k-Rainbow Domination Number of $P_3 \square P_n$

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Abstract: Let $k$ be a positive integer, and set $[k] := \{1, 2, \ldots, k\}$. For a graph $G$, a $k$-rainbow dominating function (or kRDF) of $G$ is a mapping $f : V(G) \to 2^{|k|}$ in such a way that, for any vertex $v \in V(G)$ with the empty set under $f$, the condition $\bigcup_{u \in N_G(v)} f(u) = |k|$ always holds, where $N_G(v)$ is the open neighborhood of $v$. The weight of $kRDF$ $f$ of $G$ is the summation of values of all vertices under $f$. The $k$-rainbow domination number of $G$, denoted by $\gamma_{rk}(G)$, is the minimum weight of a $kRDF$ of $G$. In this paper, we obtain the $k$-rainbow domination number of grid $P_3 \square P_n$ for $k \in \{2, 3, 4\}$.

Keywords: $k$-rainbow dominating function; $k$-rainbow domination number; grids

1. Introduction

For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For a vertex $v \in V(G)$, the open neighborhood of $v$, denoted by $N_G(v)$, is the set $\{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is defined by $d_G(v) = |N_G(v)|$. We let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of a graph $G$, respectively.

Let $k$ be a positive integer, and $[k] := \{1, 2, \ldots, k\}$. For a graph $G$, a $k$-rainbow dominating function (or kRDF) of $G$ is a mapping $f : V(G) \to 2^{|k|}$ in such a way that for any vertex $v \in V(G)$ with the empty set under $f$, the condition $\bigcup_{u \in N_G(v)} f(u) = |k|$ always holds. The weight of a $kRDF$ $f$ of $G$ is the value $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The $k$-rainbow domination number of $G$, denoted by $\gamma_{rk}(G)$, is the minimum weight of a $kRDF$ of $G$. A $kRDF$ $f$ of $G$ is a $\gamma_{rk}$-function if $\omega(f) = \gamma_{rk}(G)$. The $k$-rainbow domination number was introduced by Brešar, Henning, and Rall [1] was studied by several authors (see, for example [2–15]).

For graphs $F$ and $G$, we let $F \square G$ denote the Cartesian product of $F$ and $G$. Vizing [16] conjectured that for arbitrary graphs $F$ and $G$, $\gamma(F \square G) \geq \gamma(F)\gamma(G)$. This conjecture is still open, and the domination number or its related invariants of $F \square G$ are extensively studied with the motivation from Vizing’s conjecture.

Concerning the $k$-rainbow domination number of $F \square G$, one problem naturally arises: Given two graphs $F$ and $G$ under some conditions, determine $\gamma_{rk}(F \square G)$ for all $k$. In [3], the authors determined $\gamma_{rk}(P_2 \square P_n)$ for $k = 3, 4, 5$.
In this paper, we examine grid graphs $P_m \Box P_n$, and determine the value $\gamma_{rk}(P_m \Box P_n)$ for $k \in \{2, 3, 4\}$ and all $n$, where $P_m$ is the path of order $m$.

2. 2-Rainbow Domination Number of $P_m \Box P_n$

We write $V(P_m \Box P_n) = \{v_i, u_i, w_i \mid 0 \leq i \leq n - 1\}$ and let $E(P_m \Box P_n) = \{v_iu_i, u_iw_i \mid 0 \leq i \leq n - 1\} \cup \{v_iv_{i+1}, u_iu_{i+1}, w_iw_{i+1} \mid 0 \leq i \leq n - 1\}$ (see Figure 1). A 2RDF $f$ is given in three lines, where in the first line there are values of the function $f$ for vertices $\{v_0, v_1, \ldots, v_{n-1}\}$, in the second line of the vertices $\{u_0, u_1, \ldots, u_{n-1}\}$, and in the third line of the vertices $\{w_0, w_1, \ldots, w_{n-1}\}$ (see Figure 2).

Furthermore, we use 0, 1, 2, 3 to encode the sets $\emptyset, \{1\}, \{2\}, \{1, 2\}$.

![Figure 1. The grid graph $P_3 \Box P_{16}$](image)

![Figure 2. A 2RDF of $P_3 \Box P_n$](image)

To provide a complete answer, we need the following fact that can easily be proved as an exercise.

