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Convergence Analysis of Weighted-Newton Methods of Optimal Eighth Order in Banach Spaces

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Abstract: We generalize a family of optimal eighth order weighted-Newton methods to Banach spaces and study their local convergence. In a previous study, the Taylor expansion of higher order derivatives is employed which may not exist or may be very expensive to compute. However, the hypotheses of the present study are based on the first Fréchet-derivative only, thereby the application of methods is expanded. New analysis also provides the radius of convergence, error bounds and estimates on the uniqueness of the solution. Such estimates are not provided in the approaches that use Taylor expansions of derivatives of higher order. Moreover, the order of convergence for the methods is verified by using computational order of convergence or approximate computational order of convergence without using higher order derivatives. Numerical examples are provided to verify the theoretical results and to show the good convergence behavior.

Keywords: weighted-Newton methods; convergence; Banach spaces; Fréchet-derivative

MSC: 49M15; 41A25; 65H10; 65J10

1. Introduction

In this work, we generate a sequence $\{x_n\}$ for approximating a locally unique solution α of the nonlinear equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on a closed convex subset D of Banach space B_1 with values in a Banach space B_2. In computational sciences, many problems can be written in the form (1). See, for example [1,2]. The solutions of such equations are rarely attainable in closed form. This shows why most methods for solving these equations are usually iterative in nature. The important part in the construction of an iterative method is to study its convergence analysis. In general, the convergence domain is small. Therefore, it is important to enlarge the convergence domain without using extra hypotheses. Knowledge of the radius of convergence is useful because it gives us the degree of difficulty for obtaining initial points. Another important problem is to find more precise error estimates on $\|x_{n+1} - x_n\|$ or $\|x_n - \alpha\|$. Many authors have studied convergence analysis of iterative methods, see, for example [1–7].

The most widely used iterative method for solving (1) is the quadratically convergent Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \;\; n = 0, 1, 2, \ldots, \tag{2}$$

where $F'(x)^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$. In order to accelerate the convergence, researchers have also obtained modified Newton's or Newton-like methods (see [4,6,8–17]) and references therein.

There are numerous higher order iterative methods for solving a scalar equation $f(x) = 0$ (see, for example [2]. Contrary to this fact, higher order methods are rare for multi-dimensional cases, that is, for approximating the solution of $F(x) = 0$. One possible reason is that the construction of higher order methods for solving systems is a difficult task. Another factual reason is that not every method developed for single equations can be generalized to solve systems of nonlinear equations. Recently, a family of optimal eighth order methods for solving a scalar equation $f(x) = 0$ has been proposed in [16], which is given by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= \phi_4(x_n, y_n), \\ x_{n+1} &= z_n - \frac{f[z_n, y_n]}{2f[z_n, y_n] - f[z_n, x_n]} \frac{f(z_n)}{f[z_n, x_n]}, \end{aligned} \tag{3}$$

where $\phi_4(x_n, y_n)$ is any optimal fourth order scheme with the base as Newton's iteration y_n and $f[\cdot, \cdot]$ is Newton's first order divided difference. In particular, they have considered the following optimal fourth order schemes in the second step of (3):

Ostrowski method (see [12]):

$$z_n = y_n - \frac{1}{2f[y_n, x_n] - f'(x_n)} f(y_n). \tag{4}$$

Ostrowski-like method (see [12]):

$$z_n = y_n - \left(\frac{2}{f[y_n, x_n]} - \frac{1}{f'(x_n)} \right) f(y_n). \tag{5}$$

Kung-Traub method (see [15]):

$$z_n = y_n - \frac{f'(x_n) f(y_n)}{f[y_n, x_n]^2}. \tag{6}$$

Motivated by the above methods defined on the real line, we propose the methods that follow but for Banach space valued operators. It can be observed that the above family of eighth order methods can be easily extendable for solving (1). In view of this, here we study the method (3) in Banach space. The iterative methods corresponding to the fourth order schemes (4)–(6) in the Banach space setting are given as

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ z_n &= y_n - \big(2F[y_n, x_n] - F'(x_n)\big)^{-1} F(y_n), \\ x_{n+1} &= \Psi_8(x_n, y_n, z_n), \end{aligned} \tag{7}$$

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ z_n &= y_n - \big(2F[y_n, x_n]^{-1} - F'(x_n)^{-1}\big) F(y_n), \\ x_{n+1} &= \Psi_8(x_n, y_n, z_n) \end{aligned} \tag{8}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - F[y_n, x_n]^{-1}F'(x_n)F[y_n, x_n]^{-1}F(y_n), \\ x_{n+1} &= \Psi_8(x_n, y_n, z_n). \end{aligned} \tag{9}$$

In above each case, we have that

$$\Psi_8(x_n, y_n, z_n) = z_n - \left(2F[z_n, y_n] - F[z_n, x_n]\right)^{-1}F[z_n, y_n]F[z_n, x_n]^{-1}F(z_n). \tag{10}$$

Here $F[\cdot,\cdot] : D \times D \to L(B_1, B_2)$ is a first order divided difference on $D \times D$ satisfying $F[x, y](x - y) = F(x) - F(y)$ for $x \neq y$ and $F[x, x] = F'(x)$ if F is differentiable, where $L(B_1, B_2)$ stands for the space of bounded linear operators from B_1 into B_2. Methods (7)–(9) require four inverses and four function evaluations at each step.

