On the Domain of the Fibonacci Difference Matrix

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Abstract: Matrix $F^*$ derived from the Fibonacci sequence was first introduced by Kara (2013) and the spaces $l_p(F)$ and $l_\infty(F)$ ($1 \leq p < \infty$) were examined. Then, Başarır et al. (2015) defined the spaces $c_0(F)$ and $c(F)$ and Candan (2015) examined the spaces $c(F(r,s))$ and $c_0(F(r,s))$. Later, Yaşar and Kayaduman (2018) defined and studied the spaces $cs(F(s,r))$ and $bs(F(s,r))$. In this study, we built the spaces $cs(F)$ and $bs(F)$. They are the domain of the matrix $F$ on $cs$ and $bs$, where $F$ is a triangular matrix defined by Fibonacci Numbers. Some topological and algebraic properties, isomorphism, inclusion relations and norms, which are defined over them are examined. It is proven that $cs(F)$ and $bs(F)$ are Banach spaces. It is determined that they have the $\gamma$, $\beta$, $\alpha$ -duals. In addition, the Schauder base of the space $cs(F)$ are calculated. Finally, a number of matrix transformations of these spaces are found.

Keywords: matrix transformations; Fibonacci numbers; sequence spaces; Fibonacci double band matrix; $\gamma$, $\beta$, $\alpha$ -duals

1. Introduction

Cooke [1] formulated the theory of infinite matrices in the book “Infinite Matrices and Sequence Spaces”. Many researchers have investigated infinite matrices after the publication of this book in 1950. In most of these studies, the domain of infinite matrices on a sequence space was studied. In this study, we address the question: What are the properties of the domain of the Fibonacci band matrix on sequence spaces $bs$ and $cs$? The domain of the Fibonacci band matrix creates a new sequence space. We handle algebraic properties of this new space in order to determine its duals and its place among other known spaces, and to characterize the matrix transformations of this space.

One difficulty of this study is to determine whether the new space is the contraction or the expansion, or the overlap of the original space. Another difficulty is to determine the matrix transformations on this space and into this space. For the first problem, we give a few inclusion theorems. For the second problem, we use the matrix transformation between the standard sequence spaces and two theorems.

Generating a new sequence space and researching on its properties have been important in the studies on the sequence space. Some researchers examined the algebraic properties of the sequence space while others investigated its place among other known spaces and its duals, and characterized the matrix transformations on this space.

We can create a new sequence space by using the domain of infinite matrices. Ng-Lee [2] first investigated the domain of an infinite matrix in 1978. In the same period, Wang [3] created a new sequence space by using another infinite matrix. Many researchers such as Malkovsky [4], Altay, and Başar [5] followed these studies. This topic was studied intensively after 2000.
Leonardo Fibonacci invented Fibonacci numbers. He introduced Fibonacci numbers originated from a rabbit problem. These numbers create a number sequence:

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots\]

This sequence has important properties and applications in various fields. Let us indicate the Fibonacci sequence by \( (f_n) \). \( f_n \) is defined as

\[ f_n = f_{n-1} + f_{n-2} \quad (n \geq 2) \]

with \( f_0 = f_1 = 1 \). The golden ratio is

\[ \lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} = \phi. \]

Let us indicate the set of all real-valued sequences with \( w \) and list some subspaces of \( w \) called standard sequence spaces.

\[ c = \left\{ r = (r_k) \in w : \lim_{k \to \infty} |r_k - p| = 0 \text{ for some } p \in \mathbb{C} \right\}, \]

\[ c_0 = \left\{ r = (r_k) \in w : \lim_{k \to \infty} |r_k| = 0 \right\}, \]

\[ bs = \left\{ r = (r_k) \in w : \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} r_k \right| < \infty \right\}, \]

\[ cs_0 = \left\{ r = (r_k) \in w : \lim_{n \to \infty} \left| \sum_{k=0}^{n} r_k \right| = 0 \right\}, \]

\[ l_\infty = \left\{ r = (r_k) \in w : \sup_{k \in \mathbb{N}} |r_k| < \infty \right\}, \]

\[ bv = \left\{ r = (r_k) \in w : \sum_{k=0}^{\infty} |r_k - r_{k-1}| < \infty \right\}, \]

\[ bv_0 = bv \cap c_0 \]

\[ l_p = \left\{ r = (r_k) \in w : \sum_{k=0}^{\infty} |r_k|^p < \infty, \ 0 < p < \infty \right\}, \]

\[ l_1 = \left\{ r = (r_k) \in w : \sum_{k=0}^{\infty} |r_k| < \infty \right\}. \]

Now let us take real valued infinite matrix \( T = (t_{nk}) \), where \( t_{nk} \) is a real number for every \( n,k \in \mathbb{N} \). Let \( A \) and \( B \) be sequence spaces. Sequence \( Tx = (T_n(x)) \) is \( T \)-transform of \( a \) for every \( a = (a_k) \in A \). Here, \( Ta \in B \) and

\[ T_n(a) = \sum_k t_{nk}a_k \quad (1) \]

and \( T_n(a) \to t \) (\( t \) exists for every \( n \in \mathbb{N} \)). Then, \( T \) is called a matrix transformation from \( A \) to \( B \).