**Fact 1.** $\gamma_{r2}(P_3 \Box P_3) = 4$, $\gamma_{r2}(P_3 \Box P_4) = 6$, $\gamma_{r2}(P_3 \Box P_5) = 7$, $\gamma_{r2}(P_3 \Box P_6) = 8$, $\gamma_{r2}(P_3 \Box P_7) = 10$.

**Theorem 1.** For $n \geq 8$, $\gamma_{r2}(P_3 \Box P_n) = \left\lfloor \frac{5n + 3}{4} \right\rfloor$.

**Proof.** First, we present constructions of a 2RDF of $P_m \Box P_n$ of the desired weight.

1. $n \equiv 0 \pmod{8}$:
   
   0200 30010200...30010200 3001
   1010 00200010...00200010 0020
   0202 01003002...01003002 0101

2. $n \equiv 1 \pmod{8}$:
   
   0200 30010200...30010200 30010
   1010 00200010...00200010 00202
   0202 01003002...01003002 01010

3. $n \equiv 2 \pmod{8}$:
   
   0200 30010200...30010200 300101
   1010 00200010...00200010 002020
   0202 01003002...01003002 010101

4. $n \equiv 3 \pmod{8}$:
   
   0200 30010200...30010200 3001001
   1010 00200010...00200010 00202020
   0202 01003002...01003002 0101001
5. \( n \equiv 4 \pmod{8} \):
\[
\begin{align*}
0200 & \quad 30010200 \ldots 30010200 \\
1010 & \quad 00200010 \ldots 00200010 \\
0202 & \quad 01003002 \ldots 01003002 \\
01010010
\end{align*}
\]
6. \( n \equiv 5 \pmod{8} \):
\[
\begin{align*}
0200 & \quad 30010200 \ldots 30010200 \\
1010 & \quad 00200010 \ldots 00200010 \\
0202 & \quad 01003002 \ldots 01003002 \\
01003002
\end{align*}
\]
7. \( n \equiv 6 \pmod{8} \):
\[
\begin{align*}
0200 & \quad 30010200 \ldots 30010200 \\
1010 & \quad 00200010 \ldots 00200010 \\
0202 & \quad 01003002 \ldots 01003002 \\
0102
\end{align*}
\]
8. \( n \equiv 7 \pmod{8} \):
\[
\begin{align*}
0200 & \quad 30010200 \ldots 30010200 \\
1010 & \quad 00200010 \ldots 00200010 \\
0202 & \quad 01003002 \ldots 01003002
\end{align*}
\]

To show that these are also lower bounds, we prove there is a \( \gamma_2(P_3 \Box P_n) \)-function, \( f \) such that for every \( 0 \leq i < n-1 \), \( \omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \geq 1 \). Let \( n \geq 8 \) and \( f \) be a \( \gamma_2(P_3 \Box P_n) \)-function such that the cardinality of \( S = \{ i \mid 0 \leq i < n-1 \text{ and } \omega(f_i) = 0 \} \) is as small as possible. We claim that \( |S| = 0 \). Suppose, to the contrary, that \( |S| \geq 1 \) and let \( s \) be the smallest positive integer for which \( \omega(f_s) = 0 \). Then, \( \omega(f_{s-1}) + \omega(f_{s+1}) \geq 6 \). Then, we consider the following cases.

**Case 1.** \( s = 1 \) (the case \( s = n - 1 \) is similar).

Then, we have \( f(v_1) = f(u_1) = f(w_1) = \{1,2\} \) and the function \( g \) defined by \( g(u_0) = \{1\}, g(v_1) = g(w_1) = \{2\}, g(u_2) = f(u_2) \cup \{1\}, g(v_0) = g(w_0) = g(u_1) = \emptyset \) and \( g(x) = f(x) \) otherwise, is a \( 2\text{ RDF} \) of \( P_3 \Box P_n \) of weight at most \( \omega(f) \), which contradicts the choice of \( f \).

**Case 2.** \( s = 1 \) (\( s = n - 2 \) is similar).

Then, \( \omega(f_0) + \omega(f_2) \geq 6 \) and the function \( g \) defined by \( g(u_0) = g(u_2) = \{1\}, g(v_1) = g(w_1) = \{2\}, g(v_2) = f(v_2) \cup \{2\}, g(v_3) = f(v_3) \cup \{2\}, g(v_0) = g(w_0) = g(u_1) = g(v_2) = g(w_2) = \emptyset \) and \( g(x) = f(x) \) otherwise, is an \( 2\text{ RDF} \) of \( P_3 \Box P_n \) of weight at most \( \omega(f) \), which contradicts the choice of \( f \).

