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Generalized Fractional Integral Operators Pertaining to the Product of Srivastava's Polynomials and Generalized Mathieu Series

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Abstract: Fractional calculus image formulas involving various special functions are important for evaluation of generalized integrals and to obtain the solution of differential and integral equations. In this paper, the Saigo's fractional integral operators involving hypergeometric function in the kernel are applied to the product of Srivastava's polynomials and the generalized Mathieu series, containing the factor $x^{\lambda}(x^k + c^k)^{-\rho}$ in its argument. The results are expressed in terms of the generalized hypergeometric function and Hadamard product of the generalized Mathieu series. Corresponding special cases related to the Riemann–Liouville and Erdélyi–Kober fractional integral operators are also considered.

Keywords: generalized fractional integral operators; generalized Mathieu series; Srivastava's polynomial; generalized hypergeometric series

1. Introduction and Preliminaries

Recently, the integral representation of the generalized power series of Mathieu-type was studied and defined by Tomovski and Pogány [1] as follows

$$S_\tau(p;z) = \sum_{n\geq 1} \frac{2nz^n}{(p^2+n^2)^{\tau+1}} \qquad (\tau > 0,\ p \in \mathbb{R}). \tag{1}$$

Some of the special cases of Equation (1) listed as under

$$S_\tau(p;1) = S_\tau(p) \quad \text{and} \quad \tilde{S}_\tau(p;-1) = -S_\tau(p),$$

$$S_\tau(p) = \sum_{n\geq 1} \frac{2n}{(p^2+n^2)^{\tau+1}}, \qquad (\tau > 0,\ p > 0), \tag{2}$$

$$\tilde{S}_\tau(p) = \sum_{n\geq 1} (-1)^{n-1} \frac{2n}{(p^2+n^2)^{\tau+1}}, \qquad (\tau > 0,\ p > 0). \tag{3}$$

For several interesting special cases of the generalized Mathieu series and their fundamental properties, along with integral representations, one may refer to the works of Cerone and Lenard [2] and Milovanović and Pogány [3]. The Mathieu series has been broadly acknowledged in the theory of mathematical analysis (for instance, see Cerone and Lenard [2], Diananda [4] and Pogány et al. [5]). Further, one can also find numerous applications in recent articles [6–14].

The general class of polynomials was defined by Srivastava ([15], p.1, Equation (1)) in the subsequent way:

$$S_w^u[x] = \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\, x^s, \quad w = 0, 1, 2, \ldots, (w \in \mathbb{N}_0 = \mathbb{N} \cup 0; u \in \mathbb{N}), \tag{4}$$

and the coefficients $A_{w,s}$ $(w, s \geq 0)$ are arbitrary constants, real or complex.

The generalized Wright hypergeometric function is given by the series ([16–18]) (see also, [19]):

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c} (a_i,\ c_i)_{1,\ p} \\ (b_j,\ d_j)_{1,\ q} \end{array} \Big| z\right] = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i +\ c_i r)\, z^r}{\prod_{j=1}^{q} \Gamma(b_j +\ d_j r)\, r!}, \tag{5}$$

where $a_i,\ b_j \in \mathbb{C}$ and $c_i,\ d_j \in\ \Re$, $(c_i,\ d_j \neq 0; i = 1,\ 2,\ \ldots,\ p; j = 1,\ 2,\ \ldots, q)$.

Here, we recall the generalized hypergeometric function (see [20], Section 4.1(1)) as under

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r\, z^r}{(b_1)_r \cdots (b_q)_r r!}, \tag{6}$$

provided the coefficients $a_i,\ b_j \in \mathbb{C}$, however $b_j \neq 0, -1, \cdots (i = 1, \cdots, p; j = 1, \cdots, q)$. The above series converges, if $p = q + 1$ for all $|z| < 1$ and $p \leq q$ for any z. For our investigation, we express the main results in terms of series defined as Equation (6).

Moreover, if we take $\vartheta_1 = \cdots = \vartheta_p = \varphi_1 = \cdots = \varphi_q = 1$, then we have

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)}\, {}_p\psi_q\left[\begin{array}{c} (a_i,\ 1)_{1,p} \\ (b_j,\ 1)_{1,q} \end{array} \Big| z\right]. \tag{7}$$

For our purpose, we also need the concept of Hadamard product of two functions. Assume that $f(z) := \sum_{q=0}^{\infty} a_q z^q$ and $g(z) := \sum_{q=0}^{\infty} b_q z^q$ are dual power series, whose radii of convergence are given by R_f and R_g, jointly. The power series is defined in the form of Hadamard product (see [21]) as

$$(f * g)(z) := \sum_{q=0}^{\infty} a_q b_q z^q. \tag{8}$$

If R is considered as radius of convergence for the above Hadamard product series in Equation (8), it must satisfies the condition $R_f \cdot R_g \leq R$. It is impressive to note that, if one of the power series characterizes as an entire function, then the Hadamard product series also defines an entire function.