The rest of the paper is summarized as follows. In Section 2, the local convergence, including radius of convergence, computable error bounds and uniqueness results of the proposed methods, is presented. In order to verify the theoretical results of convergence analysis, some numerical examples are presented in Section 3. Finally, the methods are applied to solve systems of nonlinear equations in Section 4.

2. Local Convergence

Local convergence analysis of the methods (7)–(9) is presented by using some real functions and parameters. Let $\lambda_0 : [0, +\infty) \to [0, +\infty)$ be a continuous and increasing function satisfying $\lambda_0(0) = 0$. Suppose that equation

$$\lambda_0(t) = 1 \tag{11}$$

has positive solutions. Denote by ϱ the smallest such solution. Let $\lambda : [0, \varrho) \to [0, +\infty)$, $\mu : [0, \varrho) \to [0, +\infty)$, $\lambda_1 : [0, \varrho) \times [0, \varrho) \to [0, +\infty)$ and $\mu_1 : [0, \varrho) \times [0, \varrho) \to [0, +\infty)$ also be continuous and increasing functions satisfying $\lambda_1(0, 0) = 0$. Define functions g_1 and h_1 on the interval $[0, \varrho)$ by

$$g_1(t) = \frac{\int_0^1 \lambda((1-\theta)t)d\theta}{1-\lambda_0(t)} \quad \text{and} \quad h_1(t) = g_1(t) - 1.$$

We have that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to \varrho^-$. By applying the Bolzano's theorem on function h_1, we deduce that equation $h_1(t) = 0$ has solutions in the interval $(0, \varrho)$. Let r_1 be the smallest such zero.

Moreover, define function p and h_p on the interval $[0, \varrho)$ by

$$p(t) = 2\lambda_1(g_1(t)t, t) + \lambda_0(t)$$

and

$$h_p(t) = p(t) - 1.$$

We get $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \to \varrho^-$. Let r_p be the smallest solution of equation $h_p(t) = 0$ in the interval $(0, \varrho)$. Furthermore, define functions g_2 and h_2 on the interval $[0, r_p)$ by

$$g_2(t) = \left(1 + \frac{\int_0^1 \mu(\theta g_1(t)t)d\theta}{(1 - p(t))(1 - \lambda_0(t))}\right) g_1(t)$$

and

$$h_2(t) = g_2(t) - 1.$$

We obtain $h_2(t) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_p^-$. Let r_2 be the smallest solution of equation $h_2(t) = 0$ in the interval $(0, r_p)$. Define functions q and h_q on the interval $(0, r_p)$ and functions φ and ψ on the interval $[0, r_p)$, respectively by

$$\begin{aligned}
q(t) &= 2\lambda_1(g_2(t)t, g_1(t)t) + \lambda_1(g_2(t)t, t),\\
h_q(t) &= q(t) - 1,\\
\varphi(t) &= \lambda_1(g_2(t), t),\\
\psi(t) &= \varphi(t) - 1.
\end{aligned}$$

We get $h_q(0) = \psi(0) = -1 < 0$ and $h_q(t) \to +\infty$ as $t \to r_p^-$, $\psi(t) \to +\infty$ as $t \to r_\psi^-$. Let r_q, r_ψ be the smallest solutions of equations $h_q(t) = 0$, $\psi(t) = 0$ in the intervals $(0, r_p)$, $(0, \varrho)$, respectively. Finally, define functions g_3 and h_3 on the interval $[0, \varrho_0)$ by

$$g_3(t) = \left(1 + \frac{\mu_1(g_2(t)t, g_1(t)t)}{(1 - q(t))(1 - \lambda_1(g_2(t)t, t))}\right) g_2(t)$$

and

$$h_3(t) = g_3(t) - 1,$$

where $\varrho_0 = \min\{r_q, r_\psi\}$. We have that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \varrho_0^-$. Let r_3 be the smallest solution of equation $h_3(t) = 0$ in the interval $(0, \varrho_0)$. Set

$$r = \min\{r_i\} \qquad i = 1, 2, 3..... \tag{12}$$

to be the radius of convergence for method (7). Then, for each $t \in [0, r)$, it follows that

$$0 \le g_i(t) \le 1, \tag{13}$$

$$0 \le p(t) \le 1, \tag{14}$$

$$0 \le \varphi(t) \le 1 \tag{15}$$

and

$$0 \le q(t) \le 1. \tag{16}$$

Let $U(a, b)$ and $\bar{U}(a, b)$ stand, respectively for the open and closed balls in B_1 with center $a \in D$ and of radius $b > 0$.

The local convergence analysis of method (7), method (8) and method (9) is based on the conditions (A):

(a_1) $F : D \subset B_1 \to B_2$ is continuously Fréchet differentiable and D is a convex set. The operator $F[\cdot, \cdot] : D \times D \to L(B_1, B_2)$ is a divided difference of order one satisfying

$$F[x, y](x - y) = F(x) - F(y) \quad \text{for } x \ne y$$

and

$$F[x, x] = F'(x).$$

(a_2) There exists $\alpha \in D$ such that $F(\alpha) = 0$ and $F'(\alpha)^{-1} \in L(B_2, B_1)$.