**Case 3.** \( 2 \leq s \leq n-3 \).

Since \( \omega(f_{s-2}) \geq 1 \), then \( |f(v_{s-2})| + |f(u_{s-2})| + |f(w_{s-2})| \geq 1 \). First, let \( |f(u_{s-2})| \geq 1 \). We may assume that \( \{1\} \subseteq f(u_{s-2}) \). It is easy to see that the function \( g \) defined by \( g(v_{s-1}) = g(v_{s-1}) = g(w_{s-1}) = \{2\}, g(u_0) = \{1\}, g(u_{s-2}) = f(u_{s-2}) \cup \{1\}, g(u_{s-1}) = g(v_3) = g(v_3) = g(w_3) = g(u_{s+1}) = \emptyset \) and \( g(x) = f(x) \) otherwise, is an \( 2\text{ RDF} \) of \( P_3 \Box P_n \) of weight at most \( \omega(f) \), which contradicts the choice of \( f \). Now, let \( |f(v_{s-2})| \geq 1 \) (\( |f(v_{s-2})| \geq 1 \) is similar). We may assume that \( \{1\} \subseteq f(w_{s-2}) \). Hence, the function \( g \) defined by \( g(v_{s-2}) = f(v_{s-2}) \cup \{1\}, g(v_{s+1}) = g(u_{s-2}) = g(w_{s-1}) = \{2\}, g(u_s) = \{1\}, g(u_{s-2}) = f(u_{s-2}) \cup \{1\}, g(u_{s-1}) = g(v_2) = g(w_2) = g(u_{s+1}) = \emptyset \) and \( g(x) = f(x) \) otherwise, is an \( 2\text{ RDF} \) of \( P_3 \Box P_n \) of weight \( \omega(f) \), which is contradicting the choice of \( f \). Therefore, \( |S| = 0 \).

We can see that for every \( 0 \leq i \leq n - 2 \), if \( \omega(f_i) = \omega(f_{i+1}) = \omega(f_{i+2}) = 1 \), then \( \omega(f_{i-1}) + \omega(f_{i+3}) > 1 \). In addition, there is the function \( f \) such that, if \( \omega(f_0) = 1 \) (\( \omega(f_{n-1}) = 1 \) is similar), then \( \omega(f_1) > 1 \) and \( \omega(f_1) + \omega(f_2) + \omega(f_3) + \omega(f_4) \geq 6 \) and if \( \omega(f_0) = 2 \) (\( \omega(f_{n-1}) = 2 \) is similar), then \( \omega(f_0) + \omega(f_1) + \omega(f_2) + \omega(f_3) \geq 6 \).
If \( \omega(f_0) = 1 \) and \( \omega(f_{n-1}) = 1 \), then

\[
4\omega(f) = 4 \sum_{0 \leq i \leq n-1} \omega(f_i)
= [3\omega(f_0) + 2\omega(f_1) + \omega(f_2)] + [3\omega(f_{n-1}) + 2\omega(f_{n-2}) + \omega(f_{n-3})]
+ \sum_{i \in \{0, \ldots, n-4\} - \{1, n-3\}} (\omega(f_i) + \omega(f_{i+1}) + \omega(f_{i+2}) + \omega(f_{i+3}))
+ [\omega(f_1) + \omega(f_2) + \omega(f_3) + \omega(f_4)] + [\omega(f_{n-5}) + \omega(f_{n-4}) + \omega(f_{n-3}) + \omega(f_{n-2})]
\geq 8 + 8 + 5(n - 5) + 12
= 5(n - 3) + 18.
\]

If \( \omega(f_0) = 1 \) and \( \omega(f_{n-1}) = 2 \), then

\[
4\omega(f) = 4 \sum_{0 \leq i \leq n-1} \omega(f_i)
= [3\omega(f_0) + 2\omega(f_1) + \omega(f_2)] + [3\omega(f_{n-1}) + 2\omega(f_{n-2}) + \omega(f_{n-3})]
+ \sum_{i \in \{0, \ldots, n-4\} - \{1\}} (\omega(f_i) + \omega(f_{i+1}) + \omega(f_{i+2}) + \omega(f_{i+3}))
+ [\omega(f_1) + \omega(f_2) + \omega(f_3) + \omega(f_4)]
\geq 8 + 9 + 5(n - 4) + 6
= 5(n - 3) + 18.
\]

