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Primes in Intervals and Semicircular Elements Induced by p-Adic Number Fields $\mathbb{Q}_p$ over Primes p

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Abstract: In this paper, we study free probability on (weighted-)semicircular elements in a certain Banach $*$-probability space $(\mathfrak{LS}, \tau^0)$ induced by measurable functions on p-adic number fields $\mathbb{Q}_p$ over primes p. In particular, we are interested in the cases where such free-probabilistic information is affected by primes in given closed intervals of the set $\mathbb{R}$ of real numbers by defining suitable "truncated" linear functionals on $\mathfrak{LS}$.

Keywords: free probability; primes; p-adic number fields; Banach $*$-probability spaces; weighted-semicircular elements; semicircular elements; truncated linear functionals

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1. Introduction

In [1,2], we constructed-and-studied *weighted-semicircular elements* and *semicircular elements* induced by *p-adic number fields* $\mathbb{Q}_p$, for all $p \in \mathcal{P}$, where $\mathcal{P}$ is the set of all *primes* in the set $\mathbb{N}$ of all *natural numbers*. In this paper, we consider certain "truncated" free-probabilistic information of the weighted-semicircular laws and the semicircular law of [1]. In particular, we are interested in free distributions of certain free reduced words in our (weighted-)semicircular elements under conditions dictated by the primes p in a "suitable" *closed interval* $[t_1, t_2]$ of the set $\mathbb{R}$ of *real numbers*. Our results illustrate how the original (weighted-)semicircular law(s) of [1] is (resp., are) distorted by truncations on $\mathcal{P}$.

1.1. Preview and Motivation

Relations between *primes* and *operators* have been widely studied not only in mathematical fields (e.g., [3–6]), but also in other scientific fields (e.g., [7]). For instance, we studied how primes act on certain *von Neumann algebras* generated by p -adic and Adelic *measure spaces* in [8,9]. Meanwhile, in [10], primes are regarded as *linear functionals* acting on *arithmetic functions*, understood as *Krein-space operators* under the representation of [11]. Furthermore, in [12,13], free-probabilistic structures on *Hecke algebras* $\mathcal{H}\left(GL_2(\mathbb{Q}_p)\right)$ are studied for $p \in \mathcal{P}$. These series of works are motivated by number-theoretic results (e.g., [4,5,7]).

In [2], we constructed weighted-semicircular elements $\{Q_{p,j}\}_{j\in\mathbb{Z}}$ and corresponding semicircular elements $\{\Theta_{p,j}\}_{j\in\mathbb{Z}}$ in a certain Banach $*$-algebra $\mathfrak{LS}_p$ induced from the $*$-*algebra* $\mathcal{M}_p$ consisting of *measurable functions* on a p-adic number field $\mathbb{Q}_p$, for $p \in \mathcal{P}$. In [1], the *free product* Banach $*$-probability space $(\mathfrak{LS}, \tau^0)$ of the measure spaces $\{\mathfrak{LS}_p(j)\}_{p\in\mathcal{P},j\in\mathbb{Z}}$ of [2] were constructed over both primes and integers, and weighted-semicircular elements $\{Q_{p,j}\}_{p\in\mathcal{P},\, j\in\mathbb{Z}}$ and semicircular elements $\{\Theta_{p,j}\}_{p\in\mathcal{P},\, j\in\mathbb{Z}}$ were studied in $\mathfrak{LS}$, as *free generators*.

In this paper, we are interested in the cases where the free product linear functional τ^0 of [1] on the Banach $*$-algebra $\mathfrak{LS}$ is truncated in $\mathcal{P}$. The distorted free-distributional data from such truncations are considered. The main results characterize how the original free distributions on $\left(\mathfrak{LS},\ \tau^0\right)$ are affected by the given truncations on $\mathcal{P}$.

1.2. Overview

We briefly introduce the backgrounds of our works in Section 2. In the short Sections 3–8, we construct the Banach $*$-probability space $(\mathfrak{LS}, \tau^0)$ and study *weighted-semicircular elements* $Q_{p,j}$ and corresponding *semicircular elements* $\Theta_{p,j}$ in $(\mathfrak{LS}, \tau^0)$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

In Section 9, we define a free-probabilistic sub-structure $\mathbb{LS} = \left(\mathbb{LS},\ \tau^0\right)$ of the Banach $*$-probability space $\left(\mathfrak{LS},\ \tau^0\right)$, having possible non-zero free distributions, and study free-probabilistic properties of $\mathbb{LS}$. Then, *truncated linear functionals of* τ^0 on $\mathbb{LS}$ and truncated free-probabilistic information on $\mathbb{LS}$ are studied. The main results illustrate how our truncations distort the original free distributions on $\mathbb{LS}$ (and hence, on $\mathfrak{LS}$).

In Section 10, we study *free sums* X of $\mathbb{LS}$ having their free distribution, the (weighted-)semicircular law(s), under truncation. Note that, in general, if free sums X have more than one summand as operators, then X cannot be (weighted-)semicircular in $\mathbb{LS}$. However, certain truncations make them be.

In Section 11, we investigate a type of truncation (compared with those of Sections 9 and 10). In particular, certain truncations inducing so-called *prime-neighborhoods* are considered. The *unions* of such prime-neighborhoods provide corresponding distorted free probability on $\mathbb{LS}$ (different from that of Sections 9 and 10).

2. Preliminaries

In this section, we briefly introduce the backgrounds of our proceeding works.

2.1. Free Probability

Readers can review *free probability theory* from [14,15] (and the cited papers therein). *Free probability* is understood as the noncommutative operator-algebraic version of classical *measure theory* and *statistics*. The classical *independence* is replaced by the *freeness*, by replacing *measures* on sets with *linear functionals* on noncommutative ($*$-)algebras. It has various applications not only in pure mathematics (e.g., [16–20]), but also in related topics (e.g., see [2,8–11]). Here, we will use the *combinatorial free probability theory* of *Speicher* (e.g., see [14]).

In the text, without introducing detailed definitions and combinatorial backgrounds, *free moments* and *free cumulants* of operators will be computed. Furthermore, the *free product of $*$-probability spaces in the sense of* [14,15] is considered without detailed introduction.

Note now that one of our main objects, the $*$-algebra $\mathcal{M}_p$ of Section 3, are commutative, and hence, (traditional, or usual "noncommutative") free probability theory is not needed for studying *functional analysis* or *operator algebra theory* on $\mathcal{M}_p$, because the freeness on this commutative structure is trivial. However, we are not interested in the free-probability-depending operator-algebraic structures of commutative algebras, but in statistical data of certain elements to establish (weighted-)semicircular elements. Such data are well explained by the free-probability-theoretic terminology and language. Therefore, as in [2], we use "free-probabilistic models" on $\mathcal{M}_p$ to construct and study our (weighted-)semicircularity by using concepts, tools, and techniques from free probability theory "non-traditionally." Note also that, in Section 8, we construct "traditional" free-probabilistic structures, as in [1], from our "non-traditional" free-probabilistic structures of Sections 3–7 (like the *free group factors*; see, e.g., [15,19]).

2.2. Analysis of $\mathbb{Q}_p$

For more about p-adic and Adelic analysis, see [7]. Let $p \in \mathcal{P}$, and let $\mathbb{Q}_p$ be the *p-adic number field*. Under the p-adic addition and the p-adic multiplication of [7], the set $\mathbb{Q}_p$ forms a *field* algebraically. It is equipped with the *non-Archimedean norm* $|.|_p$, which is the inherited *p-norm* on the set $\mathbb{Q}$ of all *rational numbers* defined by:

$$|x|_p = \left|p^k \tfrac{a}{b}\right|_p = \tfrac{1}{p^k},$$

whenever $x = p^k \frac{a}{b}$ in $\mathbb{Q}$, where $k, a \in \mathbb{Z}$, and $b \in \mathbb{Z} \setminus \{0\}$. For instance,

$$\left|\tfrac{8}{3}\right|_2 = \left|2^3 \times \tfrac{1}{3}\right|_2 = \tfrac{1}{2^3} = \tfrac{1}{8},$$

and:

$$\left|\tfrac{8}{3}\right|_3 = \left|3^{-1} \times 8\right|_3 = \tfrac{1}{3^{-1}} = 3,$$

and:

$$\left|\tfrac{8}{3}\right|_q = \tfrac{1}{q^0} = 1, \text{whenever } q \in \mathcal{P} \setminus \{2, 3\}.$$

The *p-adic number field* $\mathbb{Q}_p$ is the maximal p-norm closure in $\mathbb{Q}$. Therefore, under norm topology, it forms a *Banach space* (e.g., [7]).

Let us understand the *Banach field* $\mathbb{Q}_p$ as a *measure space*,

$$\mathbb{Q}_p = \left(\mathbb{Q}_p,\ \sigma(\mathbb{Q}_p),\ \mu_p\right),$$

where $\sigma(\mathbb{Q}_p)$ is the *σ-algebra of* $\mathbb{Q}_p$ consisting of all *μ_p-measurable subsets*, where μ_p is a left-and-right additive invariant *Haar measure* on $\mathbb{Q}_p$ satisfying:

$$\mu_p(\mathbb{Z}_p) = 1,$$

where $\mathbb{Z}_p$ is the *unit disk of* $\mathbb{Q}_p$, consisting of all *p-adic integers* x satisfying $|x|_p \leq 1$. Moreover, if we define:

$$U_k = p^k \mathbb{Z}_p = \{p^k x \in r\mathbb{Q}_p : x \in \mathbb{Z}_p\}, \tag{1}$$

for all $k \in \mathbb{Z}$ (with $U_0 = \mathbb{Z}_p$), then these μ_p-measurable subsets U_k's of (1) satisfy:

$$\mathbb{Q}_p = \underset{k \in \mathbb{Z}}{\cup} U_k,$$

and:

$$\mu_p\left(U_k\right) = \frac{1}{p^k} = \mu_p\left(x + U_k\right), \forall x \in \mathbb{Q}_p, \tag{2}$$

and:

$$\cdots \subset U_2 \subset U_1 \subset U_0 = \mathbb{Z}_p \subset U_{-1} \subset U_{-2} \subset \cdots.$$

In fact, the family $\{U_k\}_{k \in \mathbb{Z}}$ forms a *basis* of the Banach topology for $\mathbb{Q}_p$ (e.g., [7]).

Define now subsets ∂_k of $\mathbb{Q}_p$ by:

$$\partial_k = U_k \setminus U_{k+1}, for\, all k \in \mathbb{Z}. \tag{3}$$

We call such μ_p-measurable subsets ∂_k the k^{th} boundaries of U_k in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. By (2) and (3), one obtains that:

$$\mathbb{Q}_p = \underset{k \in \mathbb{Z}}{\sqcup} \partial_k,$$

and:

$$\mu_p\left(\partial_k\right) = \mu_p\left(U_k\right) - \mu_p\left(U_{k+1}\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}}, \tag{4}$$

and:

$$\partial_{k_1} \cap \partial_{k_2} = \begin{cases} \partial_{k_1} & \text{if } k_1 = k_2 \\ \varnothing & \text{otherwise,} \end{cases},$$

for all $k, k_1, k_2 \in \mathbb{Z}$, where $\sqcup$ is the *disjoint union* and $\varnothing$ is the *empty set*.

Now, let $\mathcal{M}_p$ be the algebra,

$$\mathcal{M}_p = \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p)\}\right], \tag{5}$$

where χ_S are the usual *characteristic functions* of $S \in \sigma\left(\mathbb{Q}_p\right)$.

Then the algebra $\mathcal{M}_p$ of (5) forms a well-defined $*$-*algebra over* $\mathbb{C}$, with its *adjoint*,

$$\left(\sum_{S \in \sigma(G_p)} t_S \chi_S\right)^* \overset{def}{=} \sum_{S \in \sigma(G_p)} \overline{t_S}\, \chi_S,$$

where $t_S \in \mathbb{C}$, having their *conjugates* $\overline{t_S}$ in $\mathbb{C}$.

Let $\sum\limits_{S \in \sigma(G_p)} t_S \chi_S \in \mathcal{M}_p$. Then, one can define the *p-adic integral* by:

$$\int_{\mathbb{Q}_p} \left(\sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S\right) d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \mu_p(S). \tag{6}$$

Note that, by (4), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a subset Λ_S of $\mathbb{Z}$, such that:

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}, \tag{7}$$

satisfying:

$$\int_{\mathbb{Q}_p} \chi_S \, d\mu_p = \int_{\mathbb{Q}_p} \sum_{j \in \Lambda_S} \chi_{S \cap \partial_j} \, d\mu_p$$

$$= \sum_{j \in \Lambda_S} \mu_p\left(S \cap \partial_j\right)$$

by (6)

$$\leq \sum_{j \in \Lambda_S} \mu_p\left(\partial_j\right) = \sum_{j \in \Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{8}$$

by (4), for all $S \in \sigma(\mathbb{Q}_p)$, where Λ_S is in the sense of (7).

Proposition 1. *Let $S \in \sigma(\mathbb{Q}_p)$, and let $\chi_S \in \mathcal{M}_p$. Then, there exist $r_j \in \mathbb{R}$, such that:*

$$0 \leq r_j \leq 1 in \mathbb{R}, for all j \in \Lambda_S, \tag{9}$$

and:

$$\int_{\mathbb{Q}_p} \chi_S \, d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\tfrac{1}{p^j} - \tfrac{1}{p^{j+1}}\right).$$

Proof. The existence of $r_j = \frac{\mu_p(S \cap \partial_j)}{\mu_p(\partial_j)}$, for all $j \in \mathbb{Z}$, is guaranteed by (7) and (8). The p-adic integral in (9) is obtained by (8). □

3. Free-Probabilistic Model on $\mathcal{M}_p$

Throughout this section, fix a prime $p \in \mathcal{P}$, and let $\mathbb{Q}_p$ be the corresponding p-adic number field and $\mathcal{M}_p$ be the $*$-algebra (5) consisting of μ_p-measurable functions on $\mathbb{Q}_p$. Here, we establish a suitable (non-traditional) free-probabilistic model on $\mathcal{M}_p$ implying p-adic analytic data.

Let U_k be the basis elements (1) of the topology for $\mathbb{Q}_p$ with their boundaries ∂_k of (3), i.e.,

$$U_k = p^k \mathbb{Z}_p, forall k \in \mathbb{Z}, \tag{10}$$

and:

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}.$$

Define a linear functional $\varphi_p : \mathcal{M}_p \to \mathbb{C}$ by the p-adic integration (6),

$$\varphi_p\left(f\right) = \int_{\mathbb{Q}_p} f d\mu_p, forall f \in \mathcal{M}_p. \tag{11}$$

Then, by (9) and (11), one obtains:

$$\varphi_p\left(\chi_{U_j}\right) = \frac{1}{p^j} \text{ and } \varphi_p\left(\chi_{\partial_j}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}},$$

for all $j \in \mathbb{Z}$.

Definition 1. *We call the pair* $\left(\mathcal{M}_p,\ \varphi_p\right)$ *the p-adic (non-traditional) free probability space for* $p \in \mathcal{P}$*, where* φ_p *is the linear functional* (11) *on* $\mathcal{M}_p$.

Remark 1. *As we discussed in Section 2.1, we study the measure-theoretic structure* $\left(\mathcal{M}_p,\ \varphi_p\right)$ *as a free-probabilistic model on* $\mathcal{M}_p$ *for our purposes. Therefore, without loss of generality, we regard* $\left(\mathcal{M}_p,\ \varphi_p\right)$ *as a non-traditional free-probabilistic structure. In this sense, we call* $\left(\mathcal{M}_p,\ \varphi_p\right)$ *the p-adic free probability space for p. The readers can understand* $\left(\mathcal{M}_p,\ \varphi_p\right)$ *as the pair of a commutative* $*$*-algebra* $\mathcal{M}_p$ *and a linear functional* φ_p*, having as its name the p-adic free probability space.*

Let ∂_k be the k^{th} boundary $U_k \setminus U_{k+1}$ of U_k in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then, for $k_1, k_2 \in \mathbb{Z}$, one obtains that:

$$\chi_{\partial_{k_1}} \chi_{\partial_{k_2}} = \chi_{\partial_{k_1} \cap \partial_{k_2}} = \delta_{k_1,k_2} \chi_{\partial_{k_1}},$$

by (4), and hence,

$$\begin{aligned} \varphi_p\left(\chi_{\partial_{k_1}} \chi_{\partial_{k_2}}\right) &= \delta_{k_1,k_2} \varphi_p\left(\chi_{\partial_{k_1}}\right) \\ &= \delta_{k_1,k_2}\left(\frac{1}{p^{k_1}} - \frac{1}{p^{k_1+1}}\right), \end{aligned} \tag{12}$$

where δ is the *Kronecker delta*.

