T^n y_n\| \\ &\le \|x_n - T^n y_n\| + \alpha_n\|x_n - \mu F(T^n y_n)\| \to 0 \quad (n\to\infty).\end{aligned}$$

That is,

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{33}$$

We also note that

$$\begin{aligned}\|y_n - Ty_n\| &\le \|y_n - T^n y_n\| + \|T^n y_n - T^{n+1} y_n\| + \|T^{n+1} y_n - Ty_n\| \\ &\le \|y_n - T^n y_n\| + \|T^n y_n - T^{n+1} y_n\| + (1+\theta_1)\|T^n y_n - y_n\| \\ &= \|T^n y_n - T^{n+1} y_n\| + (2+\theta_1)\|T^n y_n - y_n\|.\end{aligned}$$

By the condition (v) and (32), we get

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0.$$

Further, noticing that

$$\|x_n - Tx_n\| \le \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \le \|y_n - Ty_n\| + (2+\theta_1)\|x_n - y_n\|,$$

we deduce from (29) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{34}$$

Step 5. Set $\overline{S} := (2I - S)^{-1}$. We aim to prove $\|x_n - \overline{S}x_n\| \to 0$ as $n \to \infty$. We show that $S : C \to C$ is pseudocontractive and ℓ-Lipschitzian such that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$, where $Sx = \lim_{n\to\infty} S_n x \; \forall x \in C$. Observe that for all $x, y \in C$, $\lim_{n\to\infty} \|S_n x - Sx\| = 0$ and $\lim_{n\to\infty} \|S_n y - Sy\| = 0$. Since each S_n is a pseudocontractive operator, we get

$$\langle Sx - Sy, x - y\rangle = \lim_{n\to\infty} \langle S_n x - S_n y, x - y\rangle \le \|x - y\|^2.$$

This presents that S is pseudocontractive. Note that $\{S_n\}_{n=0}^{\infty}$ is ℓ-uniformly Lipschitzian

$$\|Sx - Sy\| = \lim_{n\to\infty} \|S_n x - S_n y\| \le \ell\|x - y\|, \quad \forall x, y \in C.$$

This means that S is ℓ-Lipschitzian. Since the boundedness of $\{x_n\}$ and putting $D = \overline{\text{conv}}\{x_n : n \ge 0\}$ (the closure of convex hull of the set $\{x_n : n \ge 0\}$), we have $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$. Hence, by Proposition 1, we get

$$\lim_{n\to\infty} \|S_n x_n - Sx_n\| = 0. \tag{35}$$

Thus, combining (31) with (35) we have

$$\|x_n - Sx_n\| \le \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0 \quad (n \to \infty).$$

That is,

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{36}$$

Define $\overline{S} := (2I - S)^{-1}$. $\overline{S} : C \to C$ is nonexpansive, $\text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$ and $\lim_{n\to\infty} \|x_n - \overline{S}x_n\| = 0$. Indeed, put $\overline{S} := (2I - S)^{-1}$, where I is the identity mapping of H. Then, $\overline{S}$ is nonexpansive and the fixed point set $\text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$. Observe that

$$\begin{aligned} \|x_n - \overline{S}x_n\| &= \|\overline{S}\,\overline{S}^{-1} x_n - \overline{S}x_n\| \\ &\le \|\overline{S}^{-1} x_n - x_n\| \\ &= \|x_n - Sx_n\|. \end{aligned}$$

From (36), it follows that

$$\lim_{n\to\infty} \|x_n - \overline{S}x_n\| = 0. \tag{37}$$

Step 6. We aim to present

$$\limsup_{n\to\infty} \langle (I - f)x^*, x^* - x_n\rangle \le 0, \tag{38}$$

where $\{x^*\} = \text{VI}(\text{VI}(\Omega, A - g), I - f)$. Indeed, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle (I - f)x^*, x^* - x_{n_i}\rangle = \limsup_{n\to\infty} \langle (I - f)x^*, x^* - x_n\rangle.$$