(a_3) There exists function $\lambda_0 : [0, +\infty) \to [0, +\infty)$ continuous and increasing with $\lambda_0(0) = 0$ such that for each $x \in D$

$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \le \lambda_0(\|x - \alpha\|).$$

Set $D_0 = D \cap U(\alpha, \varrho)$, where ϱ is given in (11).

(a_4) There exist continuous and increasing functions $\lambda : [0,\varrho) \to [0,+\infty)$, $\lambda_1 : [0,\varrho) \times [0,\varrho) \to [0,+\infty)$, $\mu : [0,\varrho) \to [0,+\infty)$ and $\mu_1 : [0,\varrho) \times [0,\varrho) \to [0,+\infty)$ such that for each $x, y \in D_0$

$$\|F'(\alpha)^{-1}(F'(x) - F'(y))\| \le \lambda(\|x-y\|),$$

$$\|F'(\alpha)^{-1}(F[x,y] - F'(\alpha))\| \le \lambda_1(\|y-\alpha\|, \|x-\alpha\|),$$

$$\|F'(\alpha)^{-1}F'(x)\| \le \mu(\|x-y\|)$$

and

$$\|F'(\alpha)^{-1}F[x,y]\| \le \mu_1(\|x-\alpha\|, \|y-\alpha\|).$$

(a_5) $\bar{U}(\alpha, r) \subseteq D$ where r is given in (12) for method (7), by (30) for method (8) and by (31) for method (9).

(a_6) There exists $R \ge r$ such that

$$\int_0^1 \lambda_0(\theta R) d\theta < 1.$$

Set $D_1 = D \cap \bar{U}(\alpha, R)$.

Next, we first present the local convergence analysis of method (7) based on the conditions (A).

Theorem 1. *Assume that the conditions (A) hold. Then, sequence $\{x_n\}$ generated for $x_0 \in U(\alpha, r) - \{\alpha\}$ by method (7) is well defined in $U(\alpha, r)$, remains in $U(\alpha, r)$ for each $n = 0, 1, 2 \ldots\ldots$ and converges to α so that*

$$\|y_n - \alpha\| \le g_1(\|x_n - \alpha\|)\|x_n - \alpha\| \le \|x_n - \alpha\| < r, \tag{17}$$

$$\|z_n - \alpha\| \le g_2(\|x_n - \alpha\|)\|x_n - \alpha\| \le \|x_n - \alpha\| \tag{18}$$

and

$$\|x_{n+1} - \alpha\| \le g_3(\|x_n - \alpha\|)\|x_n - \alpha\| \le \|x_n - \alpha\|, \tag{19}$$

where the functions g_i are defined previously. Moreover, the solution α of equation $F(x) = 0$ is unique in D_1.

Proof. We shall show assertions (17)–(19) using mathematical induction. Let $x \in U[\alpha, \varrho)$. Then, using (a_3) and (12), we have that

$$\|F'(\alpha)^{-1}\big(F'(x_0) - F'(\alpha)\big)\| \le \lambda_0\|x - \alpha\| < \lambda_0(r) < 1. \tag{20}$$

By the Banach perturbation Lemma [2] and (20), we deduce that $F'(x)^{-1} \in L(B_2, B_1)$ and

$$\|F'(x)^{-1}F'(\alpha)\| \le \frac{1}{1 - \lambda_0\|x - \alpha\|}. \tag{21}$$

In particular for $x = x_0$, y_0 is well defined by the first substep of method (7) and (21) holds for $x = x_0$, since $x_0 \in U[\alpha, r)$. We get by the first substep of method (7) for $n = 0$, (a_2), (a_4), (13) (for $i = 1$) and (12) that

$$\begin{aligned} \|y_0 - \alpha\| &= \|x_0 - \alpha - F'(x_0)^{-1}F(x_0)\| \\ &= \left\|[F'(x_0)^{-1}F'(\alpha)][\int_0^1 F'(\alpha)^{-1}(F'(\alpha + \theta(x_0 - \alpha)) - F'(x_0))(x_0 - \alpha)d\theta]\right\| \\ &\le \frac{\int_0^1 \lambda((1-\theta)\|x_0 - \alpha\|)d\theta}{1 - \lambda_0(\|x_0 - \alpha\|)}\|x_0 - \alpha\| \\ &= g_1(\|x_0 - \alpha\|)\|x_0 - \alpha\| \le \|x_0 - \alpha\| < r, \end{aligned} \tag{22}$$

so (17) holds for $n = 0$ and $y_0 \in U(\alpha, r)$. We must show the existence of $(2F[y_0, x_0] - F'(x_0))^{-1}$ which shall imply that z_0 is well defined. Using (12), (14), (a_3) and (a_4), we get in turn that

$$\begin{aligned}\|F'(\alpha)^{-1}(2(F[y_0,x_0]-F'(\alpha))+(F'(\alpha)-F'(x_0)))\| &\le 2\|F'(\alpha)^{-1}(F[y_0,x_0]-F'(\alpha))\| \\ &+ \|F'(\alpha)^{-1}(F'(\alpha)-F'(x_0))\| \\ &\le 2\lambda_1(\|y_0-\alpha\|,\|x_0-\alpha\|)+\lambda_0(\|x_0-\alpha\|) \\ &\le 2\lambda_1(g_1(\|x_0-\alpha\|)\|x_0-\alpha\|,\|x_0-\alpha\|) \\ &+ \lambda_0(\|x_0-\alpha\|) \\ &= p(\|x_0-\alpha\|) \le p(r) < 1,\end{aligned}$$

so $(2F[y_0, x_0] - F'(x_0))^{-1}$ exists and

$$\|(2F[y_0,x_0]-F'(x_0))^{-1}F'(\alpha)\| = \frac{1}{1-p(\|x_0-\alpha\|)}. \tag{23}$$