Proposition 2. *Let* $(j_1, ..., j_N) \in \mathbb{Z}^N$*, for* $N \in \mathbb{N}$*. Then:*

$$\prod_{l=1}^{N} \chi_{\partial_{j_l}} = \delta_{(j_1,\ldots,j_N)} \chi_{\partial_{j_1}} \text{ in } \mathcal{M}_p,$$

and hence,

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{\partial_{j_l}}\right) = \delta_{(j_1,\ldots,j_N)}\left(\frac{1}{p^{j_1}} - \frac{1}{p^{j_1+1}}\right), \tag{13}$$

where:

$$\delta_{(j_1,\ldots,j_N)} = \left(\prod_{l=1}^{N-1} \delta_{j_l,j_{l+1}}\right)\left(\delta_{j_N,j_1}\right).$$

Proof. The proof of (13) is done by induction on (12). □

Thus, one can get that, for any $S \in \sigma\left(\mathbb{Q}_p\right)$,

$$\varphi_p\left(\chi_S\right) = \varphi_p\left(\sum_{j \in \Lambda_S} \chi_{S \cap \partial_j}\right) \tag{14}$$

where Λ_S is in the sense of (7).

$$\begin{aligned} &= \sum_{j \in \Lambda_S} \varphi_p \left(\chi_{S \cap \partial_j} \right) = \sum_{j \in \Lambda_S} \mu_p \left(S \cap \partial_j \right) \\ &= \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \end{aligned} \tag{15}$$

by (13), where $0 \leq r_j \leq 1$ are in the sense of (9) for all $j \in \Lambda_S$.

Furthermore, if $S_1, S_2 \in \sigma\left(\mathbb{Q}_p\right)$, then:

$$\begin{aligned} \chi_{S_1}\chi_{S_2} &= \left(\sum_{k \in \Lambda_{S_1}} \chi_{S_1 \cap \partial_k} \right) \left(\sum_{j \in \Lambda_{S_2}} \chi_{S_2 \cap \partial_j} \right) \\ &= \sum_{(k,j) \in \Lambda_{S_1} \times \Lambda_{S_2}} \delta_{k,j} \chi_{(S_1 \cap S_2) \cap \partial_j} \\ &= \sum_{j \in \Lambda_{S_1, S_2}} \chi_{(S_1 \cap S_2) \cap \partial_j}, \end{aligned} \tag{16}$$

where:

$$\Lambda_{S_1,S_2} = \Lambda_{S_1} \cap \Lambda_{S_2}.$$

Proposition 3. *Let $S_l \in \sigma(\mathbb{Q}_p)$, and let $\chi_{S_l} \in \left(\mathcal{M}_p, \ \varphi_p\right)$, for $l = 1, ..., N$, for $N \in \mathbb{N}$. Let:*

$$\Lambda_{S_1,\ldots,S_N} = \bigcap_{l=1}^{N} \Lambda_{S_l} \text{ in } \mathbb{Z},$$

where Λ_{S_l} are in the sense of (7), for $l = 1, ..., N$. Then, there exist $r_j \in \mathbb{R}$, such that:

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{for } j \in \Lambda_{S_1,\ldots,S_N},$$

and:

$$\varphi_p \left(\prod_{l=1}^{N} \chi_{S_l} \right) = \sum_{j \in \Lambda_{S_1,\ldots,S_N}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right). \tag{17}$$

Proof. The proof of (17) is done by induction on (16) with the help of (15). □

4. Representations of $\left(\mathcal{M}_p, \ \varphi_p\right)$

Fix a prime p in $\mathcal{P}$, and let $\left(\mathcal{M}_p, \ \varphi_p\right)$ be the p-adic free probability space. By understanding $\mathbb{Q}_p$ as a measure space, construct the L^2-*space* H_p,

$$H_p \overset{def}{=} L^2 \left(\mathbb{Q}_p, \ \sigma(\mathbb{Q}_p), \ \mu_p \right) = L^2 \left(\mathbb{Q}_p \right), \tag{18}$$

over $\mathbb{C}$. Then, this L^2-space H_p of (18) is a well-defined *Hilbert space* equipped with its *inner product* $<, >_2$,

$$\langle h_1, \ h_2 \rangle_2 \overset{def}{=} \int_{\mathbb{Q}_p} h_1 h_2^* d\mu_p, \tag{19}$$

for all $h_1, h_2 \in H_p$.

Definition 2. *We call the Hilbert space H_p of (18), the p-adic Hilbert space.*

By the definition (18) of the p-adic Hilbert space H_p, our $*$-algebra $\mathcal{M}_p$ acts on H_p, via an *algebra-action* α^p,

$$\alpha^p(f)\,(h) = fh, forallh \in H_p, \tag{20}$$

for all $f \in \mathcal{M}_p$.

Notation: Denote $\alpha^p(f)$ of (20) by α^p_f, for all $f \in \mathcal{M}_p$. Furthermore, for convenience, denote $\alpha^p_{\chi_S}$ simply by α^p_S, for all $S \in \sigma(\mathbb{Q}_p)$. □

By (20), the linear morphism α^p is indeed a well-determined $*$-algebra-action of $\mathcal{M}_p$ acting on H_p (equivalently, every α^p_f is a $*$-homomorphism from $\mathcal{M}_p$ into the operator algebra $B(H_p)$ of all bounded operators on H_p, for all $f \in \mathcal{M}_p$), since:

$$\begin{aligned}\alpha^p_{f_1 f_2}(h) &= f_1 f_2 h = f_1 (f_2 h)\\ &= f_1\left(\alpha^p_{f_2}(h)\right) = \alpha^p_{f_1}\alpha^p_{f_2}(h),\end{aligned}$$

for all $h \in H_p$, implying that:

$$\alpha^p_{f_1 f_2} = \alpha^p_{f_1}\alpha^p_{f_2}, \tag{21}$$

for all $f_1, f_2 \in \mathcal{M}_p$; and:

$$\begin{aligned}\left\langle \alpha^p_f(h_1),\, h_2\right\rangle_2 &= \langle f h_1,\, h_2\rangle_2 = \int_{\mathbb{Q}_p} f h_1 h_2^* d\mu_p\\ &= \int_{\mathbb{Q}_p} h_1 f h_2^* d\mu_p = \int_{\mathbb{Q}_p} h_1 (h_2 f^*)^* d\mu_p\\ &= \int_{\mathbb{Q}_p} h_1 (f^* h_2)^* d\mu_p = \left\langle h_1,\, \alpha^p_{f^*}(h_2)\right\rangle_2,\end{aligned}$$

for all $h_1, h_2 \in H_p$, for all $f \in \mathcal{M}_p$, implying that:

$$\left(\alpha^p_f\right)^* = \alpha_{f^*}, forall f \in \mathcal{M}_p, \tag{22}$$

where $<,>_2$ is the inner product (19) on H_p.

Proposition 4. *The linear morphism α^p of (20) is a well-defined $*$-algebra-action of $\mathcal{M}_p$ acting on H_p. Equivalently, the pair (H_p, α^p) is a Hilbert-space representation of $\mathcal{M}_p$.*

Proof. The proof is done by (21) and (22). □

Definition 3. *The Hilbert-space representation $(H_p,\, \alpha^p)$ is said to be the p-adic representation of $\mathcal{M}_p$.*

Depending on the p-adic representation (H_p, α^p) of $\mathcal{M}_p$, one can construct the C^**-subalgebra* M_p of the operator algebra $B(H_p)$.

Definition 4. *Define the C^*-subalgebra M_p of the operator algebra $B(H_p)$ by:*

$$M_p \overset{def}{=} \overline{\alpha^p(\mathcal{M}_p)} = \overline{\mathbb{C}\left[\alpha^p_f\ : f \in \mathcal{M}_p\right]}, \tag{23}$$

where $\overline{X}$ mean the operator-norm closures of subsets X of $B(H_p)$. Then, this C^*-algebra M_p is called the p-adic C^*-algebra of the p-adic free probability space $(\mathcal{M}_p,\, \varphi_p)$.

5. Free-Probabilistic Models on M_p

Throughout this section, let us fix a prime $p \in \mathcal{P}$, and let $(\mathcal{M}_p,\, \varphi_p)$ be the corresponding p-adic free probability space. Let $(H_p,\, \alpha^p)$ be the p-adic representation of $\mathcal{M}_p$, and let M_p be the p-adic C^*-algebra (23) of $(\mathcal{M}_p,\, \varphi_p)$.

We here construct suitable free-probabilistic models on M_p. In particular, we are interested in a system $\{\varphi^p_j\}_{j\in\mathbb{Z}}$ of *linear functionals on* M_p, determined by the j^{th} boundaries $\{\partial_j\}_{j\in\mathbb{Z}}$ of $\mathbb{Q}_p$.

Define a linear functional $\varphi^p_j : M_p \to \mathbb{C}$ by a linear morphism,

$$\varphi^p_j(a) \overset{def}{=} \left\langle a\left(\chi_{\partial_j}\right),\, \chi_{\partial_j}\right\rangle_2, \tag{24}$$

for all $a \in M_p$, for all $j \in \mathbb{Z}$, where $<,>_2$ is the inner product (19) on the p-adic Hilbert space H_p of (18).

Remark that if $a \in M_p$, then:

$$a = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \, \alpha_S^p, \text{ in } M_p$$

(with $t_S \in \mathbb{C}$), where $\sum$ is a finite or infinite (i.e., limit of finite) sum(s) under the C^*-topology for M_p. Thus, the linear functionals φ_j^p of (24) are well defined on M_p, for all $j \in \mathbb{Z}$, i.e., for any fixed $j \in \mathbb{Z}$, one has that:

$$\begin{aligned} \left|\varphi_j^p(a)\right| &= \left| \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \left\langle \chi_{S \cap \partial_j}, \, \chi_{\partial_j} \right\rangle_2 \right| \\ &= \left| \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \mu_p \left(\chi_{S \cap \partial_j}\right) \right| \\ &\leq \mu_p\left(\partial_j\right) \left| \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \right| \leq \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) \|a\|, \end{aligned} \tag{25}$$

where:

$$\|a\| = \sup\left\{\|a(h)\|_2 : h \in H_p \text{ with } \|h\|_2 = 1\right\}$$

is the C^*-norm on M_p (inherited by the operator norm on the operator algebra $B(H_p)$), and $\|.\|_2$ is the Hilbert-space norm,

$$\|f\|_2 = \sqrt{\langle f, \, f \rangle_2}, \forall f \in H_p,$$

induced by the inner product $<,>_2$ of (19). Therefore, for any fixed integer $j \in \mathbb{Z}$, the corresponding linear functional φ_j^p of (24) is bounded on M_p.

Definition 5. *Let $j \in \mathbb{Z}$, and let φ_j^p be the linear functional (24) on the p-adic C^*-algebra M_p. Then, the pair $\left(M_p, \, \varphi_j^p\right)$ is said to be the j^{th} p-adic (non-traditional) C^*-probability space.*

Remark 2. *As in Section 4, the readers can understand the pairs $\left(M_p, \, \varphi_j^p\right)$ simply as structures consisting of a commutative C^*-algebra M_p and linear functionals φ_j^p on M_p, whose names are j^{th} p-adic C^*-probability spaces for all $j \in \mathbb{Z}$, for $p \in \mathcal{P}$.*

Fix $j \in \mathbb{Z}$, and take the corresponding j^{th} p-adic C^*-probability space $\left(M_p, \, \varphi_j^p\right)$. For $S \in \sigma\left(\mathbb{Q}_p\right)$ and a generating operator α_S^p of M_p, one has that:

$$\begin{aligned} \varphi_j^p(\alpha_S^p) &= \left\langle \alpha_S^p(\chi_{\partial_j}), \, \chi_{\partial_j} \right\rangle_2 = \left\langle \chi_{S \cap \partial_j}, \, \chi_{\partial_j} \right\rangle_2 \\ &= \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} \chi_{\partial_j}^* d\mu_p = \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} \chi_{\partial_j} d\mu_p \end{aligned} \tag{26}$$

by (19)

$$\begin{aligned} &= \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} d\mu_p = \mu_p\left(S \cap \partial_j\right) \\ &= r_S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \end{aligned} \tag{27}$$

for some $0 \leq r_S \leq 1$ in $\mathbb{R}$, for $S \in \sigma\left(\mathbb{Q}_p\right)$.

Proposition 5. *Let $S \in \sigma\left(\mathbb{Q}_p\right)$ and $\alpha_S^p = \alpha_{\chi_S}^p \in \left(M_p, \, \varphi_j^p\right)$, for a fixed $j \in \mathbb{Z}$. Then, there exists $r_S \in \mathbb{R}$, such that:*

$$0 \leq r_S \leq 1 \text{ in } \mathbb{R},$$

and:

$$\varphi_j^p\left(\left(\alpha_S^p\right)^n\right) = r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \text{for all } n \in \mathbb{N}. \tag{28}$$

Proof. Remark that the generating operator α_S^p is a projection in M_p, in the sense that:

$$\left(\alpha_S^p\right)^* = \alpha_S^p = \left(\alpha_S^p\right)^2, \text{ in } M_p,$$

so,

$$\left(\alpha_S^p\right)^n = \alpha_S^p, \text{ for all } n \in \mathbb{N}.$$

Thus, for any $n \in \mathbb{N}$, we have:

$$\varphi_j^p\left(\left(\alpha_S^p\right)^n\right) = \varphi_j^p(\alpha_S^p) = r_S\left(\tfrac{1}{p^j} - \tfrac{1}{p^{j+1}}\right),$$

for some $0 \leq r_S \leq 1$ in $\mathbb{R}$, by (27). □

As a corollary of (28), one obtains the following corollary.

Corollary 1. *Let ∂_k be the k^{th} boundaries (10) of $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then:*

$$\varphi_j^p\left(\left(\alpha_{\partial_k}^p\right)^n\right) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) \tag{29}$$

for all $n \in \mathbb{N}$, for all $j \in \mathbb{Z}$.

Proof. The formula (29) is shown by (28). □

6. Semigroup C^*-Subalgebras $\mathfrak{S}_p$ of M_p

Let M_p be the p-adic C^*-algebra (23) for an arbitrarily-fixed $p \in \mathcal{P}$. Take operators:

$$P_{p,j} = \alpha_{\partial_j}^p \in M_p, \tag{30}$$

where ∂_j are the j^{th} boundaries (10) of $\mathbb{Q}_p$, for the fixed prime p, for all $j \in \mathbb{Z}$.

Then, these operators $P_{p,j}$ of (30) are *projections* on the p-adic Hilbert space H_p in M_p, i.e.,

$$P_{p,j}^* = P_{p,j} = P_{p,j}^2,$$

for all $j \in \mathbb{Z}$. We now restrict our interest to these projections $P_{p,j}$ of (30).

Definition 6. *Fix $p \in \mathcal{P}$. Let $\mathfrak{S}_p$ be the C^*-subalgebra:*

$$\mathfrak{S}_p = C^*\left(\{P_{p,j}\}_{j\in\mathbb{Z}}\right) = \overline{\mathbb{C}\left[\{P_{p,j}\}_{j\in\mathbb{Z}}\right]} \, of \, M_p, \tag{31}$$

where $P_{p,j}$ are projections (30), for all $j \in \mathbb{Z}$. We call this C^*-subalgebra $\mathfrak{S}_p$ the p-adic boundary (C^*-)subalgebra of M_p.

The p-adic boundary subalgebra $\mathfrak{S}_p$ of the p-adic C^*-algebra M_p satisfies the following structure theorem.

Proposition 6. *Let $\mathfrak{S}_p$ be the p-adic boundary subalgebra (31) of the p-adic C^*-algebra M_p. Then:*

$$\mathfrak{S}_p \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\oplus}\left(\mathbb{C} \cdot P_{p,j}\right) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus|\mathbb{Z}|}, \tag{32}$$

in M_p.

Proof. The proof of (32) is done by the mutual orthogonality of the projections $\{P_{p,j}\}_{j\in\mathbb{Z}}$ in M_p. Indeed, one has:

$$P_{p,j_1}P_{p,j_2} = \alpha^p_{\partial_{j_1}}\alpha^p_{\partial_{j_2}} = \alpha^p_{\partial_{j_1}\cap\partial_{j_2}} = \delta_{j_1,j_2}P_{p,j_1},$$

in $\mathfrak{S}_p$, for all $j_1, j_2 \in \mathbb{Z}$. $\square$

Define now linear functionals φ^p_j (for a fixed prime p) by:

$$\varphi^{(p)}_j = \varphi^p_j \mid_{\mathfrak{S}_p} on\mathfrak{S}_p, \tag{33}$$

where φ^p_j in the right-hand side of (33) are the linear functionals (24) on M_p, for all $j \in \mathbb{Z}$.