If \( \omega(f_0) = 2 \) and \( \omega(f_{n-1}) = 2 \), then

\[
4\omega(f) = 4 \sum_{0 \leq i \leq n-1} \omega(f_i)
= [3\omega(f_0) + 2\omega(f_1) + \omega(f_2)] + [3\omega(f_{n-1}) + 2\omega(f_{n-2}) + \omega(f_{n-3})]
+ \sum_{i \in \{0, \ldots, n-4\} - \{1\}} [\omega(f_i) + \omega(f_{i+1}) + \omega(f_{i+2}) + \omega(f_{i+3})]
+ [\omega(f_1) + \omega(f_2) + \omega(f_3) + \omega(f_4)]
\geq 9 + 9 + 5(n - 3)
= 5(n - 3) + 18.
\]

Thus, \( \omega(f) = \lceil \frac{5n+3}{2} \rceil \). \( \square \)

3. 3-Rainbow Domination Number of \( P_3 \Box P_n \)

As in the previous section, a 3RDF is given in three lines and we use 0, 1, 2, 3 to encode the sets \( \emptyset, \{1\}, \{2\}, \{3\} \).

To provide a complete answer, we need the following fact.

Fact 2. \( \gamma_{3\mathcal{R}}(P_3 \Box P_3) = 5 \), \( \gamma_{3\mathcal{R}}(P_3 \Box P_4) = 8 \).

Theorem 2. For \( n \geq 5 \),

\[
\gamma_{3\mathcal{R}}(P_3 \Box P_n) = \begin{cases} 
(3n + 1)/2 & \text{if } n \equiv 1 \pmod{2}, \\
(3n + 2)/2 & \text{if } n \equiv 0 \pmod{2}, 
\end{cases}
\]
Proof. First, we present constructions of a $3\text{RDF}$ of $P_5 \square P_n$ of the desired weight.

1. $n \equiv 0 \pmod{4}$:
   
   $2010 \ldots 2010$ 2201
   $0303 \ldots 0303$ 0030
   $1020 \ldots 1020$ 1102

2. $n \equiv 1 \pmod{4}$:
   
   $2010 \ldots 2010$ 2
   $0303 \ldots 0303$ 0
   $1020 \ldots 1020$ 1

3. $n \equiv 2 \pmod{4}$:
   
   $2010 \ldots 2010$ 201201
   $0303 \ldots 0303$ 030030
   $1020 \ldots 1020$ 102102

4. $n \equiv 3 \pmod{4}$:
   
   $2010 \ldots 2010$ 201
   $0303 \ldots 0303$ 30
   $1020 \ldots 1020$ 102

To show that these are also lower bounds, we prove there is a $\gamma_{3}(P_5 \square P_n)$-function, $f$ that satisfies the following conditions:

1. For every $0 \leq i \leq n - 1$, $\omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \geq 1$,
2. For every $1 \leq i \leq n - 2$, if $\omega(f_i) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \geq 4$. In particular, if $\omega(f_i) = 1$, then $(\omega(f_{i-1}) + \omega(f_i)) + (\omega(f_i) + \omega(f_{i+1})) \geq 6$,
3. $\omega(f_0) \geq 2$ and $\omega(f_{n-1}) \geq 2$.

First, we show that for every $\gamma_{3}(P_5 \square P_n)$-function $f$, $\omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \geq 1$ when $0 \leq i \leq n - 1$. Let $n \geq 5$ and $f$ be a $\gamma_{3}(P_5 \square P_n)$-function and $S = \{i \mid 0 \leq i \leq n - 1$ and $\omega(f_i) = 0\}$. We claim that $|S| = 0$. Assume to the contrary that $|S| \geq 1$. Then, we consider the following cases.

**Case 1.** $0 \in S$ (the case $n - 1 \in S$ is similar).

Then, we have $f(v_1) = f(u_1) = f(w_1) = \{1, 2, 3\}$ and it is easy to see that the function $g$ defined by $g(v_1) = \{1\}$, $g(u_1) = \{3\}$, $g(w_0) = \{2\}$, $g(v_2) = f(v_2) \cup \{2\}$, $g(w_2) = f(w_2) \cup \{1\}$, $g(u_0) = g(v_1) = g(w_1) = \emptyset$ and $g(x) = f(x)$ otherwise, is an $3\text{RDF}$ of $P_5 \square P_n$ of weight less than $\omega(f)$, which is a contradiction.