We can write

$$F(x) = F(x) - F(\alpha) = \int_0^1 F'(\alpha+\theta(x-\alpha))(x-\alpha)d\theta. \tag{24}$$

Notice that $\|\alpha + \theta(x-\alpha) - \alpha\| = \theta\|x-\alpha\| \le r$ for each $\theta \in [0,1]$. Using (a_4) and (24), we get

$$\|F'(\alpha)^{-1}F(x)\| \le \int_0^1 \mu(\theta\|x-\alpha\|)d\theta\|x-\alpha\|. \tag{25}$$

Then, by (12), (13) (for $i = 2$), (21), (22), (23), (25) and the second substep of method (7), we obtain in turn that

$$\begin{aligned}\|z_0-\alpha\| &\le \|y_0-\alpha\| + \|(2F[y_0,x_0]-F'(x_0))^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}F(y_0)\| \\ &\le \|y_0-\alpha\| + \frac{\int_0^1 \mu(\theta\|y_0-\alpha\|)d\theta\|y_0-\alpha\|}{(1-p(\|x_0-\alpha\|))(1-\lambda_0(\|x_0-\alpha\|))} \\ &\le \Big(1+\frac{\int_0^1 \mu(\theta g_1(\|x_0-\alpha\|)\|x_0-\alpha\|)d\theta}{(1-p(\|x_0-\alpha\|))(1-\lambda_0(\|x_0-\alpha\|))}\Big)g_1(\|x_0-\alpha\|)\|x_0-\alpha\| \\ &= g_2(\|x_0-\alpha\|)\|x_0-\alpha\| \le \|x_0-\alpha\| < r,\end{aligned} \tag{26}$$

which shows (18) for $n = 0$ and $z_0 \in U(\alpha, r)$. We must show the existence of $F[z_0, x_0]^{-1}$ which shall also imply that x_1 is well defined. Using (12), (15) and (a_4), we obtain in turn that

$$\begin{aligned}\|F'(\alpha)^{-1}(F[z_0,x_0]-F'(\alpha))\| \le \lambda_1(\|z_0-\alpha\|,\|x_0-\alpha\|) &\le \lambda_1(g_2(\|x_0-\alpha\|)\|x_0-\alpha\|,\|x_0-\alpha\|) \\ = \varphi\|x_0-\alpha\| \le \varphi(r) < 1,\end{aligned}$$

so $F[z_0, x_0]^{-1}$ exists and

$$\|F[z_0,x_0]^{-1}F'(\alpha)\| \le \frac{1}{1-\varphi(\|x_0-\alpha\|)}. \tag{27}$$

Then, using the last substep of method (7), (10), (12), (13) (for $i = 3$), (18), (23), (26) and (27), we get in turn that

$$\begin{aligned}\|x_1-\alpha\| &\le \|z_0-\alpha\| + \frac{\mu_1(\|z_0-\alpha\|,\|y_0-\alpha\|)\int_0^1 \mu(\theta\|z_0-\alpha\|)d\theta\|z_0-\alpha\|}{(1-q(x_0-\alpha))(1-\lambda_1(\|z_0-\alpha\|,\|x_0-\alpha\|))} \\ &\le \Big(1+\frac{\mu_1(g_2(\|x_0-\alpha\|)\|x_0-\alpha\|,g_1(\|x_0-\alpha\|)\|x_0-\alpha\|)}{(1-q(\|x_0-\alpha\|))(1-\lambda_1(g_2(\|x_0-\alpha\|)\|x_0-\alpha\|,\|x_0-\alpha\|))}\Big) \\ &\times g_2(\|x_0-\alpha\|)\|x_0-\alpha\|,\end{aligned} \tag{28}$$

which shows (19) for $n = 0$ and $x_1 \in U(\alpha, r)$. The induction is completed if x_k, y_k, z_k, x_{k+1} replace x_0, y_0, z_0, x_1 in the preceding estimates, respectively. Then, from the estimate

$$\|x_{k+1} - \alpha\| \leq c\|x_k - \alpha\| < r, \tag{29}$$

where $c = g_3(\|x_k - \alpha\|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} x_k = \alpha$ and $x_{k+1} \in U(\alpha, r)$. Let $Q = \int_0^1 F'(\alpha + \theta(y^* - \alpha))d\theta$ for some $y^* \in D_1$ such that $F(y^*) = 0$. By (a_3) and (a_6), we have in turn that

$$\begin{aligned}\|F'(\alpha)^{-1}(Q - F'(\alpha))\| &\leq \int_0^1 \lambda_0(\|\alpha + \theta(y^* - \alpha) - \alpha\|)d\theta, \\ &\leq \int_0^1 \lambda_0(\theta\|\alpha - y^*\|)d\theta \leq \int_0^1 \lambda_0(\theta R)d\theta < 1,\end{aligned}$$

implies that Q^{-1} exists. Then, from the identity $0 = F(y^*) - F(\alpha) = Q(y^* - \alpha)$, we conclude that $\alpha = y^*$. □

