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k-Rainbow Domination Number of $P_3 \Box P_n$

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Abstract: Let k be a positive integer, and set $[k] := \{1, 2, \ldots, k\}$. For a graph G, a *k-rainbow dominating function* (or *kRDF*) of G is a mapping $f : V(G) \to 2^{[k]}$ in such a way that, for any vertex $v \in V(G)$ with the empty set under f, the condition $\bigcup_{u \in N_G(v)} f(u) = [k]$ always holds, where $N_G(v)$ is the open neighborhood of v. The weight of *kRDF* f of G is the summation of values of all vertices under f. The *k-rainbow domination number* of G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a *kRDF* of G. In this paper, we obtain the *k-rainbow domination number* of grid $P_3 \Box P_n$ for $k \in \{2, 3, 4\}$.

Keywords: *k-rainbow dominating function*; *k-rainbow domination number*; grids

1. Introduction

For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For a vertex $v \in V(G)$, the open neighborhood of v, denoted by $N_G(v)$, is the set $\{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is defined by $d_G(v) = |N_G(v)|$. We let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of a graph G, respectively.

Let k be a positive integer, and $[k] := \{1, 2, \ldots, k\}$. For a graph G, a *k-rainbow dominating function* (or *kRDF*) of G is a mapping $f : V(G) \to 2^{[k]}$ in such a way that for any vertex $v \in V(G)$ with the empty set under f, the condition $\bigcup_{u \in N_G(v)} f(u) = [k]$ always holds. The weight of a *kRDF* f of G is the value $\omega(f) := \sum_{v \in V(G)} |f(v)|$. The *k-rainbow domination number* of G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a *kRDF* of G. A *kRDF* f of G is a γ_{rk}*-function* if $\omega(f) = \gamma_{rk}(G)$. The *k-rainbow domination number* was introduced by Brešar, Henning, and Rall [1] was studied by several authors (see, for example [2–15]).

For graphs F and G, we let $F \Box G$ denote the Cartesian product of F and G. Vizing [16] conjectured that for arbitrary graphs F and G, $\gamma(F \Box G) \geq \gamma(F)\gamma(G)$. This conjecture is still open, and the domination number or its related invariants of $F \Box G$ are extensively studied with the motivation from Vizing's conjecture.

Concerning the *k-rainbow domination number* of $F \Box G$, one problem naturally arises: Given two graphs F and G under some conditions, determine $\gamma_{rk}(F \Box G)$ for all k. In [3], the authors determined $\gamma_{rk}(P_2 \Box P_n)$ for $k = 3, 4, 5$.

In this paper, we examine grid graphs $P_3 \Box P_n$, and determine the value $\gamma_{rk}(P_3 \Box P_n)$ for $k \in \{2,3,4\}$ and all n, where P_m is the path of order m.

2. 2-Rainbow Domination Number of $P_3 \Box P_n$

We write $V(P_3 \Box P_n) = \{v_i, u_i, w_i \mid 0 \leq i \leq n-1\}$ and let $E(P_3 \Box P_n) = \{v_i u_i, u_i w_i \mid 0 \leq i \leq n-1\} \cup \{v_i v_{i+1}, u_i u_{i+1}, w_i w_{i+1} \mid 0 \leq i \leq n-1\}$ (see Figure 1). A 2RDF f is given in three lines, where in the first line there are values of the function f for vertices $\{v_0, v_1, \ldots, v_{n-1}\}$, in the second line of the vertices $\{u_0, u_1, \ldots, u_{n-1}\}$, and in the third line of the vertices $\{w_0, w_1, \ldots, w_{n-1}\}$ (see Figure 2). Furthermore, we use $0, 1, 2, 3$ to encode the sets $\emptyset, \{1\}, \{2\}, \{1,2\}$.

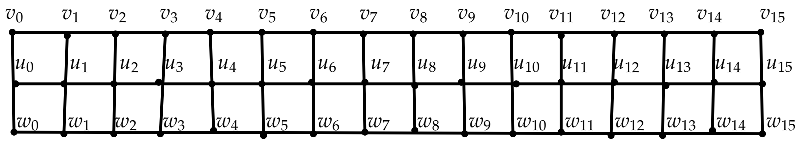

Figure 1. The grid graph $P_3 \Box P_{16}$.