Following the work of Saxena and Parmar [22], our aim is to study the novel combination of the Saigo's fractional integral operators involving the product of Srivastava's polynomials and the generalized Mathieu series. The results are general in nature and expressed in terms of the generalized hypergeometric function and Hadamard product of the generalized Mathieu series. We also include certain special cases of our results as corresponding image formulas for Riemann–Liouville and Erdélyi–Kober fractional integral operators.

2. Generalized Fractional Integration of the Mathieu Series

Let ϑ, φ and η be complex numbers, and further let $x \in \mathbb{R}_+ = (0, \infty)$. Following Saigo [23], the fractional integral $(\Re(\vartheta) > 0)$ and the fractional derivative $(\Re(\vartheta) < 0)$ of the function $f(x)$ on $\mathbb{R}_+$ are defined by

$$\left(I_{0+}^{\vartheta,\varphi,\eta} f\right)(x) = \frac{x^{-\vartheta-\varphi}}{\Gamma(\vartheta)} \int_0^x (x-t)^{\vartheta-1} {}_2F_1(\vartheta+\varphi, -\eta; \vartheta; 1-t/x) f(t)dt, (\Re(\vartheta) > 0); \tag{9}$$

$$\left(D_{0+}^{\vartheta,\varphi,\eta}f\right)(x)=\left(I_{0+}^{-\vartheta,-\varphi,\vartheta+\eta}f\right)(x)$$
$$=\frac{d^k}{dx^k}\left(I_{0+}^{-\vartheta+k,-\varphi-k,\vartheta+\eta-k}f\right)(x),\quad \Re(\vartheta)\le 0, k=[\Re(\vartheta)]+1, \tag{10}$$

$$\left(I_{-}^{\vartheta,\varphi,\eta}f\right)(x)=\frac{1}{\Gamma(\vartheta)}\int_x^{\infty}(t-x)^{\vartheta-1}t^{-\vartheta-\varphi}{}_2F_1(\vartheta+\varphi,-\eta;\vartheta;1-x/t)f(t)dt,(\Re(\vartheta)>0); \tag{11}$$

$$\left(D_{-}^{\vartheta,\varphi,\eta}f\right)(x)=\left(I_{-}^{-\vartheta,-\varphi,\vartheta+\eta}f\right)(x)$$
$$=(-1)^k\frac{d^k}{dx^k}\left(I_{-}^{-\vartheta+k,-\varphi-k,\vartheta+\eta}f\right)(x),\quad \Re(\vartheta)\le 0, k=[\Re(\vartheta)]+1. \tag{12}$$

It can be easily seen that the Riemann–Liouville and Erdélyi–Kober fractional integral operators are special cases of Saigo's operators. The symbol $\Gamma\left[\begin{smallmatrix} a\,b\,c\,\ldots \\ d\,e\,f\,\ldots\end{smallmatrix}\right]$ represents the fraction of the product of gamma functions $\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\ldots}{\Gamma(d)\,\Gamma(e)\,\Gamma(f)\,\ldots}$.

Now, we begin with the following statement:

If

$$f(t)=t^{\lambda}(t^k+c^k)^{-\rho}\; S_w^u[yt^{\mu}(t^k+c^k)^{-v}]\; S_{\tau}\left(p;zt^h(t^k+c^k)^{-\delta}\right) \tag{13}$$

then, we have the following relations:

Theorem 1. *Let $\Re(\vartheta)>0$, $\lambda>0$, $k=1,2,3,\ldots,$wherec is a positive number and ρ is a complex number, then there holds the relation*

$$\left(I_{0+}^{\vartheta,\varphi,\eta}f\right)(x)=\sum_{s=0}^{[w/u]}\sum_{n\ge1}\frac{2n\,(-w)_{us}\,A_{w,s}\;y^s\,z^n}{(p^2+n^2)^{\tau+1}\,s!\,c^R}x^{T-\varphi}$$
$$\times\frac{\Gamma(T+1)\Gamma(T+\eta-\varphi+1)}{\Gamma(T-\varphi+1)\Gamma(T+\vartheta+\eta+1)}$$
$$\times\;{}_{2k+1}F_{2k}\left[\begin{matrix}\rho+vs-\delta n,\Delta(k,T+1),\Delta(k,T-\varphi+\eta+1); \\ \Delta(k,T+1-\varphi),\Delta(k,T+\vartheta+\eta+1);\end{matrix}\;-\frac{x^k}{c^k}\right], \tag{14}$$

where $T=\lambda+\mu s+hn$ and $R=k\rho+kvs-k\delta n$.