Next, we shall show the local convergence of method (8) in an analogous way but functions g_2, φ, g_3 shall be replaced by $\bar{g}_2$, φ_1, $\bar{g}_3$ and which are given by

$$\begin{aligned}
\bar{g}_2(t) &= \Big(1 + \frac{\mu(t)\int_0^1 \mu(\theta g_1(t)t)d\theta}{(1 - \lambda_1(g_1(t)t, t))^2}\Big)g_1(t), \\
\bar{h}_2(t) &= \bar{g}_2(t) - 1, \\
\varphi_1(t) &= \lambda_1(g_1(t)t, t), \\
\psi_1(t) &= \varphi_1(t) - 1, \\
\bar{g}_3(t) &= \Big(1 + \frac{\mu_1(\bar{g}_2(t)t, g_1(t)t)}{(1 - q(t))(1 - \lambda_1(\bar{g}_2(t)t, t))}\Big)\bar{g}_2(t), \\
\bar{h}_3(t) &= \bar{g}_3(t) - 1.
\end{aligned}$$

We shall use the same notation for r_1 as in (12) but notice that $\bar{r}_2$ and $\bar{r}_3$ correspond to the smallest positive solutions of equations $\bar{h}_2(t) = 0$ and $\bar{h}_3(t) = 0$, respectively. Set

$$\bar{r} = \min\{r_1, \bar{r}_2, \bar{r}_3\}. \tag{30}$$

The local convergence analysis of method (8) is given by the following theorem:

Theorem 2. *Assume that the conditions (A) hold. Then, the conclusions of Theorem 1 also hold for method (8) with functions $\bar{g}_2$, $\bar{g}_3$ and $\bar{r}$ replacing g_2, g_3 and r, respectively.*

Proof. We have that

$$\|y_n - \alpha\| \leq g_1(\|x_n - \alpha\|)\|x_n - \alpha\| \leq \|x_n - \alpha\| < \bar{r}$$

as in Theorem 1 and using the second and third substep of method (8) we get (as in Theorem 1) that

$$\begin{aligned}\|z_n - \alpha\| &\leq \|y_n - \alpha\| + \frac{\mu(\|x_n - \alpha\|)\int_0^1 \mu(\theta\|y_n - \alpha\|)d\theta\|y_n - \alpha\|}{(1 - \lambda_1(\|y_n - \alpha\|, \|x_n - \alpha\|))^2} \\ &\leq \bar{g}_2(\|x_n - \alpha\|)\|x_n - \alpha\| \leq \|x_n - \alpha\|\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-\alpha\| &\leq \Big(1+\frac{\mu_1(\bar{g}_2(\|x_n-\alpha\|)\|x_n-\alpha\|, g_1(\|x_n-\alpha\|)\|x_n-\alpha\|)}{(1-q(\|x_n-\alpha\|))(1-\lambda_1(\bar{g}_2(\|x_n-\alpha\|)\|x_n-\alpha\|, \|x_n-\alpha\|))}\Big)\\ &\quad\times \bar{g}_2(\|x_n-\alpha\|)\|x_n-\alpha\|\\ &\leq \bar{g}_3(\|x_n-\alpha\|)\|x_n-\alpha\| \leq \|x_n-\alpha\|.\end{aligned}$$

□

We define

$$\begin{aligned}\bar{\bar{g}}_2(t) &= \Big(1+2\frac{\int_0^1 \mu(\theta(g_1(t)t)d\theta}{(1-\lambda_1(g_1(t)t,t))}+\frac{\int_0^1 \mu(\theta(g_1(t)t)d\theta}{1-\lambda_0(t)}\Big)g_1(t),\\ \bar{\bar{h}}_2(t) &= \bar{\bar{g}}_2(t)-1,\\ \bar{\bar{g}}_3(t) &= \Big(1+\frac{\mu_1(\bar{\bar{g}}_2(t)t, g_1(t)t)}{(1-q(t))(1-\lambda_1(\bar{\bar{g}}_2(t)t,t))}\Big)\bar{\bar{g}}_2(t),\\ \bar{\bar{h}}_3(t) &= \bar{\bar{g}}_3(t)-1.\end{aligned}$$

Denote by $\bar{\bar{r}}_2, \bar{\bar{r}}_3$, the smallest positive solutions of equations $\bar{\bar{h}}_2(t)=0$ and $\bar{\bar{h}}_3(t)=0$. Set

$$\bar{\bar{r}} = \min\{r_1, \bar{\bar{r}}_2, \bar{\bar{r}}_3\}. \tag{31}$$

Then, we have:

Theorem 3. *Assume that the conditions (A) hold. Then, the conclusions of Theorem 1 also hold for method (9) with functions $\bar{\bar{g}}_2$, $\bar{\bar{g}}_3$ and $\bar{\bar{r}}$ replacing g_2, g_3 and r, respectively.*