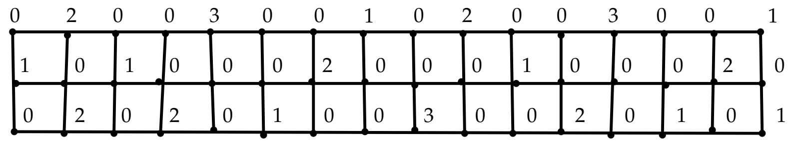

Figure 2. A 2RDF of $P_3 \Box P_n$.

To provide a complete answer, we need the following fact that can easily be proved as an exercise.

Fact 1. $\gamma_{r2}(P_3 \Box P_3) = 4$, $\gamma_{r2}(P_3 \Box P_4) = 6$, $\gamma_{r2}(P_3 \Box P_5) = 7$, $\gamma_{r2}(P_3 \Box P_6) = 8$, $\gamma_{r2}(P_3 \Box P_7) = 10$.

Theorem 1. *For $n \geq 8$,* $\gamma_{r2}(P_3 \Box P_n) = \left\lceil \dfrac{5n+3}{4} \right\rceil$.

Proof. First, we present constructions of a 2RDF of $P_3 \Box P_n$ of the desired weight.

1. $n \equiv 0 \pmod 8$:

   0200 30010200 ... 30010200 3001

   1010 00200010 ... 00200010 0020

   0202 01003002 ... 01003002 0101
2. $n \equiv 1 \pmod 8$:

   0200 30010200 ... 30010200 30010

   1010 00200010 ... 00200010 00202

   0202 01003002 ... 01003002 01010
3. $n \equiv 2 \pmod 8$:

   0200 30010200 ... 30010200 300101

   1010 00200010 ... 00200010 002020

   0202 01003002 ... 01003002 010101
4. $n \equiv 3 \pmod 8$:

   0200 30010200 ... 30010200 3001001

   1010 00200010 ... 00200010 0020220

   0202 01003002 ... 01003002 0101001

5. $n \equiv 4 \pmod 8$:

   0200 30010200 . . . 30010200 30010010

   1010 00200010 . . . 00200010 00202202

   0202 01003002 . . . 01003002 01010010
6. $n \equiv 5 \pmod 8$:

   0200 30010200 . . . 30010200 300102020

   1010 00200010 . . . 00200010 002000101

   0202 01003002 . . . 01003002 010030020
7. $n \equiv 6 \pmod 8$:

   0200 30010200 . . . 30010200 30

   1010 00200010 . . . 00200010 01

   0202 01003002 . . . 01003002 01
8. $n \equiv 7 \pmod 8$:

   0200 30010200 . . . 30010200 301

   1010 00200010 . . . 00200010 002

   0202 01003002 . . . 01003002 010

To show that these are also lower bounds, we prove there is a $\gamma_{r2}(P_3\Box P_n)$-function, f such that for every $0 \le i \le n-1$, $\omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \ge 1$. Let $n \ge 8$ and f be a $\gamma_{r2}(P_3\Box P_n)$-function such that the cardinality of $S = \{i \mid 0 \le i \le n-1 \text{ and } \omega(f_i) = 0\}$ is as small as possible. We claim that $|S| = 0$. Suppose, to the contrary, that $|S| \ge 1$ and let s be the smallest positive integer for which $\omega(f_s) = 0$. Then, $\omega(f_{s-1}) + \omega(f_{s+1}) \ge 6$. Then, we consider the following cases.

Case 1. $s = 1$ (the case $s = n-1$ is similar).
Then, we have $f(v_1) = f(u_1) = f(w_1) = \{1,2\}$ and the function g defined by $g(u_0) = \{1\}$, $g(v_1) = g(w_1) = \{2\}$, $g(u_2) = f(u_2) \cup \{1\}$, $g(v_0) = g(w_0) = g(u_1) = \emptyset$ and $g(x) = f(x)$ otherwise, is a 2RDF of $P_3\Box P_n$ of weight at most $\omega(f)$, which contradicts the choice of f.