The result in Equation (14) is valid for $\Re(\vartheta)>0$, $\Re(\lambda+\mu s+hn)>0$. In addition, c is a positive number and ρ,μ,v,h,δ are complex numbers, $k=1,2,3,\ldots,u$ is an arbitrary positive integer and the coefficients $A_{w,s}$(w, s ≥ 0) are arbitrary constants, real or complex. Here, $\Delta(k,\vartheta)$ represents the sequence of parameters

$$\frac{\vartheta}{k},\frac{\vartheta+1}{k},\ldots,\frac{\vartheta+k-1}{k},$$

and ${}_{2k+1}F_{2k}(\cdot)$ is the generalized hypergeometric function, defined in [24].

Proof. Let ℓ be the left-hand side of result in Equation (14). Using Equations (4) and (1) and applying Equations (13)–(9), we have

$$\ell=\sum_{s=0}^{[w/u]}\sum_{n\ge1}\frac{2n\,(-w)_{us}\,A_{w,s}\;y^s\,z^n}{(p^2+n^2)^{\tau+1}\,s!}$$
$$\times\frac{x^{-\vartheta-\varphi}}{\Gamma(\vartheta)}\int_0^x(x-t)^{\vartheta-1}{}_2F_1(\vartheta+\varphi,-\eta;\vartheta;1-t/x)\,t^{\lambda+\mu s+hn}(t^k+c^k)^{-(\rho+vs-\delta n)}dt. \tag{15}$$

Further, on employing the Gauss hypergeometric function and series formula, namely

$$(t^k + c^k)^{-\rho} = c^{-k\rho} \sum_{q=0}^{\infty} \frac{(\rho)_q}{q!} \left(-\frac{t^k}{c^k}\right)^q$$

in Equation (15), we get

$$= \sum_{s=0}^{[w/u]} \sum_{n\geq 1} \frac{2n\,(-w)_{us}\, A_{w,s}\; y^s\, z^n}{(p^2+n^2)^{\tau+1}\, s!}$$

$$\times \frac{x^{-\vartheta-\varphi}}{\Gamma(\vartheta)} \sum_{l=0}^{\infty} \frac{(\vartheta+\varphi)_l\,(-\eta)_l}{(\vartheta)_l\, l!} x^{-l} \sum_{q=0}^{\infty} \frac{(\rho+vs-\delta n)_q}{q!} c^{-k(\rho+vs-\delta n)} \left(-\frac{1}{c^k}\right)^q$$

$$\times \int_0^x t^{\lambda+\mu s+hn+kq} (x-t)^{\vartheta+l-1} dt.$$

On interchanging the order of integration and summation, which is valid under the given conditions, evaluating the inner integral by means of the formula

$$\int_0^x t^{\lambda+kq}\,(x-t)^{\vartheta+p-1}\,dt = x^{\vartheta+\lambda+p+kq} \frac{\Gamma(\vartheta+p)\,\Gamma(\lambda+kq+1)}{\Gamma(\vartheta+p+\lambda+kq+1)}, \tag{16}$$

and performing some simplification, we get

$$= \sum_{s=0}^{[w/u]} \sum_{n\geq 1} \frac{2n\,(-w)_{us}\, A_{w,s}\; y^s\, z^n}{(p^2+n^2)^{\tau+1}\, s!\, c^R} x^{T-\varphi}$$

$$\times \sum_{l=0}^{\infty} \frac{(\vartheta+\varphi)_l\,(-\eta)_l}{(T+\vartheta+kq+1)_l\, l!} \sum_{q=0}^{\infty} \frac{(\rho+vs-\delta n)_q}{q!} \frac{\Gamma(T+1+kq)}{\Gamma(T+\vartheta+kq+1)} \left(-\frac{x^k}{c^k}\right)^q. \tag{17}$$