Proof. Notice that from the second and third substep of method (9) we obtain

$$\begin{aligned}\|z_n-\alpha\| &\leq \|y_n-\alpha\| + \|2F[y_n,x_n]^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}F(y_n)\|\\ &\leq \|y_n-\alpha\| + 2\frac{\int_0^1 \mu(\theta\|y_n-\alpha\|)d\theta\|y_n-\alpha\|}{(1-\lambda_1(\|y_n-\alpha\|,\|x_n-\alpha\|))}+\frac{\int_0^1 \mu(\theta\|y_n-\alpha\|)d\theta}{(1-\lambda_0(\|x_n-\alpha\|))}\|y_n-\alpha\|\\ &\leq \bar{g}_2(\|x_n-\alpha\|)\|x_n-\alpha\| \leq \|x_n-\alpha\| \leq \bar{\bar{r}}\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-\alpha\| &\leq \Big(1+\frac{\mu_1(\bar{\bar{g}}_2(\|x_n-\alpha\|)\|x_n-\alpha\|, g_1(\|x_n-\alpha\|)\|x_n-\alpha\|)}{(1-q(\|x_n-\alpha\|))(1-\lambda_1(\bar{\bar{g}}_2(\|x_n-\alpha\|)\|x_n-\alpha\|, \|x_n-\alpha\|))}\Big)\\ &\quad\times \bar{\bar{g}}_2(\|x_n-\alpha\|)\|x_n-\alpha\|.\end{aligned}$$

□

Remark 1. *Methods (7)–(9) are not effected, when we use the conditions of the Theorems 1–3 instead of stronger conditions used in ([16], Theorem 1). Moreover, we can compute the computational order of convergence (COC) [18] defined by*

$$COC = ln\left(\frac{\|x_{n+1}-\alpha\|}{\|x_n-\alpha\|}\right)\Big/ ln\left(\frac{\|x_n-\alpha\|}{\|x_{n-1}-\alpha\|}\right), \tag{32}$$

or the approximate computational order of convergence (ACOC) [9], given by

$$ACOC = ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right)\Big/ ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right). \tag{33}$$

In this way, we obtain in practice the order of convergence.

3. Numerical Examples

Here, we shall demonstrate the theoretical results which we have shown in Section 2. We use the divided difference given by $F[x,y] = \frac{1}{2}(F'(x) + F'(y))$ or $F[x,y] = \int_0^1 (F'(y + \tau(x-y))d\tau$.

Example 1. *Suppose that the motion of an object in three dimensions is governed by system of differential equations*

$$\begin{aligned} f_1'(x) - f_1(x) - 1 &= 0, \\ f_2'(y) - (e-1)y - 1 &= 0, \\ f_3'(z) - 1 &= 0. \end{aligned} \tag{34}$$

with x, y, $z \in D$ for $f_1(0) = f_2(0) = f_3(0) = 0$. Then, the solution of the system is given for $v = (x,y,z)^T$ by function $F := (f_1, f_2, f_3) : D \to \mathbb{R}^3$ defined by

$$F(u) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T. \tag{35}$$

The Fréchet-derivative is given by

$$F'(u) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{36}$$

Then for $\alpha = (0,0,0)^T$ we have that $\lambda(t) = et$, $\lambda_0(t) = (e-1)t$, $\lambda_1(s,t) = \frac{s+t}{2}$, $\mu(t) = 2$ and $\mu_1(s,t) = \frac{s-t}{2}$. The parameters r_1, r_2, r_3, $\bar{r}_2$, $\bar{r}_3$, $\bar{\bar{r}}_2$ and $\bar{\bar{r}}_3$ using methods (7)–(9) are given in Table 1.

Table 1. Numerical results for Example 1.

Method (7)	Method (8)	Method (9)
$r_1 = 0.324947$	$r_1 = 0.324947$	$r_1 = 0.324947$
$r_2 = 0.119823$	$\bar{r}_2 = 0.107789$	$\bar{\bar{r}}_2 = 0.083622$
$r_3 = 0.115973$	$\bar{r}_3 = 0.103461$	$\bar{\bar{r}}_3 = 0.080798$
$r = 0.115973$	$\bar{r} = 0.103461$	$\bar{\bar{r}} = 0.080798$

Theorems 1–3 guarantee the convergence of (7)–(9) to $\alpha = 0$ provided that $x_0 \in U(\alpha, r)$. This condition yields very close initial approximation.

Example 2. *Let $B_1 = C[0,1]$, be the space of continuous functions defined on the interval $[0,1]$. We shall utilize the max norm. Let $D = \bar{U}(0,1)$. Define function G on D by*

$$G(\varphi_2)(x) = \phi(x) - 10\int_0^1 x\theta\varphi_2(\theta)^3 d\theta.$$

We get that

$$G'(\varphi_2(\xi))(x) = \xi(x) - 30\int_0^1 x\theta\varphi_2(\theta)^2\xi(\theta)d\theta, \quad for\ \ each\ \ \xi \in D.$$

Then for $\alpha = 0$ we have that $\lambda(t) = 30t$, $\lambda_0(t) = 15t$, $\lambda_1(s,t) = \frac{s+t}{2}$, $\mu(t) = 1.85$ and $\mu_1(s,t) = \frac{s-t}{2}$. The parameters r_1, r_2, r_3, $\bar{r}_2$, $\bar{r}_3$, $\bar{\bar{r}}_2$ and $\bar{\bar{r}}_3$ using (7)–(9) are given in Table 2.