We suppose a subsequence $x_{n_i} \rightharpoonup \bar{x} \in C$. Observe that G and $\overline{S}$ have the nonexpansivity and that T has the asymptotically nonexpansivity. Since $(I - G)x_n \to 0$, $(I - T)x_n \to 0$ and $(I - \overline{S})x_n \to 0$, by Lemma 7, we have that $\bar{x} \in \text{Fix}(G) = \text{GSVI}(C, B_1, B_2)$, $\bar{x} \in \text{Fix}(T)$ and $\bar{x} \in \text{Fix}(\overline{S}) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$. Then, $\bar{x} \in \Omega = \bigcap_{n=0}^{\infty} \text{Fix}(S_n) \cap \text{GSVI}(C, B_1, B_2) \cap \text{Fix}(T)$. We present that $\bar{x} \in \text{VI}(\Omega, A - g)$. As a fact, let $y \in \Omega$ be a arbitrarily fixed point. Then, it follows from (6), (8), and the monotonicity of $A - g$ that

$$\begin{aligned} &\|y_n - y\|^2 \le \|(z_n - y) - \alpha_n(Az_n - g(x_n))\|^2 \\ &= \|z_n - y\|^2 + 2\alpha_n\langle Az_n - g(x_n), y - z_n\rangle + \alpha_n^2\|Az_n - g(x_n)\|^2 \\ &\le \|x_n - y\|^2 + 2\alpha_n\langle Az_n - g(z_n), y - z_n\rangle + 2\alpha_n l\|z_n - x_n\|\|y - z_n\| + \alpha_n^2\|Az_n - g(x_n)\|^2 \\ &\le \|x_n - y\|^2 + 2\alpha_n\langle Ay - g(y), y - z_n\rangle + 2\alpha_n l\|z_n - x_n\|\|y - z_n\| + \alpha_n^2\|Az_n - g(x_n)\|^2, \end{aligned}$$

which implies that, for all $n \geq 0$,

$$\begin{aligned} 0 &\leq \frac{\|x_n-y\|^2-\|y_n-y\|^2}{\alpha_n} + 2\langle (A-g)y, y-z_n\rangle + 2l\|z_n - x_n\|\|y-z_n\| + \alpha_n\|Az_n - g(x_n)\|^2 \\ &\leq \frac{(\|x_n-y\|+\|y_n-y\|)\|x_n-y_n\|}{\alpha_n} + 2\langle (A-g)y, y-z_n\rangle + 2l\|z_n - x_n\|\|y-z_n\| + \alpha_n\|Az_n - g(x_n)\|^2. \end{aligned}$$

From (29), it is easy to see that $x_{n_i} \rightharpoonup \bar{x}$ leads to $z_{n_i} \rightharpoonup \bar{x}$. Since $\lim_{n\to\infty}\alpha_n = 0$ and $\|x_n - y_n\| = o(\alpha_n)$ (due to the assumption), we have

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty}\{\frac{(\|x_n-y\|+\|y_n-y\|)\|x_n-y_n\|}{\alpha_n} + 2\langle (A-g)y, y-z_n\rangle \\ &\quad + 2l\|z_n - x_n\|\|y-z_n\| + \alpha_n\|Az_n - g(x_n)\|^2\} \\ &= \liminf_{n\to\infty} 2\langle (A-g)y, y-z_n\rangle \leq \lim_{i\to\infty} 2\langle (A-g)y, y-z_{n_i}\rangle = 2\langle (A-g)y, y-\bar{x}\rangle. \end{aligned}$$

It follows that

$$\langle (A-g)y, y-\bar{x}\rangle \geq 0, \quad \forall y \in \Omega.$$

Accordingly, Lemma 5 and the Lipschitz continuity and monotonicity of $A-g$ grant that

$$\langle (A-g)\bar{x}, y-\bar{x}\rangle \geq 0, \quad \forall y \in \Omega;$$

that is, $\bar{x} \in \mathrm{VI}(\Omega, A-g)$. Consequently, from $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega, A-g), I-f)$, we have

$$\limsup_{n\to\infty}\langle (I-f)x^*, x^*-x_n\rangle = \lim_{i\to\infty}\langle (I-f)x^*, x^*-x_{n_i}\rangle = \langle (I-f)x^*, x^*-\bar{x}\rangle \leq 0. \tag{39}$$