Now let us take infinite matrix \( T \) and sequence space \( \delta \) to define domain of infinite matrix \( T \). The domain of the matrix \( T \) on \( \delta \) is characterized by

\[ \delta_T = \{ x = (x_k) \in w : Tx \in \delta \}. \quad (2) \]
Many researchers have studied the domain of a matrix on a sequence space. For more detailed information on these new sequence spaces, see references [6–26].

The Fibonacci difference matrix $F$ was first introduced by Kara [27] in 2013. $F$ is derived from $(f_n)$. In this study, Kara [27] defined the spaces $l_p(F)$ and $l_\infty(F); (1 \leq p < \infty)$. After this study the $c_0(F(r,s))$ and $c(F(r,s))$ were introduced by Candan [28], in 2015, where $F(r,s)$ is a generalized Fibonacci matrix. Candan and Kara [19] introduced and examined $l_p(F(r,s)); (1 \leq p < \infty)$. In 2018, Yaşar and Kayaduman [29] defined and examined $cs(F(s,r))$ and $bs(F(s,r))$ and Kayaduman and Ya¸sar [30] studied spaces $bs(N^\beta)$ and $cs(N^\alpha)$, where $N^\delta$ is a Nörlund matrix.

In this study, spaces $cs(F)$ and $bs(F)$ are introduced and the related notations are given in Section 2. In addition, some topological and algebraic properties, isomorphism, inclusion relations and norms which are defined over them are examined. The $\gamma$, $\beta$, $\alpha$ -duals of these spaces are determined in Section 3. The Schauder base of space $cs(F)$ are calculated. Finally, many matrix transformations of these spaces are found. In the last section, the results and previous studies and the working hypotheses are discussed.

A detailed literature review was performed before this study was started. Scans were made on related articles, magazines, and books. As a result of these scans, the part related to our subject was synthesized and the results were noted. These results were then applied to our problem area. Finally, the results of this study were obtained.

2. Results

2.1. The Domain of Fibonacci Difference Matrix $F$ on Bounded and Convergent Series

In this section, $cs(F)$ and $bs(F)$ are introduced. Related notations are given. In addition, some topological and algebraic properties, isomorphism, inclusion relations, and norms defined over them are examined.

For similar studies, see references [19] and [27–34].

Let spaces $cs(F)$ and $bs(F)$ be the domain of the matrix $F$ on $cs$ and $bs$, where $F = [f_{nk}]$ infinite matrix is defined by $f_{nk}$

$$f_{nk} = \begin{cases} -f_{n+1,k} & k = n - 1 \\ f_{n+1,k} & k = n \\ 0 & 0 \leq k < n - 1 \text{ or } n < k \end{cases}$$

for all $k, n \in \mathbb{N}$. Then we introduce $cs(F)$ and $bs(F)$ as

$$cs(F) = \left\{ x = (x_k) \in w : \left( \sum_{k=0}^{n} \left( \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right) \right) \in c \right\}$$

$$bs(F) = \left\{ x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left( \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right) \right| < \infty \right\}.$$
Let the inverse matrix of $F$ be $F^{-1}$. For all $k, n \in \mathbb{N}$, $F^{-1} = \{F^{-1}_{nk}\}$ is found as

$$f^{-1}_{nk} = \begin{cases} \frac{f^2_{k+1}}{f_{k+1}}, & 0 \leq k \leq n \\ 0, & k > n. \end{cases}$$

(3)

Let us take sequence $x = (x_n)$. If $y = Fx$, then we calculate as

$$y_n = (Fx)_n = \begin{cases} x_0, & n = 0 \\ \frac{f_{n+1}}{f_n} x_n - \frac{f_{n+1}}{f_{n-1}} x_{n-1}, & n \geq 1. \end{cases}$$

(4)

Herefrom, if we calculate inverse of $F$, then we find that $x = F^{-1}y$ and

$$x_n = \sum_{k=0}^{n} \frac{f^2_{n+1}}{f_{k}f_{k+1}} y_k.$$  

(5)

Now, let us give some theorems related to our study.

**Theorem 1.** $bs(F)$ is a linear space.

**Proof.** The proof is left to the reader since it is easy to show. □

**Theorem 2.** $cs(F)$ is a linear space.

**Proof.** The proof is left to the reader since it is easy to show. □

**Theorem 3.** $bs(F)$ is a normed space with:

$$\|x\| = \sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} \left( \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right) \right|.$$  

(6)

**Proof.** The proof is left to the reader since it is easy to show. □

**Theorem 4.** $cs(F)$ is a normed space with Equation (6).

**Proof.** The proof is left to the reader since it is easy to show. □

**Theorem 5.** $bs(F)$ is isomorphic to $bs$.

**Proof.** Let us take $T: bs(F) \rightarrow bs$ mentioned Equation (4) by $x \rightarrow y = Tx = Fx$. It is easy to see that $T$ is linear and injective.