7. Weighted-Semicircular Elements

Let M_p be the p-adic C^*-algebra, and let $\mathfrak{S}_p$ be the p-adic boundary subalgebra (31) of M_p, satisfying the structure theorem (32). Fix $p \in \mathcal{P}$. Recall that the generating projections $P_{p,j}$ of $\mathfrak{S}_p$ satisfy:

$$\varphi^{(p)}_j\left(P_{p,j}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}}, \forall j \in \mathbb{Z}, \tag{34}$$

by (33) (also see (28) and (29)).

Now, let ϕ be the *Euler totient function*, an *arithmetic function:*

$$\phi : \mathbb{N} \to \mathbb{C},$$

defined by:

$$\phi(n) = \left|\{k \in \mathbb{N} : k \leq n,\ \gcd(n,k) = 1\}\right|, \tag{35}$$

for all $n \in \mathbb{N}$, where $|X|$ mean the cardinalities of sets X and gcd is the *greatest common divisor*.

It is well known that:

$$\phi(n) = n\left(\prod_{q\in\mathcal{P},\, q|n}\left(1-\tfrac{1}{q}\right)\right),$$

for all $n \in \mathbb{N}$, where "$q \mid n$" means "q divides n." For instance,

$$\phi(p) = p - 1 = p\left(1 - \frac{1}{p}\right), \forall p \in \mathcal{P}. \tag{36}$$

Thus:

$$\begin{aligned}\varphi^{(p)}_j\left(P_{p,j}\right) &= \left(\tfrac{1}{p^j} - \tfrac{1}{p^{j+1}}\right) = \tfrac{1}{p^j}\left(1-\tfrac{1}{p}\right)\\ &= \tfrac{p}{p^{j+1}}\left(1-\tfrac{1}{p}\right) = \tfrac{\phi(p)}{p^{j+1}},\end{aligned}$$

by (34), (35), and (36), for all $P_{p,j} \in \mathfrak{S}_p$. More generally,

$$\varphi^{(p)}_j\left(P_{p,k}\right) = \delta_{j,k}\left(\frac{\phi(p)}{p^{j+1}}\right), \forall p \in \mathcal{P}, k \in \mathbb{Z}. \tag{37}$$

Now, for a fixed prime p, define new linear functionals τ^p_j on $\mathfrak{S}_p$, by linear morphisms satisfying that:

$$\tau^p_j = \frac{1}{\phi(p)}\varphi^{(p)}_j, on\mathfrak{S}_p, \tag{38}$$

for all $j \in \mathbb{Z}$, where φ^p_j are in the sense of (33).

Then, one obtains new (non-traditional) C^*-probabilistic structures,

$$\{\mathfrak{S}_p(j) = \left(\mathfrak{S}_p,\ \tau_j^p\right) : p \in \mathcal{P}, j \in \mathbb{Z}\}, \tag{39}$$

where τ_j^p are in the sense of (38).

Proposition 7. *Let $\mathfrak{S}_p(j) = (\mathfrak{S}_p,\ \tau_j^p)$ be in the sense of (39), and let $P_{p,k}$ be generating operators of $\mathfrak{S}_p(j)$, for $p \in \mathcal{P}$, $j \in \mathbb{Z}$. Then:*

$$\tau_j^p\left(P_{p,k}^n\right) = \frac{\delta_{j,k}}{p^{j+1}}, for all n \in \mathbb{N}. \tag{40}$$

Proof. The formula (40) is proven by (37) and (38). Indeed, since $P_{p,k}$ are projections in $\mathfrak{S}_p(j)$,

$$\tau_j^p\left(P_{p,k}^n\right) = \tau_j^p\left(P_{p,k}\right) = \delta_{j,k}\left(\tfrac{1}{p^{j+1}}\right),$$

for all $n \in \mathbb{N}$, for all $p \in \mathcal{P}$, and $j, k \in \mathbb{Z}$. □

7.1. Semicircular and Weighted-Semicircular Elements

Let $(A,\ \varphi)$ be an arbitrary (traditional or non-traditional) *topological $*$-probability space* (C^*-probability space, or W^*-probability space, or Banach $*$-probability space, etc.), consisting of a (noncommutative, resp., commutative) topological $*$-algebra A (C^*-algebra, resp., W^*-algebra, resp., Banach $*$-algebra, etc.), and a (bounded or unbounded) linear functional φ on A.

Definition 7. *Let a be a self-adjoint element in (A, φ). It is said to be even in (A, φ), if all odd free moments of a vanish, i.e.,*

$$\varphi\left(a^{2n-1}\right) = 0, for all n \in \mathbb{N}. \tag{41}$$

Let a be a "self-adjoint," and "even" element of (A, φ) satisfying (41). Then, it is said to be semicircular in (A, φ), if:

$$\varphi(a^{2n}) = c_n, for all n \in \mathbb{N}, \tag{42}$$

where c_k are the k^{th} Catalan number,

$$c_k = \frac{1}{k+1}\begin{pmatrix} 2k \\ k \end{pmatrix} = \frac{1}{k+1}\frac{(2k)!}{(k!)^2} = \frac{(2k)!}{k!(k+1)!},$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It is well known that, if $k_n(...)$ is the *free cumulant on A in terms of a linear functional φ* (in the sense of [14]), then a self-adjoint element a is *semicircular* in (A, φ), if and only if:

$$k_n\left(\underbrace{a,\ a,\ ......,\ a}_{n\text{-times}}\right) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{43}$$

for all $n \in \mathbb{N}$ (e.g., see [14]). The above equivalent free-distributional data (43) of the semicircularity (42) are obtained by the *Möbius inversion of* [14].

Motivated by (43), one can define the *weighted-semicircularity*.

Definition 8. *Let $a \in (A, \varphi)$ be a self-adjoint element. It is said to be weighted-semicircular in (A, φ) with its weight t_0 (in short, t_0-semicircular), if there exists $t_0 \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, such that:*

$$k_n\left(\underbrace{a,\ a,\ ....,\ a}_{n\text{-times}}\right) = \begin{cases} t_0 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{44}$$

for all $n \in \mathbb{N}$, where $k_n(\ldots)$ is the free cumulant on A in terms of φ.

By the definition (44) and by the Möbius inversion of [14], one obtains the following free-moment characterization (45) of the weighted-semicircularity (44): A self-adjoint element a is t_0-*semicircular* in (A, φ), if and only if there exists $t_0 \in \mathbb{C}^{\times}$, such that:

$$\varphi(a^n) = \omega_n \, t_0^{\frac{n}{2}} c_{\frac{n}{2}},$$

where:

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{45}$$

for all $n \in \mathbb{N}$, where c_m are the m^{th} Catalan numbers for all $m \in \mathbb{N}_0$.

Thus, from below, we use the weighted-semicircularity (44) and its characterization (45) alternatively.

7.2. *Tensor Product Banach $*$-Algebra $\mathfrak{L}\mathfrak{S}_p$*

Let $\mathfrak{S}_p(k) = \left(\mathfrak{S}_p, \tau_k^p\right)$ be a (non-traditional) C^*-probability space (39), for $p \in \mathcal{P}$, $k \in \mathbb{Z}$. Define *bounded linear transformations* $\mathbf{c}_p$ and $\mathbf{a}_p$ "acting on the p-adic boundary subalgebra $\mathfrak{S}_p$ of M_p," by linear morphisms satisfying,

$$\mathbf{c}_p\left(P_{p,j}\right) = P_{p,j+1},$$

and:

$$\mathbf{a}_p\left(P_{p,j}\right) = P_{p,j-1}, \tag{46}$$

on $\mathfrak{S}_p$, for all $j \in \mathbb{Z}$.

By (46), these linear transformations $\mathbf{c}_p$ and $\mathbf{a}_p$ are bounded under the operator-norm induced by the C^*-norm on $\mathfrak{S}_p$. Therefore, the linear transformations $\mathbf{c}_p$ and $\mathbf{a}_p$ are regarded as Banach-space operators "acting on $\mathfrak{S}_p$," by regarding $\mathfrak{S}_p$ as a Banach space (under its C^*-norm). i.e., $\mathbf{c}_p$ and $\mathbf{a}_p$ are elements of the *operator space* $B\left(\mathfrak{S}_p\right)$ consisting of all bounded operators on the Banach space $\mathfrak{S}_p$.

Definition 9. *The Banach-space operators $\mathbf{c}_p$ and $\mathbf{a}_p$ of (46) are called the p-creation, respectively, the p-annihilation on $\mathfrak{S}_p$, for $p \in \mathcal{P}$. Define a new Banach-space operator $l_p \in B\left(\mathfrak{S}_p\right)$ by:*

$$l_p = \mathbf{c}_p + \mathbf{a}_p on \mathfrak{S}_p. \tag{47}$$

We call it the p-radial operator on $\mathfrak{S}_p$.

Let l_p be the p-radial operator $\mathbf{c}_p + \mathbf{a}_p$ of (47) on $\mathfrak{S}_p$. Construct a *closed subspace* $\mathfrak{L}_p$ of $B(\mathfrak{S}_p)$ by:

$$\mathfrak{L}_p = \overline{\mathbb{C}[l_p]} \subset B(\mathfrak{S}_p), \tag{48}$$

where $\overline{Y}$ means the operator-norm-topology closure of every subset Y of the operator space $B(\mathfrak{S}_p)$.

By the definition (48), $\mathfrak{L}_p$ is not only a closed subspace, but also a well-defined Banach algebra embedded in the vector space $B(\mathfrak{S}_p)$. On this Banach algebra $\mathfrak{L}_p$, define the adjoint $(*)$ by:

$$\sum_{k=0}^{\infty} s_k l_p^k \in \mathfrak{L}_p \longmapsto \sum_{k=0}^{\infty} \overline{s_k} l_p^k \in \mathfrak{L}_p, \tag{49}$$

where $s_k \in \mathbb{C}$ with their conjugates $\overline{s_k} \in \mathbb{C}$.

Then, equipped with the adjoint (49), this Banach algebra $\mathfrak{L}_p$ of (48) forms a *Banach $*$-algebra* inside $B(\mathfrak{S}_p)$.

Definition 10. *Let $\mathfrak{L}_p$ be a Banach $*$-algebra (48) in the operator space $B(\mathfrak{S}_p)$ for $p \in \mathcal{P}$. We call it the p-radial (Banach-$*$-)algebra on $\mathfrak{S}_p$.*

Let $\mathfrak{L}_p$ be the p-radial algebra (48) on $\mathfrak{S}_p$. Construct now the tensor product Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p$ by:

$$\mathfrak{L}\mathfrak{S}_p = \mathfrak{L}_p \otimes_{\mathbb{C}} \mathfrak{S}_p, \tag{50}$$

where $\otimes_{\mathbb{C}}$ is the *tensor product of Banach $*$-algebras* (Remark that $\mathfrak{S}_p$ is a C^*-algebra and $\mathfrak{L}_p$ is a Banach $*$-algebra; and hence, the tensor product Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p$ of (50) is well-defined.).

Take now a generating element $l_p^k \otimes P_{p,j}$, for some $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $j \in \mathbb{Z}$, where $P_{p,j}$ is in the sense of (30) in $\mathfrak{S}_p$, with axiomatization:

$$l_p^0 = 1_{\mathfrak{S}_p}, \text{ the identity operator on } \mathfrak{S}_p,$$

in $B(\mathfrak{S}_p)$, satisfying:

$$1_{\mathfrak{S}_p}\left(P_{p,j}\right) = P_{p,j}, \text{ for all } P_{p,j} \in \mathfrak{S}_p,$$

for all $j \in \mathbb{Z}$.

By (50) and (32), the elements $l_p^k \otimes P_{p,j}$ indeed generate $\mathfrak{L}\mathfrak{S}_p$ under linearity, because:

$$\left(l_p \otimes P_{p,j}\right)^k = l_p^k \otimes P_{p,j},$$

for all $k \in \mathbb{N}_0$, and $j \in \mathbb{Z}$, for $p \in \mathcal{P}$, and their self-adjointness. We now focus on such generating operators of $\mathfrak{L}\mathfrak{S}_p$.

Define a linear morphism:

$$E_p : \mathfrak{L}\mathfrak{S}_p \to \mathfrak{S}_p$$

by a linear transformation satisfying that:

$$E_p\left(l_p^k \otimes P_{p,j}\right) = \frac{\left(p^{j+1}\right)^{k+1}}{\left[\frac{k}{2}\right]+1} l_p^k(P_{p,j}), \tag{51}$$

for all $k \in \mathbb{N}_0$, $j \in \mathbb{Z}$, where $\left[\frac{k}{2}\right]$ is the *minimal integer greater than or equal to* $\frac{k}{2}$, for all $k \in \mathbb{N}_0$; for example,

$$\left[\tfrac{3}{2}\right] = 2 = \left[\tfrac{4}{2}\right].$$

By the cyclicity (48) of the tensor factor $\mathfrak{L}_p$ of $\mathfrak{L}\mathfrak{S}_p$, and by the structure theorem (32) of the other tensor factor $\mathfrak{S}_p$ of $\mathfrak{L}\mathfrak{S}_p$, the above morphism E_p of (51) is a well-defined bounded surjective linear transformation.

Now, consider how our p-radial operator l_p of (47) works on $\mathfrak{S}_p$. Observe first that: if $\mathbf{c}_p$ and $\mathbf{a}_p$ are the p-creation, respectively, the p-annihilation on $\mathfrak{S}_p$, then:

$$\mathbf{c}_p\mathbf{a}_p\left(P_{p,j}\right) = P_{p,j} = \mathbf{a}_p\mathbf{c}_p\left(P_{p,j}\right),$$

for all $j \in \mathbb{Z}$, $p \in \mathcal{P}$, and hence:

$$\mathbf{c}_p\mathbf{a}_p = 1_{\mathfrak{S}_p} = \mathbf{a}_p\mathbf{c}_p on \mathfrak{S}_p. \tag{52}$$

Lemma 1. *Let $\mathbf{c}_p$, $\mathbf{a}_p$ be the p-creation, respectively, the p-annihilation on $\mathfrak{S}_p$. Then:*

$$\mathbf{c}_p^n\mathbf{a}_p^n = \left(\mathbf{c}_p\mathbf{a}_p\right)^n = 1_{\mathfrak{S}_p} = \left(\mathbf{a}_p\mathbf{c}_p\right)^n = \mathbf{a}_p\mathbf{c}_p,$$

and:

$$\mathbf{c}_p^{n_1}\mathbf{a}_p^{n_2} = \mathbf{a}_p^{n_2}\mathbf{c}_p^{n_1} on \mathfrak{S}_p, \tag{53}$$

for all $n, n_1, n_2 \in \mathbb{N}_0$.

Proof. The formula (53) holds by (52). □

By (53), one can get that:

$$l_p^n = (\mathbf{c}_p + \mathbf{a}_p)^n = \Sigma_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} \mathbf{c}_p^k \mathbf{a}_p^{n-k} \text{ on } \mathfrak{S}_p,$$

with identities;

$$\mathbf{c}_p^0 = 1_{\mathfrak{S}_p} = \mathbf{a}_p^0, \tag{54}$$

for all $n \in \mathbb{N}$, where:

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}, \forall k \leq n \in \mathbb{N}_0.$$

Thus, one obtains the following proposition.

Proposition 8. *Let $l_p \in \mathfrak{L}_p$ be the p-radial operator on $\mathfrak{S}_p$. Then:*

$$l_p^{2m-1} \textit{ does not contain } 1_{\mathfrak{S}_p} - \textit{term, and} \tag{55}$$

$$l_p^{2m} \textit{ contains its } 1_{\mathfrak{S}_p} - \textit{term}, \begin{pmatrix} 2m \\ m \end{pmatrix} \cdot 1_{\mathfrak{S}_p}, \tag{56}$$

for all $m \in \mathbb{N}$.