Let $s$ be the smallest positive integer for which $\omega(f_s) = 0$. Then, $s \geq 1$ and $\omega(f_{s-1}) + \omega(f_{s+1}) \geq 9$.

**Case 2.** $s = 1$ ($s = n - 2$ is similar).

Then, the function $g$ defined by $g(v_0) = g(u_0) = g(w_0) = \{1\}$, $g(v_1) = \{2\}$, $g(w_1) = \{1\}$, $g(v_2) = \{3\}$, $g(v_3) = f(v_3) \cup \{1\}$, $g(w_3) = f(w_3) \cup \{2\}$, $g(u_1) = g(v_2) = g(w_2) = \emptyset$ and $g(x) = f(x)$ otherwise, is an $3\text{RDF}$ of $P_5 \square P_n$ of weight less than $\omega(f)$, which is a contradiction.

**Case 3.** $2 \leq s \leq n - 3$.

The function $g$ defined by $g(u_{s-1}) = g(u_{s+1}) = \{3\}$, $g(v_s) = \{2\}$, $g(w_s) = \{1\}$, $g(v_{s-2}) = f(v_{s-2}) \cup \{1\}$, $g(w_{s-2}) = f(w_{s-2}) \cup \{2\}$, $g(v_{s+2}) = f(v_{s+2}) \cup \{1\}$, $g(w_{s+2}) = f(w_{s+2}) \cup \{2\}$, $g(v_{s-1}) = g(v_{s+1}) = g(u_s) = g(w_{s-1}) = g(w_{s+1}) = \emptyset$ and $g(x) = f(x)$ otherwise, is an $3\text{RDF}$ of $P_5 \square P_n$ of weight less than $\omega(f)$, which is a contradiction. Therefore, $|S| = 0$. 

Now, let $f$ be a $\gamma_{T}(P_{3} \Box P_{n})$-function. It is easy to see that, if $\omega(f_{i}) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \geq 4$ when $1 \leq i \leq n - 2$.

Finally, we show that there is $\gamma_{T}(P_{3} \Box P_{n})$-function $f$ such that $\omega(f_{0}) \geq 2$ ($\omega(f_{n-1}) \geq 2$ is similar). Let $f$ be a $\gamma_{T}(P_{3} \Box P_{n})$-function such that $\omega(f_{0}) = 1$. If $|f(v_{0})| = 1$ ($|f(w_{0})| = 1$ is similar), then $|f(w_{0})| = |f(u_{1})| \geq 2$ and $|f(w_{1})| = 3$. We may assume that $\{1,2\} \subseteq f(u_{1})$. It is easy to see that the function $g$ defined by $g(w_{0}) = \{3\}, g(w_{2}) = \{3\}, g(w_{1}) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 3RDF of $P_{3} \Box P_{n}$ of weight less than $\omega(f)$, which is a contradiction. Now, let $|f(u_{0})| = 1$.

Then, $|f(v_{0})| = |f(v_{1})| = 0, |f(v_{2})| \geq 2$ and $|f(w_{1})| \geq 2$. It is easy to see that the function $g$ defined by $g(w_{0}) = \{1\}, g(w_{2}) = \{2\}, g(u_{1}) = \{3\}, g(v_{2}) = f(v_{2}) \cup \{1\}, g(w_{2}) = f(w_{2}) \cup \{2\}$, $g(u_{1}) = g(u_{2}) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 3RDF of $P_{3} \Box P_{n}$ of weight $\omega(f)$.

Hence, there is a $\gamma_{T}(P_{3} \Box P_{n})$-function, $f$ that satisfies the following conditions:

1. For every $0 \leq i \leq n - 1$, $\omega(f_{i}) \geq 1$;
2. For every $1 \leq i \leq n - 2$, if $\omega(f_{i}) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \geq 4$; and
3. $\omega(f_{0}) \geq 2$ and $\omega(f_{n-1}) \geq 2$.