We must find $T$ is surjective. Let $y = (y_n) \in bs$. By using Equation (5) and Equation (6), we see

$$\|x\| = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left( \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right) \right| = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} y_k \right| = \|y\|_{bs} < \infty.$$  

We see that $x \in bs(F)$. Hence, $T$ is surjective. In addition, $bs(F)$ and $bs$ izometric because $\|x\|_{bs(F)} = \|y\|_{bs}$. □
Theorem 6. \( cs(F) \) is isomorphic to \( cs \).

**Proof.** The proof can be made similar to Theorem 5, so it is left to the reader. \( \square \)

Theorem 7. \( bs(F) \) is a Banach space with Equation (6).

**Proof.** It is easy to see the norm conditions are ensured. Let a Cauchy sequence \( x^i = (x^i_k) \) in \( bs(F) \) for each \( i \in \mathbb{N} \). For all \( k \in \mathbb{N} \), we have

\[
y^i_k = \frac{f_k}{f_{k+1}} x^i_k - \frac{f_{k+1}}{f_k} x^i_{k-1}
\]

from Equation (4). For all \( \epsilon > 0 \) there is \( n_0 = n_0(\epsilon) \) such that

\[
\|x^i - x^m\|_{bs(F)} = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left( \frac{f_k}{f_{k+1}} (x^i_k - x^m_k) - \frac{f_{k+1}}{f_k} (x^i_{k-1} - x^m_{k-1}) \right) \right|
\]

for all \( i, m \geq n_0 \), \( y_i \rightarrow y \) (\( i \rightarrow \infty \)) such that \( y \in bs \) exists, since \( bs \) is complete. Since \( bs \) and \( bs(F) \) are isomorphic, \( bs(F) \) is complete. It hereby is a Banach space. \( \square \)

Theorem 8. \( cs(F) \) is a Banach space with Equation (6).

**Proof.** It is easy to see the norm conditions are ensured. Let a Cauchy sequence \( x^i = (x^i_k) \) in \( cs(F) \) for each \( i \in \mathbb{N} \). For all \( k \in \mathbb{N} \), we have

\[
y^i_k = \frac{f_k}{f_{k+1}} x^i_k - \frac{f_{k+1}}{f_k} x^i_{k-1}
\]

from Equation (4). For all \( \epsilon > 0 \), there is \( n_0 = n_0(\epsilon) \) such that

\[
\|x^i - x^m\|_{cs(F)} = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left( \frac{f_k}{f_{k+1}} (x^i_k - x^m_k) - \frac{f_{k+1}}{f_k} (x^i_{k-1} - x^m_{k-1}) \right) \right|
\]

for all \( i, m \geq n_0 \), \( y_i \rightarrow y \) (\( i \rightarrow \infty \)) such that \( y \in cs \) exists, since \( cs \) is complete. Since \( cs \) and \( cs(F) \) are isomorphic, \( cs(F) \) is complete. It hereby is a Banach space. \( \square \)

Now, let \( R = (r_{nk}) \) infinite matrix. Let us list the following:

\[
\sup_{n \in \mathbb{N}} \sum_{k} |r_{nk}| < \infty, \tag{7}
\]

\[
\lim_{k} r_{nk} = 0 \quad \text{for each} \quad n \in \mathbb{N}, \tag{8}
\]

\[
\sup_{m} \sum_{k} \sum_{n=0}^{m} (r_{nk} - r_{n,k+1}) < \infty, \tag{9}
\]

\[
\lim_{n} \sum_{k} r_{nk} = p \quad \text{for each} \quad k \in \mathbb{N}, \quad p \in \mathbb{C}, \tag{10}
\]

\[
\sup_{n} \sum_{k} |r_{nk} - r_{n,k+1}| < \infty, \tag{11}
\]

\[
\lim_{n} r_{nk} = a_k \quad \text{for each} \quad k \in \mathbb{N}, \quad a_k \in \mathbb{C}, \tag{12}
\]
Lemma 9. Let us suppose infinite matrix \( R = (r_{nk}) \). Then,