Proof. The proofs of (55) and (56) are done by straightforward computations by (53) and (54). See [1] for more details. □

7.3. Weighted-Semicircular Elements $Q_{p,j}$ in $\mathfrak{L}\mathfrak{S}_p$

Fix $p \in \mathcal{P}$, and let $\mathfrak{L}\mathfrak{S}_p = \mathfrak{L}_p \otimes_{\mathbb{C}} \mathfrak{S}_p$ be the tensor product Banach $*$-algebra (50) and E_p be the linear transformation (51) from $\mathfrak{L}\mathfrak{S}_p$ onto $\mathfrak{S}_p$. Throughout this section, fix a generating element:

$$Q_{p,j} = l_p \otimes P_{p,j} of \mathfrak{L}\mathfrak{S}_p, \tag{57}$$

for $j \in \mathbb{Z}$, where $P_{p,j}$ is a projection (30) generating $\mathfrak{S}_p$. Observe that:

$$Q_{p,j}^n = (l_p \otimes P_{p,j})^n = l_p^n \otimes P_{p,j}, \tag{58}$$

for all $n \in \mathbb{N}$, for all $j \in \mathbb{Z}$.

If $Q_{p,j} \in \mathfrak{L}\mathfrak{S}_p$ is in the sense of (57) for $j \in \mathbb{Z}$, then:

$$E_p\left(Q_{p,j}^n\right) = E_p\left(l_p^n \otimes P_{p,j}\right) = \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1} l_p^n\left(P_{p,j}\right), \tag{59}$$

by (51) and (58), for all $n \in \mathbb{N}$.

Now, for a fixed $j \in \mathbb{Z}$, define a linear functional $\tau_{p,j}^0$ on $\mathfrak{L}\mathfrak{S}_p$ by:

$$\tau_{p,j}^0 = \tau_j^p \circ E_p on \mathfrak{L}\mathfrak{S}_p, \tag{60}$$

where $\tau_j^p = \frac{1}{\phi(p)} \varphi_j^{(p)}$ is the linear functional (38) on $\mathfrak{S}_p$.

By the bounded-linearity of both τ_j^p and E_p, the morphism $\tau_{p,j}^0$ of (60) is a bounded linear functional on $\mathfrak{L}\mathfrak{S}_p$.

By (59) and (60), if $Q_{p,j}$ is in the sense of (57), then:

$$\tau_{p,j}^0\left(Q_{p,j}^n\right) = \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1} \tau_j^p\left(l_p^n(P_{p,j})\right), \tag{61}$$

for all $n \in \mathbb{N}$.

Theorem 1. *Let $Q_{p,j} = l_p \otimes P_{p,j} \in \left(\mathfrak{LS}_p, \tau_{p,j}^0\right)$, for a fixed $j \in \mathbb{Z}$. Then, $Q_{p,j}$ is $p^{2(j+1)}$-semicircular in $\left(\mathfrak{LS}_p,\ \tau_{p,j}^0\right)$. More precisely,*

$$\tau_{p,j}^0 \left(Q_{p,j}^n\right) = \omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{62}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (45). Equivalently, if $k_n^{0,p,j}(\ldots)$ is the free cumulant on $\mathfrak{LS}_p$ in terms of the linear functional $\tau_{p,j}^0$ of (61) on $\mathfrak{LS}_p$, then:

$$k_n^{0,p,j}\left(\underbrace{Q_{p,j},\ Q_{p,j},\ \ldots,\ Q_{p,j}}_{n\text{-times}}\right) = \begin{cases} p^{2(j+1)} & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{63}$$

for all $n \in \mathbb{N}$.

Proof. The free-moment formula (62) is obtained by (55), (56) and (61). The free-cumulant formula (63) is obtained by (62) under the Möbius inversion of [14]. See [1] for details. ☐

8. Semicircularity on $\mathfrak{LS}$

For all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, let:

$$\mathfrak{LS}_p(j) = \left(\mathfrak{LS}_p,\ \tau_{p,j}^0\right) \tag{64}$$

be a Banach $*$-probabilistic model of the Banach $*$-algebra $\mathfrak{LS}_p$ of (50), where $\tau_{p,j}^0$ is the linear functional (60).

Definition 11. *We call the pairs $\mathfrak{LS}_p(j)$ of (64) the j^{textth}* p-adic filters, *for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.*

Let $Q_{p,k} = l_p \otimes P_{p,k}$ be the k^{th} generating elements of the j^{th} p-adic filter $\mathfrak{LS}_p(j)$ of (64), for all $k \in \mathbb{Z}$, for fixed $p \in \mathcal{P}$, $j \in \mathbb{Z}$. Then, the generating elements $\{Q_{p,k}\}_{k \in \mathbb{Z}}$ of the j^{th} p-adic filter $\mathfrak{LS}_p(j)$ satisfy that:

$$k_n^{0,p,j}\left(Q_{p,k},\ \ldots,\ Q_{p,k}\right) = \begin{cases} \delta_{j,k}\ p^{2(j+1)} & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

and:

$$\tau_{p,j}^0\left(Q_{p,k}^n\right) = \delta_{j,k}\left(\omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right), \tag{65}$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, for all $n \in \mathbb{N}$, by (62) and (63), where:

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$.

For the family:

$$\left\{\mathfrak{LS}_p(j) = \left(\mathfrak{LS}_p,\ \tau_{p,j}^0\right) : p \in \mathcal{P},\ j \in \mathbb{Z}\right\}$$

of j^{th} p-adic filters of (64), one can define the *free product Banach $*$-probability space*,

$$\mathfrak{LS} \overset{denote}{=} \left(\mathfrak{LS},\ \tau^0\right) \overset{def}{=} \underset{p \in \mathcal{P},\ j \in \mathbb{Z}}{\star} \mathfrak{LS}_p(j). \tag{66}$$

as in [14,15], with:

$$\mathfrak{LS} = \underset{p \in \mathcal{P},\, j \in \mathbb{Z}}{\star} \mathfrak{LS}_p, \text{ and } \tau^0 = \underset{p \in \mathcal{P},\, j \in \mathbb{Z}}{\star} \tau^0_{p,j}.$$

Note that the pair $\mathfrak{LS} = \left(\mathfrak{LS},\ \tau^0\right)$ of (66) is a well-defined "traditional or noncommutative" Banach $*$-probability space. For more about the (free-probabilistic) *free product* of free probability spaces, see [14,15].

Definition 12. *The Banach $*$-probability space $\mathfrak{LS} = \left(\mathfrak{LS},\ \tau^0\right)$ of (66) is called the free Adelic filtration.*

Let $\mathfrak{LS}$ be the free Adelic filtration (66). Then, by (65), one can take a subset:

$$\mathcal{Q} = \left\{Q_{p,j} = l_p \otimes P_{p,j} \in \mathfrak{LS}_p(j)\right\}_{p \in \mathcal{P},\, j \in \mathbb{Z}}$$

of $\mathfrak{LS}$, consisting of "j^{th}" generating elements $Q_{p,j}$ of the "j^{th}" p-adic filters $\mathfrak{LS}_p(j)$, which are the free blocks of $\mathfrak{LS}$, for all $j \in \mathbb{Z}$, for all $p \in \mathcal{P}$.

Lemma 2. *Let $\mathcal{Q}$ be the above family in the free Adelic filtration $\mathfrak{LS}$. Then, all elements $Q_{p,j}$ of $\mathcal{Q}$ are $p^{2(j+1)}$-semicircular in the free Adelic filtration $\mathfrak{LS}$.*

Proof. Since all self-adjoint elements $Q_{p,j}$ of the family $\mathcal{Q}$ are chosen from mutually-distinct *free blocks* $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, they are $p^{2(j+1)}$-semicircular in $\mathfrak{LS}_p(j)$. Indeed, since every element $Q_{p,j} \in \mathcal{Q}$ is from a free block $\mathfrak{LS}_p(j)$, the powers $Q^n_{p,j}$ are free reduced words with their lengths-N in $\mathfrak{LS}_p(j)$ in $\mathfrak{LS}$. Therefore, each element $Q_{p,j} \in \mathcal{Q}$ satisfies that:

$$\tau^0\left(Q^n_{p,j}\right) = \tau^0_{p,j}\left(Q^n_{p,j}\right) = \omega_n p^{n(j+1)} c_{\frac{n}{2}},$$

equivalently,

$$\begin{aligned} k^0_n\left(Q_{p,j},\ \ldots,\ Q_{p,j}\right) &= k^{0,p,j}_n\left(Q_{p,j},\ \ldots,\ Q_{p,j}\right) \\ &= \begin{cases} p^{2(j+1)} & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for all $n \in \mathbb{N}$, by (62) and (63), where $k^0_n(\ldots)$ is the free cumulant on $\mathfrak{LS}$ in terms of τ^0. Therefore, all elements $Q_{p,j} \in \mathcal{Q}$ are $p^{2(j+1)}$-semicircular in $\mathfrak{LS}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$. □

Furthermore, since all $p^{2(j+1)}$-semicircular elements $Q_{p,j} \in \mathcal{Q}$ are taken from the mutually-distinct free blocks $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, they are mutually free from each other in the free Adelic filtration $\mathfrak{LS}$ of (66), for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Recall that a subset $S = \{a_t\}_{t \in \Delta}$ of an arbitrary (topological or pure-algebraic) $*$-probability space $(A,\ \varphi)$ is said to be a *free family*, if, for any pair $(t_1,\ t_2) \in \Delta^2$ of $t_1 \neq t_2$ in a countable (finite or infinite) index set Δ, the corresponding free random variables a_{t_1} and a_{t_2} are free in $(A,\ \varphi)$ (e.g., [7,14]).

Definition 13. *Let $S = \{a_t\}_{t \in \Delta}$ be a free family in an arbitrary topological $*$-probability space $(A,\ \varphi)$. This family S is said to be a free (weighted-)semicircular family, if it is not only a free family, but also a set consisting of all (weighted-)semicircular elements a_t in $(A,\ \varphi)$, for all $t \in \Delta$.*

Therefore, by the construction (66) of the free Adelic filtration $\mathfrak{LS}$, we obtain the following result.

Theorem 2. *Let $\mathfrak{LS}$ be the free Adelic filtration (66), and let:*

$$\mathcal{Q} = \{Q_{p,j} \in \mathfrak{LS}_p(j)\}_{p \in \mathcal{P},\, j \in \mathbb{Z}} \subset \mathfrak{LS}, \tag{67}$$

where $\mathfrak{LS}_p(j)$ are the j^{th} p-adic filters, the free blocks of $\mathfrak{LS}$. Then, this family $\mathcal{Q}$ of (67) is a free weighted-semicircular family in $\mathfrak{LS}$.

Proof. Let $\mathcal{Q}$ be a subset (67) in $\mathfrak{LS}$. Then, all elements $Q_{p,j}$ of $\mathcal{Q}$ are $p^{2(j+1)}$-semicircular in $\mathfrak{LS}$ by the above lemma, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$. Furthermore, all elements $Q_{p,j}$ of $\mathcal{Q}$ are mutually free from each other in $\mathfrak{LS}$, because they are contained in the mutually-distinct free blocks $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$. Therefore, the family $\mathcal{Q}$ of (67) is a free weighted-semicircular family in $\mathfrak{LS}$. □

Now, take elements:

$$\Theta_{p,j} \overset{def}{=} \frac{1}{p^{j+1}} Q_{p,j}, \forall p \in \mathcal{P}, j \in \mathbb{Z}, \tag{68}$$

in $\mathfrak{LS}$, where $Q_{p,j} \in \mathcal{Q}$, where $\mathcal{Q}$ is the free weighted-semicircular family (67) in the free Adelic filtration $\mathfrak{LS}$.

Then, by the self-adjointness of $Q_{p,j}$, these operators $\Theta_{p,j}$ of (68) are self-adjoint in $\mathfrak{LS}$, as well, because:

$$p^{j+1} \in \mathbb{Q} \subset \mathbb{R} \text{ in } \mathbb{C},$$

satisfying $\overline{p^{j+1}} = p^{j+1}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Furthermore, one obtains the following free-cumulant computation; if $k_n^0(...)$ is the free cumulant on $\mathfrak{LS}$ in terms of τ^0, then:

$$\begin{aligned} k_n^0\left(\Theta_{p,j}, ..., \Theta_{p,j}\right) &= k_n^{0,p,j}\left(\tfrac{1}{p^{j+1}} Q_{p,j}, ..., \tfrac{1}{p^{j+1}} Q_{p,j}\right) \\ &= \left(\tfrac{1}{p^{j+1}}\right)^n k_n^{0,p,j}\left(Q_{p,j}, ..., Q_{p,j}\right), \end{aligned} \tag{69}$$

by the *bimodule-map property* of the free cumulant (e.g., see [14]), for all $n \in \mathbb{N}$, where $k_n^{0,p,j}(...)$ are the free cumulants (63) on the free blocks $\mathfrak{LS}_p(j)$ in terms of the linear functionals $\tau_{p,j}^0$ of (60) on $\mathfrak{LS}_p$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Theorem 3. *Let $\Theta_{p,j} = \frac{1}{p^{j+1}} Q_{p,j}$ be free random variables (68) of the free Adelic filtration $\mathfrak{LS}$, for $Q_{p,j} \in \mathcal{Q}$. Then, the family:*

$$\Theta = \left\{\Theta_{p,j} \in \mathfrak{LS}_p(j) : p \in \mathcal{P},\ j \in \mathbb{Z}\right\} \tag{70}$$

forms a free semicircular family in $\mathfrak{LS}$.

Proof. Consider that:

$$\begin{aligned} k_n^0\left(\Theta_{p,j}, ..., \Theta_{p,j}\right) &= \left(\tfrac{1}{p^{j+1}}\right)^n k_n^{0,p,j}\left(Q_{p,j}, ..., Q_{p,j}\right) \text{ by (69)} \\ &= \begin{cases} \left(\tfrac{1}{p^{j+1}}\right)^2 k_2^{0,p,j}\left(Q_{p,j}, Q_{p,j}\right) & \text{if } n = 2 \\ \\ \left(\tfrac{1}{p^{j+1}}\right)^n k_n^{0,p,j}\left(Q_{p,j}, ..., Q_{p,j}\right) = 0 & \text{otherwise,} \end{cases} \end{aligned}$$

by the $p^{2(j+1)}$-semicircularity of $Q_{p,j} \in \mathcal{Q}$ in $\mathfrak{LS}$:

$$= \begin{cases} \left(\tfrac{1}{p^{j+1}}\right)^2 \left(p^{j+1}\right)^2 = 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{71}$$

for all $n \in \mathbb{N}$.

By the free-cumulant computation (71), these self-adjoint free random variables $\Theta_{p,j} \in \mathfrak{LS}_p(j)$ are semicircular in $\mathfrak{LS}$ by (43), for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Furthermore, the family Θ of (70) forms a free family in $\mathfrak{LS}$, because all elements $\Theta_{p,j}$ are the scalar-multiples of $Q_{p,j} \in \mathcal{Q}$, contained in mutually-distinct free blocks $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, for all $j \in \mathbb{Z}$, $p \in \mathcal{P}$.

Therefore, this family Θ of (70) is a free semicircular family in $\mathfrak{LS}$. □

Now, define a Banach $*$-subalgebra $\mathbb{LS}$ of $\mathfrak{LS}$ by:

$$\mathbb{LS} \overset{def}{=} \overline{\mathbb{C}[\mathcal{Q}]} in \mathfrak{LS}, \tag{72}$$

where $\mathcal{Q}$ is the free weighted-semicircular family (67) and $\overline{Y}$ means the Banach-topology closures of subsets Y of $\mathfrak{LS}$.

Then, one can obtain the following structure theorem for the Banach $*$-algebra $\mathbb{LS}$ of (72) in $\mathfrak{LS}$.

Theorem 4. *Let $\mathbb{LS}$ be the Banach $*$-subalgebra (72) of the free Adelic filtration $\mathfrak{LS}$ generated by the free weighted-semicircular family $\mathcal{Q}$ of (67). Then:*

$$\mathbb{LS} = \overline{\mathbb{C}\,[\Theta]} in \mathfrak{LS}, \tag{73}$$

where Θ is the free semicircular family (70) and where "$=$" means "being identically same as sets." Moreover,

$$\mathbb{LS} \overset{*\text{-iso}}{=} \underset{p \in \mathcal{P},\, j \in \mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{Q_{p,j}\}\right]} \overset{*\text{-iso}}{=} \overline{\mathbb{C}\left[\underset{p \in \mathcal{P},\, j \in \mathbb{Z}}{\star} \{Q_{p,j}\}\right]}, \tag{74}$$

in $\mathfrak{LS}$, where "$\overset{*\text{-iso}}{=}$" means "being Banach-$*$-isomorphic," and:

$$\overline{\mathbb{C}\left[\{Q_{p,j}\}\right]} \text{ are Banach } *\text{-subalgebras of } \mathfrak{LS}_p(j),$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, in $\mathfrak{LS}$.