Case 2. $s = 1$ ($s = n-2$ is similar).
Then, $\omega(f_0) + \omega(f_2) \ge 6$ and the function g defined by $g(u_0) = g(u_2) = \{1\}$, $g(v_1) = g(w_1) = \{2\}$, $g(v_3) = f(v_3) \cup \{2\}$, $g(w_3) = f(w_3) \cup \{2\}$, $g(v_0) = g(w_0) = g(u_1) = g(v_2) = g(w_2) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 2RDF of $P_3\Box P_n$ of weight at most $\omega(f)$, which contradicts the choice of f.

Case 3. $2 \le s \le n-3$.
Since $\omega(f_{s-2}) \ge 1$, then $|f(v_{s-2})| + |f(u_{s-2})| + |f(w_{s-2})| \ge 1$. First, let $|f(u_{s-2})| \ge 1$. We may assume that $\{1\} \subseteq f(u_{s-2})$. It is easy to see that the function g defined by $g(v_{s-1}) = g(v_{s+1}) = g(w_{s-1}) = g(w_{s+1}) = \{2\}$, $g(u_s) = \{1\}$, $g(u_{s+2}) = f(u_{s+2}) \cup \{1\}$, $g(u_{s-1}) = g(v_s) = g(w_s) = g(u_{s+1}) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 2RDF of $P_3\Box P_n$ of weight at most $\omega(f)$, which contradicts the choice of f. Now, let $|f(w_{s-2})| \ge 1$ ($|f(v_{s-2})| \ge 1$ is similar). We may assume that $\{1\} \subseteq f(w_{s-2})$. Hence, the function g defined by $g(v_{s-2}) = f(v_{s-2}) \cup \{1\}$, $g(v_{s+1}) = g(u_{s-1}) = g(w_{s+1}) = \{2\}$, $g(u_s) = \{1\}$, $g(u_{s+2}) = f(u_{s+2}) \cup \{1\}$, $g(u_{s-1}) = g(v_s) = g(w_s) = g(w_{s-1}) = g(u_{s+1}) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 2RDF of $P_3\Box P_n$ of weight $\omega(f)$, which is contradicting the choice of f. Therefore, $|S| = 0$.

We can see that for every $0 \le i \le n-2$, if $\omega(f_i) = \omega(f_{i+1}) = \omega(f_{i+2}) = 1$, then $\omega(f_{i-1}), \omega(f_{i+3}) > 1$. In addition, there is the function f such that, if $\omega(f_0) = 1$ ($\omega(f_{n-1}) = 1$ is similar), then $\omega(f_1) > 1$ and $\omega(f_1) + \omega(f_2) + \omega(f_3) + \omega(f_4) \ge 6$ and if $\omega(f_0) = 2$ ($\omega(f_{n-1}) = 2$ is similar), then $\omega(f_0) + \omega(f_1) + \omega(f_2) + \omega(f_3) \ge 6$.

If $\omega(f_0) = 1$ and $\omega(f_{n-1}) = 1$, then

$$\begin{aligned}
4\omega(f) &= 4 \sum_{0\le i\le n-1} \omega(f_i) \\
&= [3\omega(f_0) + 2\omega(f_1) + \omega(f_2)] + [3\omega(f_{n-1}) + 2\omega(f_{n-2}) + \omega(f_{n-3})] \\
&\quad + \sum_{i\in\{0,\dots,n-4\}-\{1,n-5\}} (\omega(f_i) + \omega(f_{i+1}) + \omega(f_{i+2}) + \omega(f_{i+3})) \\
&\quad +[\omega(f_1) + \omega(f_2) + \omega(f_3) + \omega(f_4)] + [\omega(f_{n-5}) + \omega(f_{n-4}) + \omega(f_{n-3}) + \omega(f_{n-2})] \\
&\ge 8 + 8 + 5(n-5) + 12 \\
&= 5(n-3) + 18.
\end{aligned}$$