1. \( R = (r_{nk}) \in (bs, l_\infty) \) iff Equations (11) and (8) hold [35].
2. \( R = (r_{nk}) \in (cs, c) \) iff Equations (11) and (12) hold [36].
3. \( R = (r_{nk}) \in (bs, l_1) \) iff Equations (13) and (8) hold [37].
4. \( R = (r_{nk}) \in (cs, l_1) \) iff Equation (14) holds [35].
5. \( R = (r_{nk}) \in (bs, c) \) iff Equations (8), (15) and (16) hold [37].
6. \( R = (r_{nk}) \in (cs, l_\infty) \) iff Equations (17) and (11) hold [35].
7. \( R = (r_{nk}) \in (bs, c_0) \) iff Equations (18) and (8) hold [35].
8. \( R = (r_{nk}) \in (bs, cs_0) \) iff Equations (19) and (8) hold [38].
9. \( R = (r_{nk}) \in (bs, cs) \) iff Equations (20) and (8) hold [38].
10. \( R = (r_{nk}) \in (bs, bv) \) iff Equations (21) and (8) hold [38].
11. \( R = (r_{nk}) \in (bs, bs) \) iff Equations (8) and (9) hold [38].
12. \( R = (r_{nk}) \in (cs, cs) \) iff Equations (9) and (10) hold [39].
13. \( R = (r_{nk}) \in (bs, b_{\infty}) \) iff Equations (21), (18) and (21) hold [35].
14. \( R = (r_{nk}) \in (cs, c_0) \) iff Equations (11) and (12) hold with \( a_k = 0 \) for all \( k \in \mathbb{N} \) [40].
15. \( R = (r_{nk}) \in (cs, bs) \) iff Equations (9) and (22) hold [38].
16. \( R = (r_{nk}) \in (cs, cs_0) \) iff Equations (9) and (23) hold with \( a_k = 0 \) for all \( k \in \mathbb{N} \) [38].
17. \( R = (r_{nk}) \in (cs, bv) \) iff Equation (24) holds [38].
18. \( R = (r_{nk}) \in (cs, b_{\infty}) \) iff Equations (24) and (12) hold with \( a_k = 0 \) for all \( k \in \mathbb{N} \) [35].

The collection of all finite subsets of \( \mathbb{N} \) denoted by \( \mathcal{F} \).

\[
\sup_{N, k \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in N} (r_{nk} - r_{n,k+1}) \right| < \infty, \tag{13}
\]

\[
\sup_{N, k \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in N} (r_{nk} - r_{n,k-1}) \right| < \infty, \tag{14}
\]

\[
\lim_{n} (r_{nk} - r_{n,k+1}) = a \text{ for each } k \in \mathbb{N}, \quad a \in \mathbb{C}, \tag{15}
\]

\[
\lim_{n} \sum_{k} |r_{nk} - r_{n,k+1}| = \sum_{k} \lim_{n} (r_{nk} - r_{n,k+1}), \tag{16}
\]

\[
\sup_{n} \left| \lim_{k} r_{nk} \right| < \infty, \tag{17}
\]

\[
\lim_{n} \sum_{k} |r_{nk} - r_{n,k+1}| = 0 \text{ uniformly in } n, \tag{18}
\]

\[
\lim_{m} \sum_{k} \left| \sum_{n=0}^{m} (r_{nk} - r_{n,k+1}) \right| = 0, \tag{19}
\]

\[
\lim_{m} \sum_{k} \left| \sum_{n=0}^{m} (r_{nk} - r_{n,k+1}) \right| = 0, \tag{20}
\]

\[
\sup_{N, k \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in N} (r_{nk} - r_{n,k+1} - (r_{n-1,k} - r_{n-1,k+1})) \right| < \infty, \tag{21}
\]

\[
\sup_{m \in \mathbb{N}} \left| \lim_{k} \sum_{n=0}^{m} r_{nk} \right| < \infty, \tag{22}
\]

\[
\exists a_k \in \mathbb{C} \ni \sum_{n} r_{nk} = a_k \text{ for each } k \in \mathbb{N}, \tag{23}
\]

\[
\sup_{N, k \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in N} (r_{nk} - r_{n-1,k} - r_{n,k-1} - r_{n,k}) \right| < \infty. \tag{24}
\]
Theorem 10. $bs(F) \cap \bigcap bs$ is valid.

Proof. Suppose $x \in bs$. If we show that $F$ is an element of $(bs,bs)$ then $x$ is element of $bs(F)$. For this, $F$ must provide Equations (8) and (9). Since $\lim_{k} f_{nk} = 0$ for each $n \in \mathbb{N}$, Equation (8) is provided.

If we examine Equation (9), we find

$$\sup_{m} \sum_{k} \left| \sum_{n=0}^{m} (f_{nk} - f_{n,k+1}) \right| = 0.$$

$\square$

Theorem 11. $bs(F) \cap \bigcap \ell_{\infty}$ is not valid.

Proof. Suppose $x = (x_{k}) = (f_{2k+1})$. Then $y = Fx = (1,0,0,...) \in bs$. On the other hand, $f_{2k+1} \to \infty$ as $k \to \infty$. It is clear $x \in bs(F)$, but $x \notin \ell_{\infty}$. This result completes the proof. $\square$

Theorem 12. $cs(F) \cap \bigcap cs$ is valid.

Proof. Suppose $x \in cs$. If we show that $F$ is element of $(cs,cs)$ then $x$ is element of $cs(F)$. For this, $F$ must provide Equations (10) and (9). Equation (9) has been provided from the Theorem 10. If we look at the Equation (10) then, for each $k \in \mathbb{N},$

$$\lim_{n} \sum_{k} f_{nk} = \lim_{n} \left( \frac{f_{n} - f_{n+1}}{f_{n+1} - f_{n}} \right) = \frac{1}{\phi} - \phi = l$$

such that $l \in \mathbb{C}$ exists. $\square$

Theorem 13. $cs(F) \cap \bigcap c$ is not valid.