Here, ($\star$) in the first $*$-isomorphic relation of (74) is the (free-probability-theoretic) free product of [14,15], and ($\star$) in the second $*$-isomorphic relation of (74) is the (pure-algebraic) free product (generating noncommutative algebraic free words in the family $\mathcal{Q}$).

Proof. Let $\mathbb{LS}$ be the Banach $*$-subalgebra (72) of $\mathfrak{LS}$. Then, all generating operators $Q_{p,j} \in \mathcal{Q}$ of $\mathbb{LS}$ are contained in mutually-distinct free blocks $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, and hence, the Banach $*$-subalgebras $\overline{\mathbb{C}\left[\{Q_{p,j}\}\right]}$ of $\mathfrak{LS}$ are contained in the free blocks $\mathfrak{LS}_p(j)$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$. Therefore, as embedded sub-structures of $\mathfrak{LS}$, they are free from each other. Equivalently,

$$\mathbb{LS} \overset{*\text{-iso}}{=} \underset{p \in \mathcal{P},\, j \in \mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{Q_{p,j}\}\right]} in \mathfrak{LS}, \tag{75}$$

by (66).

Since every free block $\overline{\mathbb{C}\left[\{Q_{p,j}\}\right]}$ of the Banach $*$-algebra $\mathbb{LS}$ of (75) is generated by a single self-adjoint (weighted-semicircular) element, every operator T of $\mathbb{LS}$ is a limit of linear combinations of free words in the free family $\mathcal{Q}$ of (67), which form noncommutative free "reduced" words (in the sense of [14,15]), as operators in $\mathbb{LS}$ of (75). Note that every (pure-algebraic) free word in $\mathcal{Q}$ has a unique free reduced word in $\mathbb{LS}$, under operator-multiplication on $\mathfrak{LS}$ (and hence, on $\mathbb{LS}$). Therefore, the $*$-isomorphic relation (75) guarantees that:

$$\mathbb{LS} \overset{*\text{-iso}}{=} \overline{\mathbb{C}\left[\underset{p \in \mathcal{P},\, j \in \mathbb{Z}}{\star} \{Q_{p,j}\}\right]}, \tag{76}$$

where the free product ($\star$) in (76) is pure-algebraic.

Remark that, indeed, the relation (76) holds well, because all weighted-semicircular elements of $\mathcal{Q}$ are self-adjoint; if:

$$T = \prod_{l=1}^{N} Q_{p_l, j_l}^{n_l} \in \mathbb{LS}$$

is a free (reduced) word (as an operator), then:

$$T^{*} = \prod_{l=1}^{N} Q_{p_{N-l+1}, j_{N-l+1}}^{n_{N-l+1}} \in \mathbb{LS}$$

is a free word of $\mathbb{LS}$ in $\mathcal{Q}$, as well. Therefore, by (75) and (76), the structure theorem (74) holds true.

Note now that $Q_{p,j} \in \mathcal{Q}$ satisfy:

$$Q_{p,j} = p^{j+1}\Theta_{p,j}, \text{ for all } p \in \mathcal{P},\, j \in \mathbb{Z},$$

where $\Theta_{p,j}$ are semicircular elements in the family Θ of (70). Therefore, the free blocks of (75) satisfy that:

$$\overline{\mathbb{C}\left[\{Q_{p,j}\}\right]} = \overline{\mathbb{C}\left[\{p^{j+1}\Theta_{p,j}\}\right]} = \overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]}, \tag{77}$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Thus, one can get that:

$$\mathbb{LS} \overset{*\text{-iso}}{=} \underset{p\in\mathcal{P},\, j\in\mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]}, \tag{78}$$

by (75) and (77).

With similar arguments of (75), we have:

$$\mathbb{LS} = \overline{\mathbb{C}[\Theta]}, set-theoretically, \tag{79}$$

by (78).

Therefore, the identity (73) holds true by (79). □

As a sub-structure, one can restrict the linear functional τ^0 of (66) on $\mathfrak{LS}$ to that on $\mathbb{LS}$, i.e., one can obtain the Banach $*$-probability space,

$$\left(\mathbb{LS},\ \tau^0 \overset{denote}{=} \tau^0 \mid_{\mathbb{LC}}\right). \tag{80}$$

Definition 14. *Let* $(\mathbb{LS}, \tau^0)$ *be the Banach* $*$*-probability space* (80)*. Then, we call* $(\mathbb{LS}, \tau^0)$ *the semicircular (free Adelic sub-)filtration of* $\mathfrak{LS}$*.*

Note that, by (66), all elements of the semicircular filtration $(\mathbb{LS}, \tau^0)$ provide "possible" non-vanishing free distributions in the free Adelic filtration $\mathfrak{LS}$. Especially, all free reduced words of $\mathfrak{LS}$ in the generator set $\{Q_{p,j}\}_{p\in P, j\in\mathbb{Z}}$ have non-zero free distributions only if they are contained in $(\mathbb{LS}, \tau^0)$. Therefore, studying free-distributional data on $(\mathbb{LS}, \tau^0)$ is to study possible non-zero free-distributional data on $\mathfrak{LS}$.

9. Truncated Linear Functionals on $\mathbb{LS}$

In *number theory*, one of the most interesting, but difficult topics is to find a number of primes or a density of primes contained in closed intervals $[t_1, t_2]$ of the real numbers $\mathbb{R}$ (e.g., [3,6,21,22]). Since the theory is deep, we will not discuss more about it here. Hhowever, motivated by the theory, we consider certain "suitable" *truncated linear functionals* on our semicircular filtration $(\mathbb{LS}, \tau^0)$ of (80) in the free Adelic filtration $\mathfrak{LS}$ of (66).

Notation: From below, we will use the following notations to distinguish their structural differences;

$\mathbb{LS} \overset{denote}{=}$ the Banach $*$-subalgebra (72) of $\mathfrak{LS}$,

$\mathbb{LS}_0 \overset{denote}{=}$ the semicircular filtration $(\mathbb{LS},\ \tau^0)$ of (80).

□

9.1. Linear Functionals $\{\tau_{(t)}\}_{t\in\mathbb{R}}$ on $\mathbb{LS}$

Let $\mathbb{LS}_0$ be the semicircular filtration $(\mathbb{LS}, \tau^0)$ of the free Adelic filtration $\mathfrak{LS}$. Furthermore, let $\mathcal{Q}$ and Θ be the free weighted-semicircular family (67), respectively, the free semicircular family (70) of $\mathfrak{LS}$, generating $\mathbb{LS}$ by (73) and (74). We here truncate τ^0 on $\mathbb{LS}$ for a fixed real number $t \in \mathbb{R}$.

First, recall and remark that:

$$\tau^0 = \underset{p\in\mathcal{P},\, j\in\mathbb{Z}}{\star} \tau^0_{p,j} \text{ on } \mathbb{LS},$$

by (66) and (80). Therefore, one can sectionize τ^0 over $\mathcal{P}$, as follows;

$$\tau^0 = \underset{p\in\mathcal{P}}{\star} \tau^0_p \text{ on } \mathbb{LS},$$

with:

$$\tau^0_p = \underset{j\in\mathbb{Z}}{\star} \tau^0_{p,j} on \mathbb{LS}_p, for all p \in \mathcal{P}, \tag{81}$$

where:

$$\mathbb{LS}_p \overset{def}{=} \underset{j\in\mathbb{Z}}{\star} \overline{\mathbb{C}[\{\Theta_{p,j}\}]} \subset \mathbb{LS} \subset \mathfrak{LS}, \tag{82}$$

for each $p \in \mathcal{P}$, under (74).

From below, we understand the Banach $*$-subalgebras $\mathbb{LS}_p$ of $\mathbb{LS}$ as free-probabilistic sub-structures,

$$\mathbb{LS}_{(p)} \overset{denote}{=} \left(\mathbb{LS}_p,\, \tau^0_p\right), for all p \in \mathcal{P}. \tag{83}$$

Lemma 3. *Let $\mathbb{LS}_{p_l}$ be in the sense of (82) in the semicircular filtration $\mathbb{LS}_0$, for $l = 1, 2$. Then, $\mathbb{LS}_{p_1}$ and $\mathbb{LS}_{p_2}$ are free in $\mathbb{LS}_0$, if and only if $p_1 \neq p_2$ in $\mathcal{P}$.*

Proof. The proof is directly done by (81) and (82). Indeed,

$$\begin{aligned} \mathbb{LS} &= \underset{p\in\mathcal{P},\, j\in\mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]} \\ &= \underset{p\in\mathcal{P}}{\star} \left(\underset{j\in\mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{\Theta_{p,j}\}\right]}\right) = \underset{p\in\mathcal{P}}{\star} \mathbb{LS}_p, \end{aligned}$$

by (80) and (82).

Therefore, $\mathbb{LS}_{p_1}$ and $\mathbb{LS}_{p_2}$ are free in $\mathbb{LS}_0$, if and only if $p_1 \neq p_2$ in $\mathcal{P}$. □

Fix now $t \in \mathbb{R}$, and define a new linear functional $\tau_{(t)}$ on $\mathbb{LS}$ by:

$$\tau_{(t)} \overset{def}{=} \begin{cases} \underset{p\leq t}{\star} \tau^0_p & \text{on } \underset{p\leq t}{\star} \mathbb{LS}_p \subset \mathbb{LS} \\ O & \text{on } \mathbb{LS} \setminus \left(\underset{p\leq t}{\star} \mathbb{LS}_p\right), \end{cases} \tag{84}$$

where τ^0_p are the linear functionals (81) on the Banach $*$-subalgebras $\mathbb{LS}_p$ of (82) in $\mathbb{LS}_0$, for all $p \in \mathcal{P}$, and O means the *zero linear functional* on $\mathbb{LS}$, satisfying that:

$$O(T) = 0, \text{ for all } T \in \mathbb{LS}.$$

For convenience, if there is no confusion, we simply write the definition (84) as:

$$\tau_{(t)} \overset{denote}{=} \underset{p\leq t}{\star} \tau^0_p. \tag{85}$$

By the definition (84) (with a simpler expression (85)), one can easily verify that, if $t < 2$ in $\mathbb{R}$, then the corresponding linear functional $\tau_{(t)}$ is identical to the zero linear functional O on $\mathbb{LS}$. To avoid such triviality, one may refine $\tau_{(t)}$ of (84) by:

$$\tau_{(t)} \overset{def}{=} \begin{cases} \tau_{(t)} \text{ of (84)} & \text{if } t \geq 2 \\ O & \text{if } t < 2, \end{cases} \tag{86}$$

for all $t \in \mathbb{R}$.

In the following text, $\tau_{(t)}$ mean the linear functionals in (86), satisfying (84) whenever $t \geq 2$, for all $t \in \mathbb{R}$. In fact, we are not interested in the cases where $t < 2$.

For example,

$$\tau_{\left(\frac{\sqrt{3}}{2}\right)} = O, \tau_{(2.1003)} = \tau_2^0, and \tau_{(6)} = \tau_2^0 \star \tau_3^0 \star \tau_5^0,$$

on $\mathbb{LS}$, under (85), etc.

Theorem 5. *Let $Q_{p,j} \in \mathcal{Q}$ and $\Theta_{p,j} \in \Theta$ in the semicircular filtration $\mathbb{LS}_0$, for $p \in \mathcal{P}$, $j \in \mathbb{Z}$, and let $t \in \mathbb{R}$ and $\tau_{(t)}$, the corresponding linear functional (86) on $\mathbb{LS}$. Then:*

$$\tau_{(t)}\left(Q_{p,j}^n\right) = \begin{cases} \omega_n p^{2(j+1)} c_{\frac{n}{2}} & \text{if } t \geq p \\ 0 & \text{if } t < p, \end{cases}$$

and:

$$\tau_{(t)}\left(\Theta_{p,j}^n\right) = \begin{cases} \omega_n c_{\frac{n}{2}} & \text{if } t \geq p \\ 0 & \text{if } t < p, \end{cases} \tag{87}$$

for all $n \in \mathbb{N}$.

Proof. By the $p^{2(j+1)}$-semicircularity of $Q_{p,j} \in \mathcal{Q}$, the semicircularity of $\Theta_{p,j} \in \Theta$ in the semicircular filtration $\mathbb{LS}_0$, and by the definition (86), if $t \geq p$ in $\mathbb{R}$, then:

$$\begin{aligned} \tau_{(t)}\left(Q_{p,j}^n\right) &= \tau_p^0\left(Q_{p,j}^n\right) = \tau_{p,j}^0\left(Q_{p,j}^n\right) \\ &= \omega_n p^{2(j+1)} c_{\frac{n}{2}}, \end{aligned}$$

and:

$$\begin{aligned} \tau_{(t)}\left(\Theta_{p,j}^n\right) &= \tau_p^0\left(\Theta_{p,j}^n\right) = \tau_{p,j}^0\left(\Theta_{p,j}^n\right) \\ &= \omega_n c_{\frac{n}{2}}, \end{aligned}$$

by (62), (71), and (81), for all $n \in \mathbb{N}$.

If $t < p$, then:

$$\tau_{(t)} = \underset{2 \leq q < t < p \text{ in } \mathcal{P}}{\star} \tau_q^0 \text{ or } O, \text{ on } \mathbb{LS}.$$

Therefore, in such cases,

$$\tau_{(t)}\left(Q_{p,j}^n\right) = \tau_{(t)}\left(\Theta_{p,j}^n\right) = 0, \text{ for all } n \in \mathbb{N},$$

by (84), (85), and (86).

Therefore, the free-distributional data (87) for the linear functional $\tau_{(t)}$ hold on $\mathbb{LS}$. $\square$

The above theorem shows how the original free-probabilistic information on the semicircular filtration $\mathbb{LS}_0$ is affected by the new free-probabilistic models on $\mathbb{LS}$, under "truncated" linear functionals $\tau_{(t)}$ of τ^0 on $\mathbb{LS}$, for $t \in \mathbb{R}$.

Definition 15. *Let $\tau_{(t)}$ be the linear functionals (86) on $\mathbb{LS}$, for $t \in \mathbb{R}$. Then, the corresponding new Banach $*$-probability spaces,*

$$\mathbb{LS}_{(t)} \overset{denote}{=} \left(\mathbb{LS},\ \tau_{(t)}\right), \tag{88}$$

are called the semicircular t-(truncated-)filtrations of $\mathbb{LS}$ (or, of $\mathbb{LS}_0$).

Note that if t is "suitable" in the sense that "$\tau_{(t)} \neq O$ on $\mathbb{LS}$," then the free-probabilistic structure $\mathbb{LS}_{(t)}$ of (88) is meaningful.

Notation and Assumption 9.1 (**NA 9.1**, from below): In the following, we will say "$t \in \mathbb{R}$ is suitable," if the semicircular t-filtration "$\mathbb{LS}_{(t)}$ of (88) is meaningful," in the sense that: $\tau_{(t)} \neq O$ fully on $\mathbb{LS}$. □

Now, let us consider the following concepts.

Definition 16. *Let $(A_k,\ \varphi_k)$ be Banach $*$-probability spaces (or C^*-probability spaces, or W^*-probability spaces, etc.), for $k = 1, 2$. A Banach $*$-probability space $(A_1,\ \varphi_1)$ is said to be free-homomorphic to a Banach $*$-probability space $(A_2,\ \varphi_2)$, if there exists a bounded $*$-homomorphism:*

$$\Phi : A_1 \to A_2,$$

such that:

$$\varphi_2\left(\Phi(a)\right) = \varphi_1\left(a\right),$$

for all $a \in A_1$. Such a $*$-homomorphism Φ is called a free-homomorphism.

If Φ is both a $*$-isomorphism and a free-homomorphism, then Φ is said to be a free-isomorphism, and we say that $(A_1,\ \varphi_1)$ and $(A_2,\ \varphi_2)$ are free-isomorphic. Such a free-isomorphic relation is nothing but the equivalence in the sense of Voiculescu (e.g., [15]).

By (87), we obtain the following free-probabilistic-structural theorem.