If $n$ is odd, then

$$2\omega(f) = 2 \sum_{0 \leq i \leq n-1} \omega(f_{i}) = \omega(f_{0}) + \omega(f_{n-1}) + \sum_{0 \leq i \leq n-2} (\omega(f_{i}) + \omega(f_{i+1})) \geq 4 + 3(n - 1).$$

Then, $\omega(f) = \frac{3n+1}{2}$ when $n$ is odd. Now, let $n$ is even. Then, there is $s \neq n - 1$ such that $\omega(f_{s}) + \omega(f_{s+1}) \geq 4$. Hence,

$$2\omega(f) = 2 \sum_{0 \leq i \leq n-1} \omega(f_{i}) = \omega(f_{0}) + \omega(f_{n-1}) + \omega(f_{n-1}) + \sum_{0 \leq i \leq n-2, i \neq s} (\omega(f_{i}) + \omega(f_{i+1})) \geq 8 + 3(n - 2).$$

Therefore, $\omega(f) = \frac{3n+2}{2}$ when $n$ is even. $\square$

4. 4-Rainbow Domination Number of $P_{3} \Box P_{n}$

As above, a 4RDF is given in three lines and we use 0, 1, 2, 5 to encode the sets $\emptyset, \{1\}, \{2\}, \{3,4\}$. To provide a complete answer, we need the following fact.

**Fact 3.** $\gamma_{T4}(P_{3} \Box P_{3}) = 6, \gamma_{T4}(P_{3} \Box P_{4}) = 9.$

**Theorem 3.** For $n \geq 5$, $\gamma_{T4}(P_{3} \Box P_{n}) = 2n$.

**Proof.** First, we show that $\gamma_{T4}(P_{3} \Box P_{n}) \leq 2n$. To do this, we present constructions of a 4RDF of $P_{3} \Box P_{n}$ of the desired weight.

1. $n \equiv 0 \pmod{4}$:
   
   $2010 \ldots 2010$ 2201
   $0505 \ldots 0505$ 0050
   $1020 \ldots 1020$ 1102
2. $n \equiv 1 \pmod{4}$:
   
   \begin{align*}
   2010 \ldots 2010 \ 2 \\
   0505 \ldots 0505 \ 0 \\
   1020 \ldots 1020 \ 1
   \end{align*}

3. $n \equiv 2 \pmod{4}$:
   
   \begin{align*}
   2010 \ldots 2010 \ 201201 \\
   0505 \ldots 0505 \ 050050 \\
   1020 \ldots 1020 \ 102102
   \end{align*}

4. $n \equiv 3 \pmod{4}$:
   
   \begin{align*}
   2010 \ldots 2010 \ 201 \\
   0505 \ldots 0505 \ 050 \\
   1020 \ldots 1020 \ 102
   \end{align*}

To prove the inverse inequality, we show that every $\gamma_{4}(P_3 \Box P_n)$-function $f$ satisfies the following conditions:

1. For every $0 \leq i \leq n - 1$, $\omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \geq 1$;
2. For every $1 \leq i \leq n - 2$, if $\omega(f_i) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \geq 6$; and
3. $\omega(f_0) \geq 2$ and $\omega(f_{n-1}) \geq 2$.

First, we show that for every $\gamma_{4}(P_3 \Box P_n)$-function $f$, $\omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \geq 1$ when $0 \leq i \leq n - 1$. Let $n \geq 5$ and $f$ be a $\gamma_{4}(P_3 \Box P_n)$-function and $S = \{|i| \ 0 \leq i \leq n - 1 \text{ and } \omega(f_i) = 0\}$. We claim that $|S| = 0$. Assume to the contrary that $|S| \geq 1$. Then, we consider the following cases.

**Case 1.** $0 \in S$ (the case $n - 1 \in S$ is similar).

Then, we have $f(v_1) = f(u_1) = f(w_1) = \{1, 2, 3, 4\}$ and the function $g$ defined by $g(v_0) = \{1\}$, $g(u_1) = \{3, 4\}$, $g(w_0) = \{2\}$, $g(v_2) = f(v_2) \cup \{2\}$, $g(w_2) = f(w_2) \cup \{1\}$, $g(u_0) = g(v_1) = g(w_1) = \emptyset$ and $g(x) = f(x)$ otherwise, is an $4RDF$ of $P_3 \Box P_n$ of weight less than $\omega(f)$, which is a contradiction.