Proof. Let $x = (x_{k}) = (f_{2k+1})$. Then $y = Fx = (1,0,0,...) \in cs$. On the other hand, $f_{2k+1} \to \infty$ as $k \to \infty$. It is clear $x \in cs(F)$, but $x \notin c$. This result completes the proof. $\square$

Theorem 14. $cs(F) \subset bs(F)$ is valid.

Proof. If $x \in cs(F)$, $y = Fx \in cs$. Hence, $\sum_{k} Fx \in c$. Since $c \subset \ell_{\infty}$, $\sum_{k} Fx \in l_{\infty}$. Hence, $Fx \in bs$. That is, $x \in bs(F)$. This result completes the proof. $\square$

Let us take normed space $A$ and let $(a_{k}) \in A$. If there is only one scalar sequence $(v_{k})$ such that $y = \sum_{k=0}^{\infty} v_{k}a_{k}$ and $\lim_{n \to \infty} \| y - \sum_{k=0}^{n} v_{k}a_{k} \| = 0$ then $(a_{k})$ is called a Schauder base for $A$.

Now, let us give corollary related to Schauder basis.

Corollary 15. Let a sequence $u^{(k)} = \left\{ u_{n}^{(k)} \right\}_{n \in \mathbb{N}}$ in $cs(F)$ be for each $k \in \mathbb{N}$ and

$$u_{n}^{(k)} = \begin{cases} \frac{f_{k+1}}{f_{k}}, & 0 \leq k \leq n \\ 0, & n < k. \end{cases}$$

Then $\left\{ u_{n}^{(k)} \right\}_{n \in \mathbb{N}}$ is a base for $cs(F)$. Every $x \in cs(F)$ can write as a single $x = \sum_{k=0}^{\infty} y_{k}u_{k}$ such that $y_{k} = (Fx)_{k}$. 
2.2. The Duals of cs(F) and bs(F) and Matrix Transformations

Let us give the two lemmas to use in the next stage.

Lemma 16. Let infinite matrix $C = (c_{nk}) \frac{1}{2}$ and $a = (a_n) \in w$. Let us take $C = aF^{-1}$, that is,

$$c_{nk} = \begin{cases} a_n f_{nk}^{-1}, & 0 \leq k \leq n \\ 0, & n < k \end{cases}$$

for all $k, n \in \mathbb{N}, \delta \in \{cs, bs\}$. Then, $a \in \{\delta(F^{-1})\}^\delta$ iff $C \in \{\delta, I_1\}$.

Proof. Let $x = (x_n)$ and $a = (a_n)$ elements of $w$. $y = (y_n)$ such that $y = Fx$ which is defined in Equation (4). If we use to Equation (4), then

$$a_n x_n = a_n(F^{-1}y)_n = (Cy)_n. \quad (25)$$

If we consider Lemma 9, Lemma 16 and Lemma 17 together, the following is obtained;

Corollary 18. Let us take $r = (r_k) \in w$ and infinite matrix $A = (a_{nk})$ and $B = (b_{nk})$ such that

$$a_{nk} = \begin{cases} r_n f_{nk}^{-1}, & 0 \leq k \leq n \\ 0, & n < k \end{cases} \quad \text{and} \quad b_{nk} = \sum_{j=k}^{n} \frac{f_{jk}^2}{f_{jk}^2 + 1} r_j.$$

If we take $d_1, d_2, d_3, d_4, d_5, d_6, d_7$ and $d_8$ as follows:

\begin{align*}
    d_1 &= \left\{ r = (r_k) \in w : \sup_{N,k \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} (a_{nk} - a_{n,k+1}) \right| < \infty \right\}, \\
    d_2 &= \left\{ r = (r_k) \in w : \sup_{N,k \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} (a_{nk} - a_{n,k-1}) \right| < \infty \right\}, \\
    d_3 &= \left\{ r = (r_k) \in w : \lim_{k \to \infty} c_{nk} = 0 \right\}, \\
    d_4 &= \left\{ r = (r_k) \in w : \exists \alpha \in C : \lim_{n \to \infty} (b_{nk} - b_{n,k+1}) = \alpha \text{ for all } k \in \mathbb{N} \right\}, \\
    d_5 &= \left\{ r = (r_k) \in w : \lim_{n \to \infty} \sum_{k} |b_{nk} - b_{n,k+1}| = \sum_{k} |\lim_{n \to \infty} (b_{nk} - b_{n,k+1})| \right\}, \\
    d_6 &= \left\{ r = (r_k) \in w : \exists \alpha \in C : \lim_{n \to \infty} b_{nk} = \alpha \text{ for all } k \in \mathbb{N} \right\}, \\
    d_7 &= \left\{ r = (r_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k} |b_{nk} - b_{n,k+1}| < \infty \right\}, \\
    d_8 &= \left\{ r = (r_k) \in w : \sup_{n \in \mathbb{N}} \left| \lim_{k \to \infty} b_{nk} \right| < \infty \right\}.
\end{align*}
Then,

(1) \((bs(F))^0 = d_1\)

(2) \((cs(F))^0 = d_2\)

(3) \((bs(F))^0 = d_3 \cap d_4 \cap d_5\)

(4) \((cs(F))^0 = d_6 \cap d_7\)

(5) \((bs(F))^1 = d_3 \cap d_7\)

(6) \((cs(F))^1 = d_9 \cap d_8\).