If $\omega(f_0) = 1$ and $\omega(f_{n-1}) = 2$, then

$$\begin{aligned}
4\omega(f) &= 4 \sum_{0\le i\le n-1} \omega(f_i) \\
&= [3\omega(f_0) + 2\omega(f_1) + \omega(f_2)] + [3\omega(f_{n-1}) + 2\omega(f_{n-2}) + \omega(f_{n-3})] \\
&\quad + \sum_{i\in\{0,\dots,n-4\}-\{1\}} (\omega(f_i) + \omega(f_{i+1}) + \omega(f_{i+2}) + \omega(f_{i+3})) \\
&\quad +[\omega(f_1) + \omega(f_2) + \omega(f_3) + \omega(f_4)] \\
&\ge 8 + 9 + 5(n-4) + 6 \\
&= 5(n-3) + 18.
\end{aligned}$$

If $\omega(f_0) = 2$ and $\omega(f_{n-1}) = 2$, then

$$\begin{aligned}
4\omega(f) &= 4 \sum_{0\le i\le n-1} \omega(f_i) \\
&= [3\omega(f_0) + 2\omega(f_1) + \omega(f_2)] + [3\omega(f_{n-1}) + 2\omega(f_{n-2}) + \omega(f_{n-3})] \\
&\quad + \sum_{i\in\{0,\dots,n-4\}-\{1\}} [\omega(f_i) + \omega(f_{i+1}) + \omega(f_{i+2}) + \omega(f_{i+3})] \\
&\quad +[\omega(f_1) + \omega(f_2) + \omega(f_3) + \omega(f_4)] \\
&\ge 9 + 9 + 5(n-3) \\
&= 5(n-3) + 18.
\end{aligned}$$

Thus, $\omega(f) = \lceil \frac{5n+3}{4} \rceil$. $\square$

3. 3-Rainbow Domination Number of $P_3\square P_n$

As in the previous section, a 3RDF is given in three lines and we use $0, 1, 2, 3$ to encode the sets $\emptyset, \{1\}, \{2\}, \{3\}$.

To provide a complete answer, we need the following fact.

Fact 2. $\gamma_{r3}(P_3\square P_3) = 5$, $\gamma_{r3}(P_3\square P_4) = 8$.

Theorem 2. *For* $n \ge 5$,

$$\gamma_{r3}(P_3\square P_n) = \begin{cases} (3n+1)/2 & \text{if} \quad n \equiv 1 \text{ (mode 2)}, \\ (3n+2)/2 & \text{if} \quad n \equiv 0 \text{ (mode 2)}, \end{cases}$$

Proof. First, we present constructions of a 3RDF of $P_3 \Box P_n$ of the desired weight.

1. $n \equiv 0 \pmod 4$:

   2010 . . . 2010 2201

   0303 . . . 0303 0030

   1020 . . . 1020 1102
2. $n \equiv 1 \pmod 4$:

   2010 . . . 2010 2

   0303 . . . 0303 0

   1020 . . . 1020 1
3. $n \equiv 2 \pmod 4$:

   2010 . . . 2010 201201

   0303 . . . 0303 030030

   1020 . . . 1020 102102
4. $n \equiv 3 \pmod 4$:

   2010 . . . 2010 201

   0303 . . . 0303 030

   1020 . . . 1020 102

To show that these are also lower bounds, we prove there is a $\gamma_{r3}(P_3 \Box P_n)$-function, f that satisfies the following conditions:

1. For every $0 \leq i \leq n-1$, $\omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \geq 1$,
2. For every $1 \leq i \leq n-2$, if $\omega(f_i) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \geq 4$. In particular, if $\omega(f_i) = 1$, then $(\omega(f_{i-1}) + \omega(f_i)) + (\omega(f_i) + \omega(f_{i+1})) \geq 6$,
3. $\omega(f_0) \geq 2$ and $\omega(f_{n-1}) \geq 2$.

First, we show that for every $\gamma_{r3}(P_3 \Box P_n)$-function f, $\omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \geq 1$ when $0 \leq i \leq n-1$. Let $n \geq 5$ and f be a $\gamma_{r3}(P_3 \Box P_n)$-function and $S = \{i \mid 0 \leq i \leq n-1 \text{ and } \omega(f_i) = 0\}$. We claim that $|S| = 0$. Assume to the contrary that $|S| \geq 1$. Then, we consider the following cases.