Step 7. Finally, we prove $x_n \to x^*$ as $n \to \infty$. Indeed, from (4) we get

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &= \beta_n\langle x_n - x^*, x_{n+1}-x^*\rangle + \alpha_n\langle f(x_n)-x^*, x_{n+1}-x^*\rangle \\ &\quad + (1-\alpha_n-\beta_n)\langle w_n - x^*, x_{n+1}-x^*\rangle \\ &= \alpha_n[\langle f(x_n)-f(x^*), x_{n+1}-x^*\rangle + \langle f(x^*)-x^*, x_{n+1}-x_n\rangle \\ &\quad + \langle f(x^*)-x^*, x_n-x^*\rangle] + \beta_n\langle x_n-x^*, x_{n+1}-x^*\rangle \\ &\quad + (1-\alpha_n-\beta_n)[\langle w_n - x_n, x_{n+1}-x^*\rangle + \langle x_n - x^*, x_{n+1}-x^*\rangle] \\ &\leq \alpha_n[\delta\|x_n-x^*\|\|x_{n+1}-x^*\| + \|f(x^*)-x^*\|\|x_{n+1}-x_n\| \\ &\quad + \langle f(x^*)-x^*, x_n-x^*\rangle] + \beta_n\|x_n-x^*\|\|x_{n+1}-x^*\| \\ &\quad + \|w_n-x_n\|\|x_{n+1}-x^*\| + (1-\alpha_n-\beta_n)\|x_n-x^*\|\|x_{n+1}-x^*\| \\ &\leq [1-\alpha_n(1-\delta)]\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\|f(x^*)-x^*\|\|x_{n+1}-x_n\| \\ &\quad + \alpha_n\langle f(x^*)-x^*, x_n-x^*\rangle + \|w_n-x_n\|\|x_{n+1}-x^*\| \\ &\leq \frac{[1-\alpha_n(1-\delta)]^2}{2}\|x_n-x^*\|^2 + \frac{1}{2}\|x_{n+1}-x^*\|^2 + \alpha_n\|f(x^*)-x^*\|\|x_{n+1}-x_n\| \\ &\quad + \alpha_n\langle f(x^*)-x^*, x_n-x^*\rangle + \|w_n-x_n\|\|x_{n+1}-x^*\|, \end{aligned}$$

which immediately yields

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &\leq 2\alpha_n\|f(x^*)-x^*\|\|x_{n+1}-x_n\| + [1-\alpha_n(1-\delta)]^2\|x_n-x^*\|^2 \\ &\quad + 2\alpha_n\langle f(x^*)-x^*, x_n-x^*\rangle + 2\|w_n-x_n\|\|x_{n+1}-x^*\| \\ &\leq [1-\alpha_n(1-\delta)]\|x_n-x^*\|^2 + \alpha_n(1-\delta)\cdot\frac{2}{1-\delta}\{\|f(x^*)-x^*\|\|x_{n+1}-x_n\| \\ &\quad + \langle f(x^*)-x^*, x_n-x^*\rangle + \frac{\|w_n-x_n\|}{\alpha_n}\cdot\|x_{n+1}-x^*\|\}. \end{aligned} \tag{40}$$

Since $\|w_n - x_n\| = o(\alpha_n)$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, and $\lim_{n\to\infty}\alpha_n = 0$, we deduce from (15), (38), and (39) that $\sum_{n=0}^{\infty}\alpha_n(1-\delta) = \infty$ and

$$\limsup_{n\to\infty}\{\|f(x^*)-x^*\|\|x_{n+1}-x_n\| + \langle f(x^*)-x^*, x_n-x^*\rangle + \frac{\|w_n-x_n\|}{\alpha_n}\cdot\|x_{n+1}-x^*\|\} \leq 0.$$

Therefore, applying Lemma 3 to relation (40), we conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

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