**Theorem 19.** Let \(\mu \in \{cs, bs\}\) and \(\lambda \subset w\). Then, \(U = (u_{nk}) \in (\mu(F), \lambda)\) iff

\[ V^m = \{v_{nk}^{(m)}\} \in (\mu, c) \text{ for all } n \in \mathbb{N} \]

(26)

where

\[ v_{nk}^{(m)} = \begin{cases} \sum_{j=k}^{m} \frac{f_j}{j!} u_{nj}, & 0 \leq k \leq m \\ 0, & m < k \end{cases} \]

(28)

and

\[ v_{nk} = \sum_{j=k}^{\infty} \frac{f_j}{j!} u_{nj} \]

(29)

for all \(k, m, n \in \mathbb{N}\).

**Proof.** Necessity part: Let us take that \(A = (a_{nk}) \in (\mu(F), \lambda)\) and \(x = (x_k) \in \mu(F)\). If we use Equation (5), then we find

\[ \sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} a_{nk} \sum_{j=k}^{m} \frac{f_j}{j!} y_j = \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{f_j}{j!} a_{nk} y_k = \sum_{k=0}^{m} a_{nk}^{(m)} y_k = D_{n}^{(m)}(y). \]

(30)

According to the hypothesis, for each \(m \in \mathbb{N}\), \(A_m(x) \in c\). Then, \(V^{(m)} \in c\) for each \(m \in \mathbb{N}\) and \(V^{(m)} \in (\mu, c)\). \(Ax = Vy\) if we consider for \(m \to \infty\) from Equation (30). As a result, we find \(V = (v_{nk}) \in (\mu, \lambda)\).

Sufficient part: Suppose that Equations (26) and (27) are satisfied and \(x = (x_k) \in \mu(F)\) be.

By using Corollary 18 and Equations (26) and (30), we have that \(y = Fx \in \mu\) and

\[ V_{n}^{(m)}(y) = \sum_{k=0}^{m} v_{nk}^{(m)} y_k = \sum_{k=0}^{m} a_{nk} x_k = A_{n}^{(m)}(x) \in c. \]

Hence, \(A = (a_{nk})_{k \in \mathbb{N}}\) exists. \(Ax = Vy\) if we consider for \(m \to \infty\) from Equation (30). Consequently, we find \(A = (a_{nk}) \in (\mu(F), \lambda)\). \(\Box\)

**Theorem 20.** Let \(\mu \in \{bs, cs\}\) and \(\lambda \subset w\) and \(U = (u_{nk})\) and \(B = (b_{nk})\) be infinite matrices. Let

\[ b_{nk} := \frac{f_n}{f_{n+1}} u_{nk} - \frac{f_{n+1}}{f_n} u_{n-1,k}. \]

(31)

Then, \(U \in (\lambda, \mu(F))\) iff \(B \in (\lambda, \mu)\).
Proof. Let $z = (z_k) \in l$ and Equation (31) exist. Then, we have
\[
\sum_{k=0}^{m} b_{nk} z_k = \sum_{k=0}^{m} \left( \frac{f_n}{f_{n+1}} a_{nk} - \frac{f_{n+1}}{f_n} a_{n-1,k} \right) z_k.
\] (32)

If we take $m \to \infty$ to Equation (32), we have that $(Bz)_n = (F(Az))_n$. Consequently, $Az \in \mu(F)$ iff $Bz \in \mu$. That is, $B \in (\lambda, \mu)$. □

Let us give almost convergent sequences space, which was first defined by Lorentz [42]. Let
t = (t_k) \in \ell_\infty. t is almost convergent to limit \ell iff \lim_{m \to \infty} \sum_{k=0}^{m} t_{n+k} = \alpha \text{ uniformly in } n.
It is denoted by \( \hat{c}^{-}\lim t = \alpha \). In addition, \( \hat{c}s \) and \( \hat{c}0 \) mean the spaces of almost convergent series and almost null sequences, respectively. \( \hat{c}0 \) and \( \hat{c} \) are
\[
\hat{c}0 = \{ x = (x_k) \in \ell_\infty : \lim_{m \to \infty} \sum_{k=0}^{m} x_{n+k} = 0 \text{ uniformly in } n \},
\]
\[
\hat{c} = \{ x = (x_k) \in \ell_\infty : \exists \ell \in \mathbb{C} \ni \lim_{m \to \infty} \sum_{k=0}^{m} x_{n+k} = \alpha \text{ uniformly in } n \}.
\]