Let $\omega(f_s) = 0$. Then, $s \geq 1$ and $\omega(f_{s-1}) + \omega(f_{s+1}) \geq 12$.

**Case 2.** $1 \leq s \leq n - 2$ is similar.

The function $g$ defined by $g(v_0) = g(u_0) = g(w_0) = \{1\}$, $g(v_1) = \{2\}$, $g(w_1) = \{1\}$, $g(v_2) = \{3, 4\}$, $g(v_3) = f(v_3) \cup \{1\}$, $g(w_3) = f(w_3) \cup \{2\}$, $g(u_1) = g(v_2) = g(w_2) = \emptyset$ and $g(x) = f(x)$ otherwise, is an $4RDF$ of $P_3 \Box P_n$ of weight less than $\omega(f)$, which is a contradiction.

**Case 3.** $2 \leq s \leq n - 3$.

Then, it is easy to see that the function $g$ defined by $g(u_{s-1}) = g(u_{s+1}) = \{3, 4\}$, $g(v_i) = \{2\}$, $g(w_i) = \{1\}$, $g(v_{s-2}) = f(v_{s-2}) \cup \{1\}$, $g(w_{s-2}) = f(w_{s-2}) \cup \{2\}$, $g(v_{s+2}) = f(v_{s+2}) \cup \{1\}$, $g(w_{s+2}) = f(w_{s+2}) \cup \{2\}$, $g(v_{s-1}) = g(v_{s+1}) = g(u_s) = g(w_{s-1}) = g(w_{s+1}) = \emptyset$ and $g(x) = f(x)$ otherwise, is an $4RDF$ of $P_3 \Box P_n$ of weight less than $\omega(f)$, which is a contradiction. Therefore, $|S| = 0$.

Now, let $f$ be a $\gamma_{4}(P_3 \Box P_n)$-function. It is easy to see that, if $\omega(f_1) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \geq 6$ when $1 \leq i \leq n - 2$.

We show that for every $\gamma_{4}(P_3 \Box P_n)$-function $f$, $\omega(f_0) \geq 2$ ($\omega(f_{n-1}) \geq 2$ is similar). Let $f$ be a $\gamma_{4}(P_3 \Box P_n)$-function such that $\omega(f_0) = 1$. If $|f(v_0)| = |f(w_0)| = 1$ (if $|f(u_0)| = 1$ is similar), then $|f(v_0)| = |f(u_0)| = 0$, $|f(u_1)| \geq 3$ and $|f(w_1)| = 4$. We may assume that $\{1, 2, 3\} \subseteq f(u_1)$. The function $g$ defined by $g(v_0) = \{4\}$, $g(w_2) = \{4\}$, $g(w_1) = \emptyset$ and $g(x) = f(x)$ otherwise, is an $4RDF$ of $P_3 \Box P_n$ of weight less than $\omega(f)$, which is a contradiction. Now, let $|f(v_0)| = 1$. Then, $|f(v_0)| = |f(v_0)| = 0,$
|f(v_1)| \geq 3 and |f(w_1)| \geq 3. The function g defined by g(w_0) = \{1\}, g(w_2) = \{2\}, g(u_1) = \{3, 4\}, 
g(v_2) = f(v_2) \cup \{1\}, g(w_2) = f(w_2) \cup \{2\}, g(u_1) = g(u_2) = \emptyset and g(x) = f(x) otherwise, is an 4RDF of P_3 \square P_n of weight less than \omega(f), which is a contradiction.

Hence, every \gamma_4(P_3 \square P_n)-function f satisfies the following conditions:

1. For every 0 \leq i \leq n - 1, \omega(f_i) \geq 1;
2. For every 1 \leq i \leq n - 2, if \omega(f_i) = 1, then \omega(f_{i-1}) + \omega(f_{i+1}) \geq 6. In particular \omega(f_{i-1}) + \omega(f_i) + \omega(f_{i+1}) \geq 8; and
3. \omega(f_0) \geq 2 and \omega(f_{n-1}) \geq 2.

Hence,

\[ 2\omega(f) = 2 \sum_{0 \leq i \leq n-1} \omega(f_i) = \sum_{0 \leq i \leq n-2} (\omega(f_i) + \omega(f_{i+1}) + \omega(f_0) + \omega(f_{n-1}) \geq 4(n - 1) + 4. \]

Hence, \omega(f) = 2n. \ \Box

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References