Table 2. Numerical results for Example 2.

Method (7)	Method (8)	Method (9)
$r_1 = 0.033333$	$r_1 = 0.033333$	$r_1 = 0.033333$
$r_2 = 0.013431$	$\bar{r}_2 = 0.011013$	$\bar{\bar{r}}_2 = 0.008039$
$r_3 = 0.013389$	$\bar{r}_3 = 0.010972$	$\bar{\bar{r}}_3 = 0.008015$
$r = 0.0133889$	$\bar{r} = 0.010972$	$\bar{\bar{r}} = 0.008015$

It is clear that the convergence of (7)–(9) is guaranteed to $\alpha = 0$ provided that $x_0 \in U(\alpha, r)$.

Example 3. *Let us consider the function* $H := (f_1, f_2, f_3) : D \to \mathbb{R}^3$ *defined by*

$$H(x) = \big(10\,x_1 + \sin(x_1 + x_2) - 1,\ 8\,x_2 - \cos^2(x_3 - x_2) - 1,\ 12\,x_3 + \sin(x_3) - 1\big)^T, \tag{37}$$

where $x = (x_1,\ x_2,\ x_3)^T$.

The Fréchet-derivative is given by

$$H'(x) = \begin{bmatrix} 10 + \cos(x_1 + x_2) & \cos(x_1 + x_2) & 0 \\ 0 & 8 + \sin 2(x_2 - x_3) & -2\sin(x_2 - x_3) \\ 0 & 0 & 12 + \cos(x_3) \end{bmatrix}.$$

With the initial approximation $x_0 = \{0, 0.5, 0.1\}^T$, we obtain the solution α of the function (37)

$$\alpha = \{0.06897\ldots, 0.24644\ldots, 0.07692\ldots\}^T.$$

Then we get that $\lambda(t) = 0.269812t$, $\lambda_0(t) = 0.269812t$, $\lambda_1(s,t) = \frac{s+t}{2}$, $\mu(t) = 13.0377$ and $\mu_1(s,t) = \frac{s-t}{2}$. The parameters r_1, r_2, r_3, $\bar{r}_2$, $\bar{r}_3$, $\bar{\bar{r}}_2$ and $\bar{\bar{r}}_3$ using methods (7)–(9) are given in Table 3.

Table 3. Numerical results for Example 3.

Method (7)	Method (8)	Method (9)
$r_1 = 2.470865$	$r_1 = 2.470865$	$r_1 = 2.470865$
$r_2 = 0.288117$	$\bar{r}_2 = 0.612891$	$\bar{\bar{r}}_2 = 0.639134$
$r_3 = 0.254805$	$\bar{r}_3 = 0.473734$	$\bar{\bar{r}}_3 = 0.461618$
$r = 0.254805$	$\bar{r} = 0.473734$	$\bar{\bar{r}} = 0.461618$

4. Applications

Lastly, we apply the methods (7)–(9) to solve systems of nonlinear equations in $\mathbb{R}^m$. The performance is also compared with some existing methods. For example, we choose Newton method (NM), sixth-order methods proposed by Grau et al. [12] and Sharma and Arora [15], and eighth-order Triple-Newton Method [14]. These methods are given as follows:

Grau-Grau-Noguera method:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - \big(2\,F[y_n, x_n] - F'(x_n)\big)^{-1}F(y_n), \\ x_{n+1} &= z_n - \big(2\,F[y_n, x_n] - F'(x_n)\big)^{-1}F(z_n). \end{aligned} \tag{38}$$

This method requires two inverses and three function evaluations.

Grau-Grau-Noguera method:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - \big(2\,F[y_n, x_n]^{-1} - F'(x_n)^{-1}\big)F(y_n), \\ x_{n+1} &= z_n - \big(2\,F[y_n, x_n]^{-1} - F'(x_n)^{-1}\big)F(z_n). \end{aligned} \tag{39}$$

It requires two inverses and three function evaluations.

Sharma-Arora Method:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - \big(3I - 2F'(x_n)^{-1}F[y_n, x_n]\big)F'(x_n)^{-1}F(y_n), \\ x_{n+1} &= z_n - \big(3I - 2F'(x_n)^{-1}F[y_n, x_n]\big)F'(x_n)^{-1}F(z_n). \end{aligned} \tag{40}$$

The method requires one inverse and three function evaluations.

Triple-Newton Method:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - F'(y_n)^{-1}F(y_n), \\ x_{n+1} &= z_n - F'(z_n)^{-1}F(z_n). \end{aligned} \tag{41}$$

This method requires three inverses and three function evaluations.

Example 4. *Let us consider the system of nonlinear equations:*

$$\begin{cases} x_i^2 x_{i+1} - 1 = 0,\ 1 \le i \le m-1, \\ x_i^2 x_1 - 1 = 0,\ i = m. \end{cases}$$

with initial value $x_0 = \{2, 2, \overset{m-times}{\cdots}, 2\}^T$ *towards the required solution* $\alpha = \{1, 1, \overset{m-times}{\cdots}, 1\}^T$ *of the systems for* $m = 8,\ 25,\ 50,\ 100$.