Now, let us take infinite matrix $R = (r_{nk})$ and list the following:
\[
\exists a_k \in \mathbb{C} \ni f - \lim r_{nk} = a_k \text{ for each } k \in \mathbb{N},
\] (33)
\[
\lim \frac{1}{q} \sum_{k=0}^{q} \Delta \left[ \sum_{j=0}^{q} (r_{jk} - a_k) \right] = 0 \text{ uniformly in } n,
\] (34)
\[
\sup_{n \in \mathbb{N}} \Delta \left[ \sum_{j=0}^{n} r_{jk} \right] < \infty,
\] (35)
\[
\exists a_k \in \mathbb{C} \ni f - \lim \sum_{j=0}^{n} r_{jk} = a_k \text{ for each } k \in \mathbb{N},
\] (36)
\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} r_{jk} < \infty,
\] (37)
\[
\exists a_k \in \mathbb{C} \ni \sum_{n} r_{nk} = a_k \text{ for each } k \in \mathbb{N},
\] (38)
\[
\lim \sum_{k} \Delta \left[ \sum_{j=0}^{n} (r_{jk} - a_k) \right] = 0,
\] (39)
\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=0}^{n} r_{jk} \right|^p < \infty, \quad q = \frac{p}{p-1},
\] (40)
\[
\sup_{m,n \in \mathbb{N}} \left| \sum_{k=0}^{m} r_{nk} \right| < \infty,
\] (41)
\[
\sup_{m,l \in \mathbb{N}} \left| \sum_{n=0}^{m} \sum_{k=0}^{l} r_{nk} \right| < \infty,
\] (42)
\[
\sup_{m, l \in \mathbb{N}} \left| \sum_{n=0}^{m} \sum_{k=0}^{l} r_{nk} \right| < \infty,
\] (43)
Lemma 21. Let infinite matrix $R = (r_{nk})$ for all $k, n \in \mathbb{N}$. Then,

1. $R = (r_{nk}) \in (\mathfrak{c}, \mathfrak{c})$ iff Equation (23) and Equations (37)–(39) hold \cite{43}.
2. $R = (r_{nk}) \in (\mathfrak{c}, \mathfrak{c})$ iff Equation (11) and Equation (33) hold \cite{44}.
3. $R = (r_{nk}) \in (\mathfrak{b}, \mathfrak{c})$ iff Equations (8), (11), (33) and (34) hold \cite{45}.
4. $R = (r_{nk}) \in (\mathfrak{b}, \mathfrak{c})$ iff Equations (8) and (34)–(36) hold \cite{45}.
5. $R = (r_{nk}) \in (\mathfrak{c}, \mathfrak{c})$ iff Equation (35) and Equation (36) hold \cite{44}.
6. $R = (r_{nk}) \in (l_{\infty}, \mathfrak{b}) = (\mathfrak{c}, \mathfrak{b})$ iff Equation (37) holds \cite{38}.
7. $R = (r_{nk}) \in (l_p, \mathfrak{b})$ iff Equation (40) holds \cite{46}.
8. $R = (r_{nk}) \in (l, \mathfrak{b})$ iff Equation (41) holds \cite{38}.
9. $R = (r_{nk}) \in (\mathfrak{b}v, \mathfrak{b})$ iff Equation (42) holds \cite{38}.
10. $R = (r_{nk}) \in (\mathfrak{b}v_0, \mathfrak{b})$ iff Equation (43) holds \cite{46}.
11. $R = (r_{nk}) \in (l_{\infty}, \mathfrak{c})$ iff Equation (44) holds \cite{38}.
12. $R = (r_{nk}) \in (\mathfrak{c}, \mathfrak{c})$ if and only Equations (10), (37) and (45) hold \cite{38}.
13. $R = (r_{nk}) \in (\mathfrak{c}0, \mathfrak{c})$ iff Equations (9) and (46) hold \cite{38}.
14. $R = (r_{nk}) \in (l_{p}, \mathfrak{c})$ iff Equations (10) and (40) hold \cite{46}.
15. $R = (r_{nk}) \in (l, \mathfrak{c})$ iff Equations (10) and (41) hold \cite{46}.
16. $R = (r_{nk}) \in (\mathfrak{b}v, \mathfrak{c})$ if and only Equations (10), (41) and (43) hold \cite{38}.
17. $R = (r_{nk}) \in (\mathfrak{b}v_0, \mathfrak{c})$ iff Equations (10) and (43) hold \cite{46}.

Now, suppose $v_{nk}$ and $v_{nk}^{(m)}$ which mentioned Equations (28) and (29) and give the following equations

\[
\lim_{m \to \infty} \sum_{n=0}^{\infty} \left| \sum_{k} r_{nk} \right| = 0, \tag{44}
\]

\[
\sum_{n} \sum_{k} r_{nk} \text{ convergent} \tag{45}
\]

\[
\lim_{m \to \infty} \sum_{n=0}^{m} (r_{nk} - r_{n,k+1}) = a \text{ for each } k \in \mathbb{N}, a \in \mathbb{C}. \tag{46}
\]
If we consider Theorems 19 and 20, Lemmas 9 and 21, then we give the below conclusions.