Now, by employing the Gauss theorem and multiplication formula (see, Rainville ([25], pp. 49, 24–29)) in Equation (17), we arrive at the right-hand side of Equation (14). □

Again, by considering the another function in the form

$$f(t) = t^{\lambda}(t^k+c^k)^{-\rho}\; S_w^u[yt^{\mu}(t^k+c^k)^{-v}]\; S_{\tau}\left(p; zt^{-h}(t^k+c^k)^{-\delta}\right) \tag{18}$$

we deduce the following result:

Theorem 2. *Let $\Re(\vartheta) > 0, \lambda > 0, k = 1, 2, 3, \ldots,$ where c is a positive number and ρ is a complex number, then there holds the relation*

$$\left(I_{-}^{\vartheta,\varphi,\eta} f\right)(x) = \sum_{s=0}^{[w/u]} \sum_{n\geq 1} \frac{2n\,(-w)_{us}\, A_{w,s}\; y^s\, z^n}{(p^2+n^2)^{\tau+1}\, s!\, c^R} x^{T'-\varphi}$$

$$\times \frac{\Gamma(\varphi-T')\Gamma(\eta-T')}{\Gamma(-T')\Gamma(\vartheta+\varphi+\eta-T')}$$

$$\times\; {}_{2k+1}F_{2k}\left[\begin{matrix} \rho+vs-\delta n, \Delta(k, T'+1), \Delta(k, T'-\vartheta-\varphi-\eta+1); \\ \Delta(k, T'-\eta+1), \Delta(k, T'-\varphi+1); \end{matrix}\; -\frac{x^k}{c^k}\right]. \tag{19}$$

Here, c is a positive number and ρ, μ, v, h, δ are complex numbers, $k = 1, 2, 3, \ldots,$ and $T' = \lambda + \mu s - hn$. The result in Equation (19) is valid for $\Re(\vartheta) > 0$, $\Re(\lambda + \mu s - hn) > 0$, u is an arbitrary positive integer and the coefficients $A_{w,s}(\mathrm{w}, \mathrm{s} \geq 0)$ are arbitrary constants, real or complex.

Proof. Suppose ℓ is the left-hand side of Equation (19), then we can write

$$\ell = \sum_{s=0}^{[w/u]} \sum_{n \geq 1} \frac{2n\,(-w)_{us}\,A_{w,s}\,y^s\,z^n}{(p^2+n^2)^{\tau+1}\,s!}$$

$$\times \frac{1}{\Gamma(\vartheta)} \int_x^\infty (t-x)^{\vartheta-1} t^{-\vartheta-\varphi} {}_2F_1(\vartheta+\varphi, -\eta; \vartheta; 1-x/t) t^{\lambda+\mu s-hn} (t^k+c^k)^{-(\rho+vs-\delta n)} dt. \tag{20}$$

Following a similar fashion as in the proof of Theorem 1, expressing the series expansion for the Gauss hypergeometric function and binomial series $\left(t^k+c^k\right)^{-\rho}$, interchanging the order of integration and summation, evaluating the inner integral by means of the formula

$$\int_x^\infty t^{\lambda-\vartheta-\varphi-n+kq}\,(t-x)^{\vartheta+n-1}\,dt = x^{\lambda-\varphi+kq} \frac{\Gamma\,(\vartheta+n)\,\Gamma\,(\varphi-\lambda-kq)}{\Gamma\,(\vartheta+\varphi+n-\lambda-kq)}, \tag{21}$$

using the relation $(\vartheta)_n = \frac{(-1)^n}{(1-\vartheta)_n}$, and further employing the Gauss theorem and multiplication formula, we easily obtain the right-hand side of Equation (19). □

In a different manner, on setting $\rho = v = \delta = 0$, $y = z = 1$ and $\lambda = \lambda - 1$ in Equations (13) and (18), and using the Hadamard product in Equation (8), then in view of Equations (1) and (5), we attain the following typical cases of Theorems 1 and 2, respectively.