Case 1. $0 \in S$ (the case $n-1 \in S$ is similar).
Then, we have $f(v_1) = f(u_1) = f(w_1) = \{1,2,3\}$ and it is easy to see that the function g defined by $g(v_0) = \{1\}$, $g(u_1) = \{3\}$, $g(w_0) = \{2\}$, $g(v_2) = f(v_2) \cup \{2\}$, $g(w_2) = f(w_2) \cup \{1\}$, $g(u_0) = g(v_1) = g(w_1) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 3RDF of $P_3 \Box P_n$ of weight less than $\omega(f)$, which is a contradiction.

Let s be the smallest positive integer for which $\omega(f_s) = 0$. Then, $s \geq 1$ and $\omega(f_{s-1}) + \omega(f_{s+1}) \geq 9$.

Case 2. $s = 1$ ($s = n-2$ is similar).
Then, the function g defined by $g(v_0) = g(u_0) = g(w_0) = \{1\}$, $g(v_1) = \{2\}$, $g(w_1) = \{1\}$, $g(u_2) = \{3\}$, $g(v_3) = f(v_3) \cup \{1\}$, $g(w_3) = f(w_3) \cup \{2\}$, $g(u_1) = g(v_2) = g(w_2) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 3RDF of $P_3 \Box P_n$ of weight less than $\omega(f)$, which is a contradiction.

Case 3. $2 \leq s \leq n-3$.
The function g defined by $g(u_{s-1}) = g(u_{s+1}) = \{3\}$, $g(v_s) = \{2\}$, $g(w_s) = \{1\}$, $g(v_{s-2}) = f(v_{s-2}) \cup \{1\}$, $g(w_{s-2}) = f(w_{s-2}) \cup \{2\}$, $g(v_{s+2}) = f(v_{s+2}) \cup \{1\}$, $g(w_{s+2}) = f(w_{s+2}) \cup \{2\}$, $g(v_{s-1}) = g(v_{s+1}) = g(u_s) = g(w_{s-1}) = g(w_{s+1}) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 3RDF of $P_3 \Box P_n$ of weight less than $\omega(f)$, which is a contradiction. Therefore, $|S| = 0$.

Now, let f be a $\gamma_{r3}(P_3\Box P_n)$-function. It is easy to see that, if $\omega(f_i) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \geq 4$ when $1 \leq i \leq n-2$.

Finally, we show that there is $\gamma_{r3}(P_3\Box P_n)$-function f such that $\omega(f_0) \geq 2$ ($\omega(f_{n-1}) \geq 2$ is similar). Let f be a $\gamma_{r3}(P_3\Box P_n)$-function such that $\omega(f_0) = 1$. If $|f(v_0)| = 1$ ($|f(w_0)| = 1$ is similar), then $|f(w_0)| = |f(u_0)| = 0$, $|f(u_1)| \geq 2$ and $|f(w_1)| = 3$. We may assume that $\{1,2\} \subseteq f(u_1)$. It is easy to see that the function g defined by $g(w_0) = \{3\}$, $g(w_2) = \{3\}$, $g(w_1) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 3RDF of $P_3\Box P_n$ of weight less than $\omega(f)$, which is a contradiction. Now, let $|f(u_0)| = 1$. Then, $|f(w_0)| = |f(v_0)| = 0$, $|f(v_1)| \geq 2$ and $|f(w_1)| \geq 2$. It is easy to see that the function g defined by $g(w_0) = \{1\}$, $g(w_2) = \{2\}$, $g(u_1) = \{3\}$, $g(v_2) = f(v_2) \cup \{1\}$, $g(w_2) = f(w_2) \cup \{2\}$, $g(u_1) = g(u_2) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 3RDF of $P_3\Box P_n$ of weight $\omega(f)$.

Hence, there is a $\gamma_{r3}(P_3\Box P_n)$-function, f that satisfies the following conditions:

1. For every $0 \leq i \leq n-1$, $\omega(f_i) \geq 1$;
2. For every $1 \leq i \leq n-2$, if $\omega(f_i) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \geq 4$; and
3. $\omega(f_0) \geq 2$ and $\omega(f_{n-1}) \geq 2$.