**Corollary 22.** Let us take $U = (u_{nk})$ mentioned in Theorem 19. Then,

1. $U = (u_{nk}) \in (bs(F), c_0)$ iff Equations (47) and (49) hold and Equation (53) holds with $l = 0$.
2. $U = (u_{nk}) \in (bs(F), c_{00})$ iff Equations (47)–(50) and Equation (55) hold.
Corollary 23. Let us take $U = (u_{nk})$ mentioned in Theorem 19. Then,

1. $U = (u_{nk}) \in (bs(F), c)$ iff Equations (47)–(50) and Equations (52) and (53) hold.
2. $U = (u_{nk}) \in (bs(F), c_0)$ iff Equations (47)–(50) and Equation (56) hold.
3. $U = (u_{nk}) \in (bs(F), l_n)$ iff Equations (47)–(51) hold.
4. $U = (u_{nk}) \in (bs(F), bs)$ iff Equations (47)–(50) and Equation (54) hold.
5. $U = (u_{nk}) \in (bs(F), l_1)$ iff Equations (47)–(50) and Equation (58) hold.
6. $U = (u_{nk}) \in (bs(F), l_0)$ iff Equations (47)–(50) and Equation (57) hold.
7. $U = (u_{nk}) \in (bs(F), bv)$ iff Equations (57) and (47)–(49) and Equation (51) hold and Equation (53) also holds with $l = 0$.

Corollary 24. Let us take $U = (u_{nk})$ mentioned Theorem 19. Then,

1. $U = (u_{nk}) \in (bs(F), \hat{c})$ iff Equations (47)–(51), (67) and (69) hold.
2. $U = (u_{nk}) \in (bs(F), \hat{c}_0)$ iff Equations (47)–(51) hold and Equations (68) and (69) also hold with $v_k = 0$ in Equation (67) and $l_k = 0$ in Equation (69).
3. $U = (u_{nk}) \in (bs(F), c)$ iff Equations (59), (60), (67) and (51) hold.
4. $U = (u_{nk}) \in (bs(F), c_0)$ iff Equations (59), (60), Equation (64) and Equation (65) hold.
5. $U = (u_{nk}) \in (bs(F), l_n)$ iff Equations (51) and (59)–(61) hold.
6. $U = (u_{nk}) \in (bs(F), bs)$ iff Equations (59), (60), (54) and (63) hold.
7. $U = (u_{nk}) \in (bs(F), l_1)$ iff Equations (59), (60) and (66) hold.
8. $U = (u_{nk}) \in (bs(F), bv)$ iff Equations (59), (60) and (68) hold.
9. $U = (u_{nk}) \in (bs(F), bv_0)$ iff Equations (59), (60) and (62) hold and Equation (68) holds with $v_k = 0$ for all $k \in \mathbb{N}$.

Corollary 25. Let us take $U = (u_{nk})$ mentioned Theorem 20. Then,

1. $U = (u_{nk}) \in (l_p, bs(F))$ iff Equation (40) holds with $b_{nk}$ instead of $r_{nk}$, where $b_{nk}$ is defined by Equation (31).
2. $U = (u_{nk}) \in (l_{10}, bs(F)) = (c, bs(F)) = (c_0, bs(F))$ iff Equation (37) holds with $b_{nk}$ instead of $r_{nk}$, where $b_{nk}$ is defined by Equation (31).
3. $U = (u_{nk}) \in (l_1, bs(F))$ iff Equation (41) holds with $b_{nk}$ instead of $r_{nk}$, where $b_{nk}$ is defined by Equation (31).
4. $U = (u_{nk}) \in (bv, bs(F))$ iff Equation (42) holds with $b_{nk}$ instead of $r_{nk}$, where $b_{nk}$ is defined by Equation (31).
5. $U = (u_{nk}) \in (bv_0, bs(F))$ iff Equation (43) holds with $b_{nk}$ instead of $r_{nk}$, where $b_{nk}$ is defined by Equation (31).
6. $U = (u_{nk}) \in (l_{10}, cs(F))$ iff Equation (44) holds with $b_{nk}$ instead of $r_{nk}$, where $b_{nk}$ is defined by Equation (31).
7. $U = (u_{nk}) \in (c, cs(F))$ iff (10), Equations (37) and (47) hold $b_{nk}$ instead of $r_{nk}$, where $b_{nk}$ is defined by Equation (31).
8. $U = (u_{nk}) \in (c_0, cs(F))$ iff Equations (9) and (46) hold with $b_{nk}$ instead of $r_{nk}$, where $b_{nk}$ is defined by Equation (31).
9. $U = (u_{nk}) \in (l_p, cs(F))$ iff Equations (10) and (40) hold with $b_{nk}$ instead of $r_{nk}$, where $b_{nk}$ is defined by Equation (31).
were examined and found. Finally, the \( \gamma \), where \( x \), were introduced by Candan [28] in 2015. In 2015, the sequence space was read and approved by both authors.

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### References

7. Aydin, C.; Başar, F. On the new sequence spaces which include the spaces \( c_0 \) and \( c \). *Hokkaido Math. J.* **2004**, *33*, 1–16. [CrossRef]


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