Theorem 6. *Let $\mathbb{LS}_q = \underset{j\in\mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{Q_{q,j}\}\right]}$ be Banach $*$-subalgebras (82) of $\mathbb{LS}$, for all $q \in \mathcal{P}$. Let $t \in \mathbb{R}$ be suitable in the sense of **NA 9.1** and $\mathbb{LS}_{(t)}$ be the corresponding semicircular t-filtration (88). Construct a Banach $*$-probability space $\mathbb{LS}^t$ by a Banach $*$-probabilistic sub-structure of the semicircular filtration $\mathbb{LS}_0$,*

$$\mathbb{LS}^t \overset{def}{=} \underset{p\leq t}{\star}\left(\mathbb{LS}_p,\ \tau_p^0\right) = \left(\underset{p\leq t}{\star}\mathbb{LS}_p,\ \underset{p\leq t}{\star}\tau_p^0\right), \tag{89}$$

where $\tau_p^0 = \underset{j\in\mathbb{Z}}{\star}\tau_{p,j}^0$ are in the sense of (81). Then:

$$\mathbb{LS}^t is free-homomorphic to \mathbb{LS}_{(t)}. \tag{90}$$

Proof. Let $\mathbb{LS}_{(t)}$ be the semicircular t-filtration (88) of $\mathbb{LS}$, and let $\mathbb{LS}^t$ be a Banach $*$-probability space (89), for a suitably fixed $t \in \mathbb{R}$.

Define a bounded linear morphism:

$$\Phi_t : \mathbb{LS}^t \to \mathbb{LS}_{(t)},$$

by the natural embedding map,

$$\Phi_t\left(T\right) = T in \mathbb{LS}_{(t)}, for\ all T \in \mathbb{LS}^t. \tag{91}$$

Then, this morphism Φ_t is an injective bounded $*$-homomorphism from $\mathbb{LS}^t$ into $\mathbb{LS}_{(t)}$, by (72), (75), (82), (89), and (91).

Therefore, one obtains that:

$$\tau_{(t)}\left(\Phi(T)\right) = \tau_{(t)}(T) = \left(\underset{p \leq t \text{ in } \mathcal{P}}{\star} \tau_p^0 \right)(T) = \tau^t(T),$$

for all $T \in \mathbb{LS}^t$, by (87).

It shows that the Banach $*$-probability space $\mathbb{LS}^t$ of (89) is free-homomorphic to the semicircular t-filtration $\mathbb{LS}_{(t)}$ of (88). Therefore, the statement (90) holds under the free-homomorphism Φ_t of (91). $\square$

The above theorem shows that the Banach $*$-probability spaces $\mathbb{LS}^t$ of (89) are free-homomorphic to the semicircular t-filtrations $\mathbb{LS}_{(t)}$ of (88), for all $t \in \mathbb{R}$. Note that it "seems" they are not free-isomorphic, because:

$$\left(\underset{q \leq t \text{ in } \mathcal{P}}{\star} \mathbb{LS}_q \right) \subsetneqq \left(\underset{p \in \mathcal{P}}{\star} \mathbb{LS}_p \right) = \mathbb{LS},$$

set-theoretically, for $t \in \mathbb{R}$. However, we are not sure at this moment that they are free-isomorphic or not, because we have the similar difficulties discussed in [19].

Remark 3. *The famous main result of [19] says that: if $L(F_n)$ are the free group factors (group von Neumann algebras) of the free groups F_n with n-generators, for all:*

$$n \in \mathbb{N}_{>1}^{\infty} = (\mathbb{N} \setminus \{1\}) \cup \{\infty\},$$

then either (I) or (II) holds true, where:

(I) $L(F_n) \overset{*\text{-iso}}{=} L(F_\infty)$, for all $n \in \mathbb{N}_{>1}^{\infty}$,

(II) $L(F_{n_1}) \overset{*\text{-iso}}{\neq} L(F_{n_2})$, if and only if $n_1 \neq n_2 \in \mathbb{N}_{>1}^{\infty}$,

where "$\overset{*\text{-iso}}{=}$" means "being W^*-isomorphic." Depending on the author's knowledge, he does not know which one is true at this moment.

We here have similar troubles. Under the similar difficulties, we are not sure at this moment that $\mathbb{LS}^t$ and $\mathbb{LS}_{(t)}$ (or $\mathbb{LS}^t$ and $\mathbb{LS}$) are $*$-isomorphic or not (and hence, free-isomorphic or not).

However, definitely, $\mathbb{LS}^t$ is free-homomorphic "into" $\mathbb{LS}_{(t)}$ in the semicircular filtration $\mathbb{LS}_0$, by the above theorem.

The above free-homomorphic relation (90) lets us understand all
"non-zero" free distributions of free reduced words of $\mathbb{LS}_{(t)}$ as those of $\mathbb{LS}^t$, for all $t \in \mathbb{R}$, by the injectivity of a free-homomorphism Φ_t of (91).

Corollary 2. *All free reduced words T of the semicircular t-filtration $\mathbb{LS}_{(t)}$ in $\mathcal{Q} \cup \Theta$, having non-zero free distributions, are contained in the Banach $*$-probability space $\mathbb{LS}^t$ of (89), whenever t is suitable. The converse holds true, as well.*

Proof. The proof of this characterization is done by (87), (89), and (90). In particular, the injectivity of the free-homomorphism Φ_t of (91) guarantees that this characterization holds. $\square$

Therefore, whenever we consider a non-zero free-distribution having free reduced words T of semicircular t-filtrations $\mathbb{LS}_{(t)}$, they are regarded as free random variables of the Banach $*$-probability spaces $\mathbb{LS}^t$ of (89), for all suitable $t \in \mathbb{R}$.

9.2. Truncated Linear Functionals $\tau_{t_1 < t_2}$ on $\mathbb{LS}$

In this section, we generalize the semicircular t-filtrations $\mathbb{LS}_{(t)}$ by defining so-called *truncated linear functionals* on the Banach $*$-algebra $\mathbb{LS}$.

Throughout this section, let $[t_1, t_2]$ be a *closed interval* in $\mathbb{R}$, satisfying:

$$|t_1 - t_2| \neq 0, \text{ for } t_1 < t_2 \in \mathbb{R}.$$

For such a fixed closed interval $[t_1, t_2]$, define the corresponding linear functional $\tau_{t_1<t_2}$ on the semicircular filtration $\mathbb{LS}$ by:

$$\tau_{t_1<t_2} \overset{def}{=} \begin{cases} \underset{t_1 \le p \le t_2 \text{ in } \mathcal{P}}{\star} \tau_p^0 & \text{on } \underset{t_1 \le p \le t_2 \text{ in } \mathcal{P}}{\star} \mathbb{LS}_p \subset \mathbb{LS} \\ O & \text{on } \mathbb{LS} \setminus \left(\underset{t_1 \le p \le t_2 \text{ in } \mathcal{P}}{\star} \mathbb{LS} \right), \end{cases} \tag{92}$$

where τ_p^0 are the linear functionals (81) on the Banach $*$-subalgebras $\mathbb{LS}_p$ of (82) in $\mathbb{LS}$, for $p \in \mathcal{P}$. Similar to Section 9.1, if there is no confusion, then we simply write the definition (92) as:

$$\tau_{t_1<t_2} \overset{denote}{=} \underset{t_1 \le p \le t_2}{\star} \tau_p^0 on \mathbb{LS}. \tag{93}$$

To make the linear functionals $\tau_{t_1<t_2}$ of (92) be non-zero-linear functionals on $\mathbb{LS}$, the interval $[t_1, t_2]$ must be taken "suitably." For example,

$$\tau_{t_1<t_2} = O, \text{ whenever } t_2 < 2,$$

and:

$$\tau_{8<10} = O, \ \tau_{14<16} = O, \text{ and } \tau_{\frac{3}{7}<\frac{3}{2}} = O, \text{ etc.},$$

but:

$$\tau_{\frac{3}{2}<8} = \tau_{(8)} = \tau_2^0 \star \tau_3^0 \star \tau_5^0 \star \tau_7^0$$

and:

$$\tau_{7<14} = \tau_7^0 \star \tau_{11}^0 \star \tau_{13}^0,$$

under (93) on $\mathbb{LS}$.

It is not difficult to check that the definition (92) of truncated linear functionals $\tau_{t_1<t_2}$ covers the definition of linear functionals $\tau_{(t)}$ of (86). In particular, $\tau_{(t)}$ is "suitable" in the sense of **NA 9.1**, then:

$$\tau_{(t)} = \tau_{2<t} = \tau_{s<t}, \text{ for all } 2 \ge s \in \mathbb{R}.$$

For our purposes, we will axiomatize:

$$\tau_{p<p} = \tau_p^0, \text{ for all } p \in \mathcal{P} \subset \mathbb{R},$$

notationally, where τ_p^0 are the linear functionals (81), for all $p \in \mathcal{P}$, under (93). Remark that the very above axiomatized notations $\tau_{p<p}$ will be used only when p are primes.

Definition 17. *Let $[t_1, t_2]$ be a given interval in $\mathbb{R}$ and $\tau_{t_1<t_2}$, the corresponding linear functional (92) on $\mathbb{LS}$. Then, we call it the $[t_1, t_2]$(-truncated)-linear functional on $\mathbb{LS}$. The corresponding Banach $*$-probability space:*

$$\mathbb{LS}_{t_1<t_2} = (\mathbb{LS}, \tau_{t_1<t_2}) \tag{94}$$

is said to be the semicircular a $[t_1, t_2]$(-truncated)-filtration.

As we discussed in the above paragraphs, the semicircular $[t_1, t_2]$-filtration $\mathbb{LS}_{t_1<t_2}$ of (94) will be "meaningful," if $t_1 < t_2$ are suitable in $\mathbb{R}$, as in **NA 9.1**.

Notation and Assumption 9.2 (**NA 9.2**, from below): In the rest of this paper, if we write "$t_1 < t_2$ are suitable," then this means "$\mathbb{LS}_{t_1<t_2}$ is meaningful," in the sense that $\tau_{t_1<t_2} \ne O$ fully on $\mathbb{LS}$, with additional axiomatization:

$$\tau_{p<p} = \tau_p^0, \text{ for } p \in \mathcal{P} \text{ in } \mathbb{R},$$

in the sense of (93). □

Theorem 7. *Let $t_1 \leq 2$ and t_2 be suitable in $\mathbb{R}$ in the sense of* ***NA 9.1****.*

$$\begin{aligned}&\textit{The semicircular}[t_1, t_2]\textit{-filtration}\mathbb{LS}_{t_1<t_2}\textit{is not only suitable in the sense of}\\ &\textbf{\textit{NA 9.2}}\textit{, but also, it is free-isomorphic to the semicircular}t_2\textit{-filtration } \mathbb{LS}_{(t_2)}\ of\ (88).\end{aligned} \tag{95}$$

$$The\ Banach\ *-probability\ space\ \mathbb{LS}^{t_2}\ of\ (89)\ is\ free-homomorphic\ to\ \mathbb{LS}_{t_1<t_2}. \tag{96}$$

Proof. Suppose $t_1 \leq 2$, and t_2 are suitable in $\mathbb{R}$ in the sense of **NA 9.1**. Then, $t_1 < t_2$ are suitable in $\mathbb{R}$ in the sense of **NA 9.2**. Therefore, both the semicircular t_2-filtration $\mathbb{LS}_{(t_2)}$ and the semicircular $[t_1, t_2]$-filtration $\mathbb{LS}_{t_1<t_2}$ are meaningful.

Since t_1 is assumed to be less than or equal to two, the linear functional $\tau_{t_1<t_2} = \tau_{(t_2)}$, by (86) and (92), including the case where $\tau_{2<2} = \tau_2^0$, in the sense of (93). Therefore,

$$\mathbb{LS}_{t_1<t_2} = (\mathbb{LS},\ \tau_{t_1<t_2}) = \left(\mathbb{LS},\ \tau_{(t_2)}\right) = \mathbb{LS}_{(t_2)}.$$

Therefore, $\mathbb{LS}_{t_1<t_2}$ and $\mathbb{LS}_{(t_2)}$ are free-isomorphic under the identity map on $\mathbb{LS}$, acting as a free-isomorphism. Therefore, the statement (95) holds.

By (90), the Banach $*$-probability space $\mathbb{LS}^{t_2}$ of (89) is free-homomorphic to $\mathbb{LS}_{(t_2)}$. Therefore, under the hypothesis, $\mathbb{LS}^{t_2}$ is free-homomorphic to $\mathbb{LS}_{t_1<t_2}$ by (95). Equivalently, the statement (96) holds. □

The above theorem characterizes the free-probabilistic structures for semicircular $[t_1, t_2]$-filtrations $\mathbb{LS}_{t_1<t_2}$, whenever $t_1 \leq 2$, and t_2 are suitable, by (95) and (96). Therefore, we now restrict our interests to the cases where:

$$t_1 \geq 2 \text{ in } \mathbb{R}.$$

Therefore, we focus on the semicircular $[t_1, t_2]$-filtration $\mathbb{LS}_{t_1<t_2}$, where:

$$2 \leq t_1 < t_2 \text{ are suitable in } \mathbb{R},$$

in the sense of **NA 9.2**.

Theorem 8. *Let $2 \leq t_1 < t_2$ be suitable in $\mathbb{R}$, and let $\mathbb{LS}_{t_1<t_2}$ be the semicircular $[t_1, t_2]$-filtration (94). Then, the Banach $*$-probability space:*

$$\mathbb{LS}^{t_1<t_2} \overset{def}{=} \underset{t_1\leq p\leq t_2 \text{ in } \mathcal{P}}{\star} \left(\mathbb{LS}_p,\ \tau_p^0\right), \tag{97}$$

equipped with its linear functional $\tau^{t_1<t_2} = \underset{t_1\leq p\leq t_2}{\star} \tau_p^0$, is free-homomorphic to $\mathbb{LS}_{t_1<t_2}$ in $\mathbb{LS}$, i.e., if $2 \leq t_1 < t_2$ are suitable in $\mathbb{R}$,

$$\mathbb{LS}^{t_1<t_2}\ is\ free\text{-}homomorphic\ to\ \mathbb{LS}_{t_1<t_2}\ in\ \mathbb{LS}_0. \tag{98}$$

Proof. Let $\mathbb{LS}^{t_1<t_2}$ be in the sense of (97) in the semicircular filtration $\mathbb{LS}_0$, i.e.,

$$\mathbb{LS}^{t_1<t_2} = \left(\underset{t_1\leq p\leq t_2}{\star} \mathbb{LS}_p,\ \tau^{t_1<t_2} = \underset{t_1\leq p\leq t_2}{\star} \tau_p^0\right),$$

as a free-probabilistic sub-structure of the semicircular filtration $\mathbb{LS}_0$.

By (94), one can define the embedding map Φ from $\mathbb{LS}^{t_1<t_2}$ into $\mathbb{LS}$, satisfying:

$$\Phi(T) = T, \text{ for all } T \in \mathbb{LS}^{t_1<t_2}.$$

Then, for any $T \in \mathbb{LS}^{t_1<t_2}$, one can get that:

$$\tau^{t_1<t_2}(T) = \tau_{t_1<t_2}(T) = \tau^0(T).$$

Therefore, the Banach $*$-probability space $\mathbb{LS}^{t_1<t_2}$ is free-homomorphic to $\mathbb{LS}_{t_1<t_2}$ in $\mathbb{LS}$. Therefore, the relation (98) holds. □

Remark again that we are not sure if $\mathbb{LS}^{t_1<t_2}$ and $\mathbb{LS}_{t_1<t_2}$ are free-isomorphic, or not, at this moment (see Remark 9.1 above). However, similar to (90), one can verify that all free reduced words T of $\mathbb{LS}^{t_1<t_2}$ have non-zero free distributions embedded in $\mathbb{LS}_{t_1<t_2}$, and conversely, all free reduced words of $\mathbb{LS}_{t_1<t_2}$ having non-zero free distributions are contained in $\mathbb{LS}^{t_1<t_2}$.

Corollary 3. *Let T be a free reduced word of the semicircular $[t_1, t_2]$-filtration $\mathbb{LS}_{t_1<t_2}$ in $\mathcal{Q} \cup \Theta$, and assume that the free distribution of T is non-zero for $\tau_{t_1<t_2}$. Then, T is an element of the Banach $*$-probability space $\mathbb{LS}^{t_1<t_2}$ of (97). The converse holds true.* □

9.3. More about Free-Probabilistic Information on $\mathbb{LS}_{t_1<t_2}$

In this section, we discuss more about free-probabilistic information in semicircular $[t_1, t_2]$-filtrations $\mathbb{LS}_{t_1<t_2}$, for $t_1 < t_2 \in \mathbb{R}$ (which are not necessarily suitable in the sense of **NA 9.2**).

First, let us mention about the following trivial cases.