Example 5. *Next, consider the extended Freudenstein and Roth function [19]:*

$$F(x) = (f_1(x), f_2(x), \ldots, f_m(x))^T,$$

where

$$\begin{aligned} f_{2i-1}(x) &= x_{2i-1} + ((5 - x_{2i})x_{2i} - 2)x_{2i} - 13,\ \textit{for}\ i = 1, 2, \ldots, \frac{m}{2}, \\ f_{2i}(x) &= x_{2i-1} + ((1 + x_{2i})x_{2i} - 14)x_{2i} - 29,\ \textit{for}\ i = 1, 2, \ldots, \frac{m}{2}, \end{aligned}$$

with initial value $x_0 = \{3, 6, \overset{m-times}{\cdots}, 3, 6\}^T$ *towards the required solution* $x^* = \{5, 4, \overset{m-times}{\cdots}, 5, 4\}^T$ *of the systems for* $m = 20,\ 50,\ 100,\ 200$.

Computations are performed in the programming package *Mathematica* using multiple-precision arithmetic. For every method, we record the number of iterations (n) needed to converge to the solution such that the stopping criterion

$$||F(x_n)|| < 10^{-350}$$

is satisfied. In order to verify the theoretical order of convergence, we calculate the approximate computational order of convergence (ACOC) using the formula (33). For the computation of divided difference we use the formula (see [12])

$$F[x,y]_{ij} = \frac{f_i(x_1,....., x_j, y_{j+1},....., y_m) - f_i(x_1,....., x_{j-1}, y_j,....., y_m)}{x_j - y_j}, \ 1 \le i,j \le m.$$

Numerical results are displayed in Tables 4 and 5, which include:

- The dimension (m) of the system of equations.
- The required number of iterations (n).
- The value of $||F(x_n)||$ of approximation to the corresponding solution of considered problems, wherein $N(-h)$ denotes $N \times 10^{-h}$.
- The approximate computational order of convergence (ACOC).

Table 4. Comparison of performance of methods for Example 4. Approximate computational order of convergence (ACOC).

Methods	(2)	(38)	(39)	(40)	(41)	(7)	(8)	(9)
$m = 8$								
n	10	4	4	4	3	3	3	3
$\|\|F(x_n)\|\|$	9.26(−253)	1.30(−304)	8.01(−206)	1.18(−168)	2.80(−126)	6.07(−258)	1.00(−185)	4.15(−171)
ACOC	2.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
$m = 25$								
n	10	4	4	4	3	3	3	3
$\|\|F(x_n)\|\|$	1.64(−252)	2.29(−304)	1.42(−205)	2.10(−168)	4.95(−126)	1.07(−257)	1.77(−185)	7.33(−171)
ACOC	2.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
$m = 50$								
n	10	4	4	4	3	3	3	3
$\|\|F(x_n)\|\|$	2.31(−252)	3.24(−304)	2.00(−205)	2.96(−168)	7.01(−126)	1.52(−257)	2.50(−185)	1.04(−170)
ACOC	2.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
$m = 100$								
n	10	4	4	4	3	3	3	3
$\|\|F(x_n)\|\|$	3.27(−252)	4.58(−304)	2.83(−205)	4.19(−168)	9.91(−126)	2.15(−257)	3.54(−185)	1.47(−170)
ACOC	2.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000

Table 5. Comparison of performance of methods for Example 5.

Methods	(2)	(38)	(39)	(40)	(41)	(7)	(8)	(9)
$m = 20$								
n	10	3	4	4	3	3	3	3
$\|\|F(x_n)\|\|$	6.42(−327)	1.15(−63)	1.49(−278)	7.64(−234)	1.42(−162)	3.71(−246)	2.82(−184)	1.41(−197)
ACOC	2.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
$m = 50$								
n	10	3	4	4	3	3	3	3
$\|\|F(x_n)\|\|$	1.01(−326)	1.82(−63)	2.35(−278)	1.21(−233)	2.25(−162)	5.87(−246)	4.46(−184)	2.24(−197)
ACOC	2.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
$m = 100$								
n	10	3	4	4	3	3	3	3
$\|\|F(x_n)\|\|$	1.43(−326)	2.57(−63)	3.32(−278)	1.71(−233)	3.18(−162)	8.31(−246)	6.31(−184)	3.16(−197)
ACOC	2.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
$m = 200$								
n	10	3	4	4	3	3	3	3
$\|\|F(x_n)\|\|$	2.03(−326)	3.64(−63)	4.70(−278)	2.41(−233)	4.50(−162)	1.17(−245)	8.92(−184)	4.47(−197)
ACOC	2.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000

From the numerical results shown in Tables 4 and 5 it is clear that the methods possess stable convergence behavior. Moreover, the small values of $||F(x_n)||$, in comparison to the other methods, show the accurate behavior of the presented methods. The computational order of convergence also supports the theoretical order of convergence. Similar numerical tests, carried out for a number of other different problems, confirmed the above conclusions to a large extent.

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