If n is odd, then

$$\begin{aligned} 2\omega(f) &= 2\sum_{0\leq i\leq n-1} \omega(f_i) \\ &= \omega(f_0) + \omega(f_{n-1}) + \sum_{0\leq i\leq n-2} (\omega(f_i) + \omega(f_{i+1})) \\ &\geq 4 + 3(n-1). \end{aligned}$$

Then, $\omega(f) = \frac{3n+1}{2}$ when n is odd. Now, let n is even. Then, there is $s \neq n-1$ such that $\omega(f_s) + \omega(f_{s+1}) \geq 4$. Hence,

$$\begin{aligned} 2\omega(f) &= 2\sum_{0\leq i\leq n-1} \omega(f_i) \\ &= \omega(f_s) + \omega(f_{s+1}) + \omega(f_0) + \omega(f_{n-1}) + \sum_{0\leq i\leq n-2, i\neq s} (\omega(f_i) + \omega(f_{i+1})) \\ &\geq 8 + 3(n-2). \end{aligned}$$

Therefore, $\omega(f) = \frac{3n+2}{2}$ when n is even. □

4. 4-Rainbow Domination Number of $P_3\Box P_n$

As above, a 4RDF is given in three lines and we use $0, 1, 2, 5$ to encode the sets $\emptyset$, $\{1\}$, $\{2\}$, $\{3,4\}$. To provide a complete answer, we need the following fact.

Fact 3. $\gamma_{r4}(P_3\Box P_3) = 6$, $\gamma_{r4}(P_3\Box P_4) = 9$.

Theorem 3. *For $n \geq 5$, $\gamma_{r4}(P_3\Box P_n) = 2n$.*

Proof. First, we show that $\gamma_{r4}(P_3\Box P_n) \leq 2n$. To do this, we present constructions of a 4RDF of $P_3\Box P_n$ of the desired weight.

1. $n \equiv 0 \pmod 4$:

   2010 . . . 2010 2201

   0505 . . . 0505 0050

   1020 . . . 1020 1102

2. $n \equiv 1 \pmod 4$:

   $2010\ldots2010\ \ 2$

   $0505\ldots0505\ \ 0$

   $1020\ldots1020\ \ 1$
3. $n \equiv 2 \pmod 4$:

   $2010\ldots2010\ \ 201201$

   $0505\ldots0505\ \ 050050$

   $1020\ldots1020\ \ 102102$
4. $n \equiv 3 \pmod 4$:

   $2010\ldots2010\ \ 201$

   $0505\ldots0505\ \ 050$

   $1020\ldots1020\ \ 102$

To prove the inverse inequality, we show that every $\gamma_{r4}(P_3\Box P_n)$-function f satisfies the following conditions:

1. For every $0 \le i \le n-1$, $\omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \ge 1$;
2. For every $1 \le i \le n-2$, if $\omega(f_i) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \ge 6$; and
3. $\omega(f_0) \ge 2$ and $\omega(f_{n-1}) \ge 2$.

First, we show that for every $\gamma_{r4}(P_3\Box P_n)$-function f, $\omega(f_i) = |f(v_i)| + |f(u_i)| + |f(w_i)| \ge 1$ when $0 \le i \le n-1$. Let $n \ge 5$ and f be a $\gamma_{r4}(P_3\Box P_n)$-function and $S = \{i \mid 0 \le i \le n-1 \text{ and } \omega(f_i) = 0\}$. We claim that $|S| = 0$. Assume to the contrary that $|S| \ge 1$. Then, we consider the following cases.

Case 1. $0 \in S$ (the case $n-1 \in S$ is similar).
Then, we have $f(v_1) = f(u_1) = f(w_1) = \{1,2,3,4\}$ and the function g defined by $g(v_0) = \{1\}$, $g(u_1) = \{3,4\}$, $g(w_0) = \{2\}$, $g(v_2) = f(v_2) \cup \{2\}$, $g(w_2) = f(w_2) \cup \{1\}$, $g(u_0) = g(v_1) = g(w_1) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 4RDF of $P_3\Box P_n$ of weight less than $\omega(f)$, which is a contradiction.