Proposition 9. *Let $\mathbb{LS}_{t_1<t_2}$ be the semicircular $[t_1, t_2]$-filtration for $t_1 < t_2$ in $\mathbb{R}$.*

If $t_2 < 2$ in $\mathbb{R}$, then all elements of $\mathbb{LS}_{t_1<t_2}$ have the zero free distribution. (99)

Let $t_1, t_2 \geq 2$ in $\mathbb{R}$. If the closed interval $[t_1, t_2]$ does not contain a prime in $\mathbb{R}$, then all elements of $\mathbb{LS}_{t_1<t_2}$ have the zero free distribution. (100)

Proof. The proofs of the statements (99) and (100) are done immediately by (90), (95), (96), and (98). □

Even though the above results (99) and (100), themselves, are trivial, they illustrate how our original (non-zero) free-distributional data on the semicircular filtration $\mathbb{LS}_0$ are distorted under our "unsuitable" truncations.

Now, suppose $t_1 < t_2$ are suitable in $\mathbb{R}$, and:

$$t_1 \to \infty \text{ in } \mathbb{R},$$

in the sense that: t_1 is big "enough" in $\mathbb{R}$. The existence of such suitable intervals $[t_1, t_2]$ in $\mathbb{R}$ is guaranteed by the *prime number theorem* (e.g., [5,6]).

More precisely, let us collect all suitable pairs (t_1, t_2) in $\mathbb{R}^2$, i.e.,

$$\{(t_1, t_2) \in \mathbb{R}^2 : t_1 < t_2 \text{ are suitable in } \mathbb{R}\},$$

and consider its boundary.

First, consider that if $p \to \infty$ in $\mathcal{P}$ (under the usual total ordering on $\mathcal{P}$, inherited by that on $\mathbb{R}$), then:

$$\lim_{p\to\infty \text{ in } \mathcal{P}} p^{2(j+1)} = \begin{cases} 0 & \text{if } j < -1 \\ 1 & \text{if } j = -1 \\ \infty, \text{ undefined} & \text{if } j > -1, \end{cases} \tag{101}$$

for an arbitrarily-fixed $j \in \mathbb{Z}$.

Theorem 9. *Let $(t_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ be monotonically "strictly"-increasing $\mathbb{R}$-sequences, satisfying:*

$$t_n < s_n \text{ are suitable in } \mathbb{R},$$

for all $n \in \mathbb{N}$. By the suitability, there exists at least one prime $p_n \in \mathcal{P}$, such that:

$$t_n \leq p_n \leq s_n, forall n \in \mathbb{N}, \tag{102}$$

where the corresponding $\mathbb{R}$*-sequence* $(p_n)_{n=1}^{\infty}$ *is monotonically increasing.*

Let $Q_{p_n,j}$ *be the corresponding* $p_n^{2(j+1)}$*-semicircular element in the free weighted-semicircular family* $\mathcal{Q}$*, as a free random variable of the semicircular* $[t_n, s_n]$*-filtration* $\mathbb{LS}_{t_n<s_n}$*, where* p_n *are the primes of* (102)*, for all* $n \in \mathbb{N}$*, for any* $j \in \mathbb{Z}$*. Then:*

$$\lim_{n\to\infty}\left(\tau_{t_n<s_n}\left(Q_{p_n,j}^k\right)\right) = \begin{cases} 0 & \text{if } j < -1 \\ \omega_k c_{\frac{k}{2}} & \text{if } j = -1 \\ \infty & \text{if } j > -1, \end{cases} \tag{103}$$

for all $k \in \mathbb{N}$*.*

Proof. Suppose p_n are the primes satisfying (102) for given suitable:

$$t_n < s_n \text{ in } \mathbb{R},$$

in the sense of **NA 9.2**, for all $n \in \mathbb{N}$. Then, for the $p_n^{2(j+1)}$-semicircular elements $Q_{p_n,j} \in \mathcal{Q}$ (in $\mathbb{LS}_0$), one has that:

$$\tau_{t_n<s_n}\left(Q_{p_n,j}^k\right) = \left(\underset{t_n \le q \le s_n \text{ in } \mathcal{P}}{\star} \tau_q^0\right)\left(Q_{p_n,j}^k\right) = \tau_{p_n}^0\left(Q_{p_n,j}^k\right)$$

by (102)

$$= \tau_{p_n,j}^0\left(Q_{p_n,j}^k\right) = \omega_k p_n^{2(j+1)} c_{\frac{k}{2}}, \tag{104}$$

for all $k \in \mathbb{N}$.

Thus, we have that:

$$\lim_{n\to\infty}\left(\tau_{t_n<s_n}\left(Q_{p_n,j}^k\right)\right) = \lim_{n\to\infty}\left(\omega_k p_n^{2(j+1)} c_{\frac{k}{2}}\right)$$

by (104);

$$= \lim_{p\to\infty}\left(\omega_k p^{2(j+1)} c_{\frac{k}{2}}\right) = \left(\omega_k c_{\frac{k}{2}}\right)\left(\lim_{p\to\infty} p^{2(j+1)}\right)$$

$$= \begin{cases} 0 & \text{if } j < -1 \\ \omega_k c_{\frac{k}{2}} & \text{if } j = -1 \\ \infty & \text{if } j > -1, \end{cases}$$

by (101), for all $k \in \mathbb{N}$. Therefore, the estimation (103) holds. □

The above estimation (103) illustrates the asymptotic free-distributional data of our $p^{2(j+1)}$-semicircular elements $\{Q_{p,j} \in \mathcal{Q}\}_{p\in\mathcal{P}}$ (for a fixed $j \in \mathbb{Z}$), under our suitable truncations, as $p \to \infty$ in $\mathcal{P}$.

Corollary 4. *Let* $t_1 < t_2$ *be suitable in* $\mathbb{R}$ *under* ***NA 9.2****,* t_1 *be suitably big (i.e.,* $t_1 \to \infty$*) in* $\mathbb{R}$*, and* $j \le -1$ *be arbitrarily fixed in* $\mathbb{Z}$*. Then, there exists* $t_0 \in \mathbb{R}$*, such that:*

$$\left|\tau_{t_1<t_2}\left(Q_{p,j}^n\right) - t_0\right| \to 0,$$

where:

$$t_0 = \begin{cases} 0 & \text{if } j < -1 \\ \omega_n c_{\frac{n}{2}} & \text{if } j = -1, \end{cases} \tag{105}$$

for all $n \in \mathbb{N}$*.*

Under the same hypothesis, if $j > -1$ *in* $\mathbb{Z}$*, then:*

$$\left|\tau_{t_1<t_2}\left(Q_{p,j}^n\right)\right| \to \infty, \tag{106}$$

for all $n \in \mathbb{N}$.

Proof. The estimations (105) and (106), for suitably big $t_1 \in \mathbb{R}$, are obtained by (103). $\square$

10. Semicircularity of Certain Free Sums in $\mathbb{LS}_{t_1<t_2}$

As in Section 9, we will let $\mathbb{LS}$ be the Banach $*$-subalgebra (72) of the free Adelic filtration $\mathfrak{LG}$, and let $\mathbb{LS}_0$ be the semicircular filtration $(\mathbb{LS}, \tau^0)$ of (80).

Let (A, φ) be an arbitrary topological $*$-probability space and $a \in (A, \varphi)$. We say a free random variable a is a *free sum* in (A, φ), if:

$$a = \sum_{l=1}^{N} x_l, \text{ with } x_l \in (A, \varphi),$$

and the summands $x_1, ..., x_N$ of a are free from each other in (A, φ), for $N \in \mathbb{N} \setminus \{1\}$.

Let $t_1 < t_2$ be suitable in $\mathbb{R}$ in the sense of **NA 9.2**, and let $\mathbb{LS}_{t_1<t_2}$ be the corresponding semicircular $[t_1, t_2]$-filtration. Now, we define free random variables X and Y of $\mathbb{LS}$,

$$X = \sum_{l=1}^{N} Q_{p_l,j_l}^{n_l} and Y = \sum_{l=1}^{N} \Theta_{p_l,j_l}^{n_l}, \tag{107}$$

for $Q_{p_l,j_l} \in \mathcal{Q}$ and $\Theta_{p_l,j_l} \in \Theta$, for all $l = 1, ..., N$, for $N \in \mathbb{N} \setminus \{1\}$.

Remark that, the operator X (or Y) of (107) is a free sum in $\mathbb{LS}$, if and only if the summands $Q_{p_l,j_l}^{n_l}$ (resp., $\Theta_{p_l,j_l}^{n_l}$), which are the free reduced words with their lengths one, are free from each other in $\mathbb{LS}$, if and only if Q_{p_l,j_l} (resp., Θ_{p_l,j_l}) are contained in the mutually-distinct free blocks $\overline{\mathbb{C}[\{Q_{p_l,j_l}\}]}$ of $\mathbb{LS}$ by (74), if and only if the pairs (p_l, j_l) are mutually distinct from each other in the Cartesian product $\mathcal{P} \times \mathbb{Z}$, for all $l = 1, ..., N$. i.e., the given operators X and Y of (107) are free sums in $\mathbb{LS}$, if and only if:

$$(p_{l_1}, j_{l_1}) \neq (p_{l_2}, j_{l_2}) in \mathcal{P} \times \mathbb{Z}, \tag{108}$$

for all $l_1 \neq l_2$ in $\{1, ..., N\}$.

Lemma 4. *Let X and Y be in the sense of* (107) *in the semicircular filtration $\mathbb{LS}_0$. Assume that the pairs (p_l, j_l) are mutually distinct from each other in $\mathcal{P} \times \mathbb{Z}$, for all $l = 1, ..., N$, for $N \in \mathbb{N} \setminus \{1\}$. Then:*

$$\tau^0(X) = \sum_{l=1}^{N} \left(\omega_{n_l} p_l^{2(j_l+1)} c_{\frac{n_l}{2}} \right),$$

and:

$$\tau^0(Y) = \sum_{l=1}^{N} \left(\omega_{n_l} c_{\frac{n_l}{2}} \right). \tag{109}$$

Proof. Let X and Y be given as above in $\mathbb{LS}_0$. By the assumption that the pairs (p_l, j_l) are mutually distinct from each other in $\mathcal{P} \times \mathbb{Z}$, these operators X and Y satisfy the condition (108); equivalently, they are free sums in $\mathbb{LS}_0$.

Therefore, one has that:

$$\begin{aligned} \tau^0(X) &= \sum_{l=1}^{N} \tau^0\left(Q_{p_l,j_l}^{n_l}\right) = \sum_{l=1}^{N} \tau_{p_l,j_l}^0\left(Q_{p_l,j_l}^{n_l}\right) \\ &= \sum_{l=1}^{N} \left(\omega_{n_l} p_l^{2(j_l+1)} c_{\frac{n_l}{2}} \right), \end{aligned}$$

by the $p_l^{2(j_l+1)}$-semicircularity of $Q_{p_l,j_l} \in \mathcal{Q}$, for all $l = 1, ..., N$.

Similarly, one can get that:

$$\tau^0(Y) = \sum_{l=1}^{N} \tau_{p_l,j_l}^0\left(\Theta_{p_l,j_l}^{n_l}\right) = \sum_{l=1}^{N} \left(\omega_{n_l} c_{\frac{n_l}{2}} \right),$$

by the semicircularity of $\Theta_{p_l,j_l} \in \Theta$, for all $l = 1, ..., N$. $\square$

Now, for the operators X and Y of (107), we consider how our truncation distorts the free-distributional data (109).

For a given closed interval $[t_1, t_2]$ in $\mathbb{R}$, where $t_1 < t_2$ are suitable in $\mathbb{R}$, we define:

$$\mathcal{P}_{[t_1,t_2]} = \{p \in \mathcal{P} : t_1 \leq p \leq t_2\} = \mathcal{P} \cap [t_1, t_2],$$

and:

$$\mathcal{P}^c_{[t_1,t_2]} = \mathcal{P} \setminus \mathcal{P}_{[t_1,t_2]}, \tag{110}$$

in $\mathcal{P}$.

By (110), the family $\{\mathcal{P}_{[t_1,t_2]}, \mathcal{P}^c_{[t_1,t_2]}\}$ forms a partition of the set $\mathcal{P}$ of all primes for the fixed interval $[t_1, t_2]$. Of course, if $t_1 < t_2$ are not suitable, then:

$$\mathcal{P}_{[t_1,t_2]} = \varnothing, \text{ and hence, } \mathcal{P} = \mathcal{P}^c_{[t_1,t_2]}.$$

Theorem 10. *Let X and Y be the operators (107), and assume they are free sums in the semicircular filtration $\mathbb{LS}_0$; and let $\mathbb{LS}_{t_1<t_2}$ be the semicircular $[t_1, t_2]$-filtration for suitable $t_1 < t_2$ in $\mathbb{R}$. Then:*

$$\tau_{t_1<t_2}(X) = \sum_{p_l \in \mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)}} \left(\omega_{n_l} p_l^{2(j_l+1)} c_{\frac{n_l}{2}}\right),$$

and:

$$\tau_{t_1<t_2}(Y) = \sum_{p_l \in \mathcal{P}_{[t_1<t_2]:(p_1,\ldots,p_N)}} \left(\omega_{n_l} c_{\frac{n_l}{2}}\right), \tag{111}$$

where:

$$\mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)} = \mathcal{P}_{[t_1,t_2]} \cap \{p_1, \ldots, p_N\} \text{ in } \mathcal{P},$$

where $\mathcal{P}_{[t_1,t_2]}$ is in the sense of (110) in $\mathcal{P}$. Clearly, if $\mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)}$ is empty in $\mathcal{P}$, then the formulas in (111) vanish.

Proof. The proof of (111) is done by (95), (96), (98), and (109). Indeed, if:

$$\mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)} = \mathcal{P}_{[t_1,t_2]} \cap \{p_1, \ldots, p_N\} \text{ in } \mathcal{P},$$

where $\mathcal{P}_{[t_1,t_2]}$ is in the sense of (110), and if:

$$\mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)} \neq \varnothing,$$

then:

$$\tau_{t_1<t_2}(X) = \sum_{p_l \in \mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)}} \tau^0_{p_l,j_l}\left(Q^{n_l}_{p_l,j_l}\right)$$

by (98)

$$= \sum_{p_l \in \mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)}} \left(\omega_{n_l} p_l^{2(j_l+1)} c_{\frac{n_l}{2}}\right),$$

by the $p^{2(j+1)}$-semicircularity of $Q_{p,j} \in \mathcal{Q}$.

Similarly, one can get that:

$$\tau_{t_1<t_2}(Y) = \sum_{p_l \in \mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)}} \left(\omega_{n_l} c_{\frac{n_l}{2}}\right),$$

by the semicircularity of $\Theta_{p,j} \in \Theta$. Therefore, the free-distributional data (111) holds, whenever:

$$\mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)} \neq \varnothing \text{ in } \mathcal{P}.$$

Definitely, if:

$$\mathcal{P}_{[t_1,t_2]:(p_1,\ldots,p_N)} = \varnothing,$$

then:

$$\tau_{t_1<t_2}(X) = O(X) = 0 = O(Y) = \tau_{t_1<t_2}(Y).$$

Therefore, the truncated free-distributional data (111) hold. □

Remark 4. *Let us compare the free-distributional data* (109) *and* (111). *One can check the differences between them dictated by the choices of* $[t_1, t_2]$ *in* $\mathbb{R}$. *Thus, the formula* (111) *also illustrates how our truncations on* $\mathcal{P}$ *distort the original free-probabilistic information on the semicircular filtration* $\mathbb{LS}_0$.

Let q_0 be a fixed prime in $\mathcal{P}$. Choose $t_0 < s_0 \in \mathbb{R}$ such that: (i) these quantities t_0 and s_0 satisfy:

$$t_0 \leq q_0 \leq s_0 \text{ in } \mathbb{R},$$

and (ii) q_0 is the only prime in the closed interval $[t_0, s_0]$ in $\mathbb{R}$.

By the *Archimedean property* on $\mathbb{R}$ (or *the axiom of choice*), the existence of such interval $[t_0, s_0]$, satisfying (i) and (ii) for the fixed prime q_0, is guaranteed; however, the choices of the quantities $t_0 < s_0$ are of course not unique.

Definition 18. *Let* $q_0 \in \mathcal{P}$, *and let* $t < s \in \mathbb{R}$ *be the real numbers satisfying the conditions (i) and (ii) of the above paragraph. Then, the suitable closed interval* $[t, s]$ *is called a* q_0*-neighborhood.*

Depending on prime-neighborhoods, one can obtain the following semicircularity condition on our semicircular truncated-filtrations.

Corollary 5. *Let* $p \in \mathcal{P}$, $[t, s]$ *be a* p*-neighborhood in* $\mathbb{R}$, *and* $\mathbb{LS}_{t<s}$ *be the corresponding semicircular* $[t, s]$*-filtration. If* X *and* Y *are free sums formed by* (107) *in the semicircular filtration* $\mathbb{LS}_0$, *then:*

$$\tau_{t<s}(X) = \textstyle\sum_{l=1}^{N} \delta_{p,p_l} \left(\omega_{n_l} p_l^{2(j_l+1)} c_{\frac{n_l}{2}} \right),$$

and:

$$\tau_{t<s}(Y) = \sum_{l=1}^{N} \delta_{p,p_l} \left(\omega_{n_l} c_{\frac{n_l}{2}} \right), \tag{112}$$

where δ *is the Kronecker delta.*

Proof. The free-distributional data (112) are a special case of (111), under the prime-neighborhood condition. Indeed, in this case,

$$\mathcal{P}_{[t,s]:(p_1,...,p_N)} = \{p\} \cap \{p_1, ..., p_N\} = \begin{cases} \{p\} & \text{or} \\ \varnothing, & \end{cases}$$

where $\mathcal{P}_{[t,s]:(p_1,...,p_N)}$ is in the sense of (111). □

More general to (112), we obtain the following result.

Proposition 10. *Let* $p \in \mathcal{P}$ *and* $[t, s]$ *be a* p*-neighborhood in* $\mathbb{R}$, *and let* $\mathbb{LS}_{t<s}$ *be the corresponding semicircular* $[t, s]$*-filtration. Then, a free random variable* $T \in \mathbb{LS}_{t<s}$ *has its non-zero free distribution, if and only if there exists a non-zero summand* T_0 *of* T, *such that:*

$$T_0 \in \mathbb{LS}_p in \mathbb{LS}_{t<s}, \tag{113}$$

where $\mathbb{LS}_p = \underset{j\in\mathbb{Z}}{\star} \overline{\mathbb{C}[\{\Theta_{p,j}\}]}$ *is a Banach* $*$*-subalgebra* (82) *of* $\mathbb{LS}$.

Proof. By (98), if $T \in \mathbb{LS}_{t<s}$ has its non-zero free distribution, then there exists a non-zero summand T_0 of T which can be a linear combination of free reduced words contained in $\underset{t \leq q \leq s \text{ in } \mathcal{P}}{\star} \mathbb{LS}_q$, and hence,

$$T_0 \in \underset{t \leq q \leq s \text{ in } \mathcal{P}}{\star} \mathbb{LS}_q, \tag{114}$$

where $\mathbb{LS}_q$ are in the sense of (82), for $q \in \mathcal{P}$.

Since $[t, s]$ is a p-neighborhood, the relation (114) is equivalent to:

$$T_0 \in \mathbb{LS}_p. \tag{115}$$

Clearly, the converse holds true as well, by (98).

Therefore, a free random variable $T \in \mathbb{LS}_{t<s}$ has its non-zero free distribution, if and only if T contains its non-zero summand $T_0 \in \mathbb{LS}_p$, by (115); equivalently, the statement (113) holds true. □

By (112) and (113), we obtain the following interesting result.

Theorem 11. *Let $X_1 = \sum_{l=1}^{N} Q_{p_l, j_l}$ and $Y_1 = \sum_{l=1}^{N} \Theta_{p_l, j_l}$ be in the sense of* (107) *in the semicircular filtration $\mathbb{LS}_0$, and assume that $(p_l,\ j_l)$ are mutually distinct in $\mathcal{P} \times \mathbb{Z}$, for $l = 1, ..., N$, for $N \in \mathbb{N} \setminus \{1\}$. Suppose we fix:*

$$p_{l_0} \in \{p_1, ..., p_N\},$$

and take a p_{l_0}-neighborhood $[t_0, s_0]$ in $\mathbb{R}$. Then:

$$X_1 is p_{l_0}^{2(j_{l_0}+1)} - semicircular\ in\ \mathbb{LS}_{t_0<s_0}, \tag{116}$$

$$Y_1\ is\ semicircular\ in\ \mathbb{LS}_{t_0<s_0}, \tag{117}$$

where $\mathbb{LS}_{t_0<s_0}$ is the semicircular $[t_0, s_0]$-filtration.

Proof. Let X_1 and Y_1 be given as above in $\mathbb{LS}$, and fix $p_{l_0} \in \{p_1, ..., p_N\}$. Note that, by the assumption, these operators X_1 and Y_1 form free sums in the semicircular filtration $\mathbb{LS}_0$, having N-many summands. Note also that they are self-adjoint in $\mathbb{LS}$ by the self-adjointness of their summands.

By (113), if an operator T has its non-zero free distribution in the semicircular $[t_0, s_0]$-filtration $\mathbb{LS}_{t_0<s_0}$, where $[t_0, s_0]$ is a p_{l_0}-neighborhood in $\mathbb{R}$, then it must have its non-zero summand T_0,

$$T_0 \in \mathbb{LS}_{p_{l_0}} \text{ in } \mathbb{LS}_{t_0<s_0}.$$

By the very construction of X_1 and Y_1, they contain their summands,

$$Q_{p_{l_0}, j_{l_0}},\ \Theta_{p_{l_0}, j_{l_0}} \in \mathbb{LS}_{p_{l_0}} \text{ in } \mathbb{LS}_{t_0<s_0}.$$

Consider now that:

$$\begin{aligned} X_1^n &= \sum_{(i_1,...,i_n)\in\{1,...,N\}^n} \left(\prod_{k=1}^{N} Q_{p_{i_k}, j_{i_k}} \right) \\ &= Q_{p_{l_0}, j_{l_0}}^n + \sum_{(i_1,...,i_n)\neq(l_0,...,l_0)} \left(\prod_{k=1}^{N} Q_{p_{k_l}, j_{k_l}} \right) \\ &= Q_{p_{l_0}, j_{l_0}}^n + [\text{Rest Terms}], \end{aligned}$$

and similarly,

$$Y_1^n = \Theta_{p_{l_0}, j_{l_0}}^n + [RestTerms]', \tag{118}$$

for all $n \in \mathbb{N}$.

It is not difficult to check that:

$$\tau_{t_0<s_0}\left([\text{Rest Terms}]\right) = 0 = \tau_{t_0<s_0}\left([\text{Rest Terms}]'\right), \tag{119}$$

by (98) and (113), where [Rest Terms], and [Rest Terms]′ are from (118).

Therefore, one obtains that:

$$\begin{aligned}\tau_{t_0<s_0}\left(X_1^n\right) &= \tau_{t_0<s_0}\left(Q_{p_{l_0},j_{l_0}}^n\right) = \tau_{p_{l_0}}^0\left(Q_{p_{l_0},j_{l_0}}^n\right) \\ &= \tau_{p_{l_0},j_{l_0}}^0\left(Q_{p_{l_0},j_{l_0}}^n\right) = \omega_n p_{l_0}^{2(j_{l_0}+1)} c_{\frac{n}{2}},\end{aligned}$$

and, similarly,

$$\tau_{t_0<s_0}\left(Y_1^n\right) = \tau_{p_{l_0},j_{l_0}}^0\left(\Theta_{p_{l_0},j_{l_0}}^n\right) = \omega_n c_{\frac{n}{2}}, \tag{120}$$

for all $n \in \mathbb{N}$, by (119).

Therefore, the free sum $X_1 \in \mathbb{LS}$ is $p_{l_0}^{2(j_{l_0}+1)}$-semicircular in $\mathbb{LS}_{t_0<s_0}$; and the free sum $Y_1 \in \mathbb{LS}$ is semicircular in $\mathbb{LS}_{t_0<s_0}$, by (120). Therefore, the statements (116) and (117) hold true. $\square$

The above theorem shows that, if there is a free sum T in the semicircular filtration $\mathbb{LS}_0$ and if we "nicely" truncate the linear functional τ^0 on $\mathbb{LS}$, then one can focus on the non-zero summand T_0 of T, whose the free distribution not only determines the truncated free distribution of T, but also follows the (weighted-)semicircular law.

11. Applications of Prime-Neighborhoods

In Section 9, we considered the semicircular truncated-filtrations $\mathbb{LS}_{t<s}$ for $t < s \in \mathbb{R}$ and studied how $[t,s]$-truncations on $\mathcal{P}$ affect, or distort, the original free-distributional data on the semicircular filtration $\mathbb{LS}_0 = \left(\mathbb{LS},\ \tau^0\right)$. As a special case, in Section 10, we introduced p-neighborhoods for primes p and considered corresponding truncated free distributions on $\mathbb{LS}$.

In this section, by using prime-neighborhoods, we provide a completely "new" model of truncated free probability on $\mathbb{LS}$ and study how the original free-distributional data on $\mathbb{LS}_0$ are distorted under this new truncation model.

Let us now regard the set $\mathcal{P}$ of all primes as a *totally ordered set* (a TOset),

$$\mathcal{P} = \{q_1 < q_2 < q_3 < q_4 < q_5 < \ldots\} \tag{121}$$

under the usual inequality $(\leq)$ on $\mathcal{P}$, i.e.,

$$q_1 = 2, q_2 = 3, q_3 = 5, q_4 = 7, q_5 = 11, \text{etc.}$$

For each $q_k \in \mathcal{P}$ of (121), determine a q_k-neighborhood B_k

$$B_k \overset{denote}{=} [t_k, s_k] in \mathbb{R}, \tag{122}$$

for all $k \in \mathbb{N}$.

Let τ_{B_k} be our truncated linear functionals $\tau_{t_k<s_k}$ of (92) on the Banach $*$-algebra $\mathbb{LS}$, i.e.,

$$\tau_{B_k} = \tau_{t_k<s_k}, for\, all k \in \mathbb{N}. \tag{123}$$

Then, by the truncated linear functionals of (123), one can have the corresponding semicircular B_k-filtrations,

$$\mathbb{LS}_{B_k} = \mathbb{LS}_{t_k<s_k} = \left(\mathbb{LS},\ \tau_{B_k}\right), \tag{124}$$

for all $k \in \mathbb{N}$.

We now focus on the system:

$$\mathbf{T} = \{\tau_{B_k}\}_{k=1}^{\infty} \tag{125}$$

of q_k-neighborhood-truncated linear functionals (123) for all $k \in \mathbb{N}$.

Let F be a "finite" subset of the TOset $\mathcal{P}$ of (121), and for such a set F, define a new linear functional τ_F on $\mathbb{LS}$ induced by the system $\mathbf{T}$ of (125), by:

$$\tau_F = \sum_{q_k \in F} \tau_{B_k} on \mathbb{LS}. \tag{126}$$

Before proceeding, let us consider the following result obtained from (113).

Lemma 5. *Let $p \in \mathcal{P}$ and $[t, s]$ be a p-neighborhood in $\mathbb{R}$, and let $\mathbb{LS}_{t<s}$ be the semicircular $[t, s]$-filtration. Let $\tau_p^0 = \underset{j \in \mathbb{Z}}{\star} \tau_{p,j}^0$ be the linear functional (81) on the Banach $*$-subalgebra $\mathbb{LS}_p$ of (82) in the semicircular filtration $\mathbb{LS}_0$. Define a linear functional τ^p on the Banach $*$-algebra $\mathbb{LS}$ by:*

$$\tau^p(T) \overset{def}{=} \begin{cases} \tau_p^0(T) & \text{if } T \in \mathbb{LS}_p \text{ in } \mathbb{LS} \\ O(T) = 0 & \text{otherwise,} \end{cases}$$

for all $T \in \mathbb{LS}$. Then, the Banach $*$-probability space $(\mathbb{LS}, \tau^p)$ is free-isomorphic to $\mathbb{LS}_{t<s}$, i.e.,

$$[t, s] \text{ is a } p - neighborhood \Rightarrow \mathbb{LS}_{t<s} \text{ and } (\mathbb{LS}, \tau^p) \text{ are } free - isomorphic. \tag{127}$$

Proof. Under the hypothesis, it is not hard to check:

$$\tau_{t<s} = \tau^p \text{ on } \mathbb{LS}.$$

Therefore, the identity map on $\mathbb{LS}$ becomes a free-isomorphism from $\mathbb{LS}_{t<s}$ onto $(\mathbb{LS}, \tau^p)$. □

If a finite subset F is a singleton subset of $\mathcal{P}$, then the free probability on $\mathbb{LS}$ determined by the corresponding linear functional τ_F of (126) is already considered in Section 10 and in (127). Therefore, we now restrict our interests to the cases where finite subsets F have more than one element in $\mathcal{P}$.

Lemma 6. *Let F be a finite subset of the TOset $\mathcal{P}$ of (121), and let τ_F be the corresponding linear functional (126) on $\mathbb{LS}$. Then:*

$$\tau_F = \sum_{q_k \in F} \tau^{q_k} on \mathbb{LS}, \tag{128}$$

where τ^{q_k} are in the sense of (127).

Proof. The proof of (128) is done by (126) and (127) because:

$$(\mathbb{LS}, \tau^{q_k}) \text{ and } \mathbb{LS}_{B_k}$$

are free-isomorphic for all $q_k \in F$. Therefore, the linear functional τ_F of (126) satisfies that:

$$\tau_F = \sum_{q_k \in F} \tau_{B_k} = \sum_{q_k \in F} \tau^{q_k} \text{ on } \mathbb{LS}.$$

□

By (113), (116), (117), and (128), one obtains the following result.

Theorem 12. *Let $T = \prod_{l=1}^{N} Q_{p_l, j_l}^{n_l}$ or $S = \prod_{l=1}^{N} \Theta_{p_l, j_l}^{n_l}$ be a free reduced word of $\mathbb{LS}$ with its length-N, for $N \in \mathbb{N}$. If:*

$$F \cap \{p_1, ..., p_N\} = \varnothing \text{ in } \mathcal{P},$$

then:

$$\tau_F(T) = 0 = \tau_F(S). \tag{129}$$

While, if $F \cap \{p_1, ..., p_N\} \neq \varnothing$ in $\mathcal{P}$, then:

$$\tau_F(T) = \sum_{q \in F \cap \{p_1, ..., p_N\}} \left(\omega_n q^{2(j+1)} c_{\frac{n}{2}} \right),$$

respectively,

$$\tau_F(S) = |F \cap \{p_1, ..., p_N\}| \left(\omega_n c_{\frac{n}{2}} \right), \tag{130}$$

where $|X|$ mean the cardinalities of sets X.

Proof. Let T and S be given free reduced words with their length-N in $\mathbb{LS}$, for $N \in \mathbb{N}$. If:

$$F \cap \{p_1, ..., p_N\} = \varnothing \text{ in } \mathcal{P},$$

then we obtain the formula (129) by (127) and (128). Indeed,

$$\tau_F\,(T) = \sum_{q \in F} \tau^q\,(T) = 0 = \sum_{q \in F} \tau^q(S) = \tau_F(S).$$

Now, assume that:

$$F \cap \{p_1, ..., p_N\} = \{p_{i_1}, ..., p_{i_k}\} \text{ in } \mathcal{P},$$

for some $k \in \mathbb{N}$, such that $1 \leq k \leq N$. Then:

$$\tau_F(T) = \left(\sum_{l=1}^{k} \tau^0_{p_{i_l}} \right)(T) = \sum_{l=1}^{k} \tau^0_{p_{i_l}}(T)$$

by (126) and (128)

$$\begin{aligned} &= \textstyle\sum_{l=1}^{k} \tau^0_{p_{i_l}} \left(Q^{n_{i_l}}_{p_{i_l}, j_{i_l}} \right) = \textstyle\sum_{l=1}^{k} \tau^0_{p_{i_l}, j_{i_l}} \left(Q^{n_{i_l}}_{p_{i_l}, j_{i_l}} \right) \\ &= \textstyle\sum_{l=1}^{k} \left(\omega_{n_l} p_{i_l}^{2(j_{i_l}+1)} c_{\frac{n_l}{2}} \right). \end{aligned} \tag{131}$$

Similarly,

$$\tau_F(S) = \textstyle\sum_{l=1}^{k} \left(\omega_{n_l} c_{\frac{n_l}{2}} \right) = k \cdot \left(\omega_{n_l} c_{\frac{n_l}{2}} \right).$$

Therefore, the free-distributional data (130) hold. □

The above free-distributional data (129) and (130) characterize the free-probabilistic information on Banach $*$-probability spaces

$$(\mathbb{LS},\ \tau_F),$$

for any finite subsets F of $\mathcal{P}$.

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