Let $\omega(f_s) = 0$. Then, $s \ge 1$ and $\omega(f_{s-1}) + \omega(f_{s+1}) \ge 12$.

Case 2. $s = 1$ ($s = n-2$ is similar).
The function g defined by $g(v_0) = g(u_0) = g(w_0) = \{1\}$, $g(v_1) = \{2\}$, $g(w_1) = \{1\}$, $g(u_2) = \{3,4\}$, $g(v_3) = f(v_3) \cup \{1\}$, $g(w_3) = f(w_3) \cup \{2\}$, $g(u_1) = g(v_2) = g(w_2) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 4RDF of $P_3\Box P_n$ of weight less than $\omega(f)$, which is a contradiction.

Case 3. $2 \le s \le n-3$.
Then, it is easy to see that the function g defined by $g(u_{s-1}) = g(u_{s+1}) = \{3,4\}$, $g(v_s) = \{2\}$, $g(w_s) = \{1\}$, $g(v_{s-2}) = f(v_{s-2}) \cup \{1\}$, $g(w_{s-2}) = f(w_{s-2}) \cup \{2\}$, $g(v_{s+2}) = f(v_{s+2}) \cup \{1\}$, $g(w_{s+2}) = f(w_{s+2}) \cup \{2\}$, $g(v_{s-1}) = g(v_{s+1}) = g(u_s) = g(w_{s-1}) = g(w_{s+1}) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 4RDF of $P_3\Box P_n$ of weight less than $\omega(f)$, which is a contradiction. Therefore, $|S| = 0$.

Now, let f be a $\gamma_{r4}(P_3\Box P_n)$-function. It is easy to see that, if $\omega(f_i) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \ge 6$ when $1 \le i \le n-2$.

We show that for every $\gamma_{r4}(P_3\Box P_n)$-function f $\omega(f_0) \ge 2$ ($\omega(f_{n-1}) \ge 2$ is similar). Let f be a $\gamma_{r4}(P_3\Box P_n)$-function such that $\omega(f_0) = 1$. If $|f(v_0)| = 1$ ($|f(w_0)| = 1$ is similar), then $|f(w_0)| = |f(u_0)| = 0$, $|f(u_1)| \ge 3$ and $|f(w_1)| = 4$. We may assume that $\{1,2,3\} \subseteq f(u_1)$. The function g defined by $g(w_0) = \{4\}$, $g(w_2) = \{4\}$, $g(w_1) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 4RDF of $P_3\Box P_n$ of weight less than $\omega(f)$, which is a contradiction. Now, let $|f(u_0)| = 1$. Then, $|f(w_0)| = |f(v_0)| = 0$,

$|f(v_1)| \geq 3$ and $|f(w_1)| \geq 3$. The function g defined by $g(w_0) = \{1\}$, $g(w_2) = \{2\}$, $g(u_1) = \{3,4\}$, $g(v_2) = f(v_2) \cup \{1\}$, $g(w_2) = f(w_2) \cup \{2\}$, $g(u_1) = g(u_2) = \emptyset$ and $g(x) = f(x)$ otherwise, is an 4RDF of $P_3 \Box P_n$ of weight less than $\omega(f)$, which is a contradiction.

Hence, every $\gamma_{r4}(P_3 \Box P_n)$-function f satisfies the following conditions:

1. For every $0 \leq i \leq n-1$, $\omega(f_i) \geq 1$;
2. For every $1 \leq i \leq n-2$, if $\omega(f_i) = 1$, then $\omega(f_{i-1}) + \omega(f_{i+1}) \geq 6$. In particular $(\omega(f_{i-1}) + \omega(f_i)) + (\omega(f_i) + \omega(f_{i+1})) \geq 8$; and
3. $\omega(f_0) \geq 2$ and $\omega(f_{n-1}) \geq 2$.

Hence,

$$\begin{aligned} 2\omega(f) &= 2 \sum_{0 \leq i \leq n-1} \omega(f_i) \\ &= \sum_{0 \leq i \leq n-2} (\omega(f_i) + \omega(f_{i+1})) + \omega(f_0) + \omega(f_{n-1}) \\ &\geq 4(n-1) + 4. \end{aligned}$$

Hence, $\omega(f) = 2n$. □

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