Corollary 1. *Let $\vartheta, \varphi, \gamma, \lambda \in \mathbb{C}$ and $\lambda > 0$, $\tau > 0$ $p \in \mathbb{R}$ be such that $\Re(\vartheta) > 0$ and $\Re(\lambda) > max[0, \Re(\varphi - \gamma)]$. Then, the following result holds true:*

$$\left(I_{0+}^{\vartheta,\varphi,\gamma} f\right)(x) = x^{\lambda+h-\varphi-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}\,A_{w,s}\,x^{\mu s}}{s!}\, S_\tau\left(p; x^h\right)$$

$$\times\, {}_3\psi_2 \left[\begin{array}{c} (\lambda+\mu s+h, h), (\lambda+\eta-\varphi+\mu s+h, h), (1,\,1); \\ (\lambda-\varphi+\mu s+h, h), (\lambda+\vartheta+\eta+\mu s+h, h); \end{array} x^h \right]. \tag{22}$$

Corollary 2. *Let $\vartheta, \varphi, \gamma, \lambda \in \mathbb{C}$ and $\lambda > 0$, $\tau > 0$ $p \in \mathbb{R}$ be such that $\Re(\vartheta) > 0$ and $\Re(\lambda) < 1 + min[\Re(\varphi), \Re(\gamma)]$. Then, the following result holds true:*

$$\left(I_{-}^{\vartheta,\varphi,\gamma} f\right)(x) = x^{\lambda+h-\varphi-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}\,A_{w,s}\,x^{\mu s}}{s!}\, S_\tau\left(p; x^{-h}\right)$$

$$\times\, {}_3\psi_2 \left[\begin{array}{c} (1-\lambda-\mu s+\varphi+h, h), (1-\lambda-\mu s+\eta+h, h), (1,\,1); \\ (1-\lambda-\mu s+h, h), (1-\lambda+\vartheta+\varphi+\eta-\mu s+h, h); \end{array} \frac{1}{x^h} \right]. \tag{23}$$

Remark 1. *If we set $w = 0$, $A_{0,0} = 1$ then $S_0^u[x] \to 1$ in Corollaries 1 and 2, we can deduce the known result given by Sexana and Parmar ([22], Equations (31) and (33)).*

3. Interesting Special Cases

(I) When $\varphi = -\vartheta$, the operators in Equations (9) and (11) coincide with the classical Riemann–Liouville fractional integrals of order $\vartheta \in \mathbb{C}$ with $x > 0$ (see, e.g., [26]) as follows

$$\left(I_{0+}^{\vartheta,-\vartheta,\eta} f\right)(x) = \left(I_{0+}^{\vartheta} f\right)(x) = \frac{1}{\Gamma(\vartheta)} \int_0^x (x-t)^{\vartheta-1} f(t) dt, \tag{24}$$

$$\left(I_{-}^{\vartheta,-\vartheta,\eta}f\right)(x)=\left(I_{-}^{\vartheta}f\right)(x)=\frac{1}{\Gamma(\vartheta)}\int_x^{\infty}(t-x)^{\vartheta-1}f(t)dt. \tag{25}$$

In view of the above, we can write

$$\left(I_{0+}^{\vartheta}f\right)(x)=\sum_{s=0}^{[w/u]}\sum_{n\geq 1}\frac{2n\,(-w)_{us}\,A_{w,s}\;y^s\,z^n}{(p^2+n^2)^{\tau+1}\,s!\,c^R}x^{T+\vartheta}$$

$$\times\frac{\Gamma(T+1)}{\Gamma(T+\vartheta+1)}{}_{k+1}F_k\left[\begin{array}{c}\rho+vs-\delta n,\Delta(k,T+1);\\ \\ \Delta(k,T+1+\vartheta);\end{array}\; -\frac{x^k}{c^k}\right] \tag{26}$$

and

$$\left(I_{-}^{\vartheta}f\right)(x)=\sum_{s=0}^{[w/u]}\sum_{n\geq 1}\frac{2n\,(-w)_{us}\,A_{w,s}\;y^s\,z^n}{(p^2+n^2)^{\tau+1}\,s!\,c^R}x^{T'+\vartheta}$$

$$\times\frac{\Gamma(-T'-\vartheta)}{\Gamma(-T')}{}_{k+1}F_k\left[\begin{array}{c}\rho+vs-\delta n,\Delta(k,T'+1);\\ \\ (k,T'+\vartheta+1);\end{array}\; -\frac{x^k}{c^k}\right]. \tag{27}$$

(II) For $\varphi=0$, the operators in Equations (9) and (11) yield the so-called Erdélyi–Kober integrals of order $\vartheta\in\mathbb{C}$ with $x>0$ (see, e.g., [26]) as under:

$$\left(I_{0+}^{\vartheta,0,\eta}f\right)(x)=\left(I_{\eta,\vartheta}^{+}f\right)(x)=\frac{x^{-\vartheta-\eta}}{\Gamma(\vartheta)}\int_0^{x}(x-t)^{\vartheta-1}t^{\eta}f(t)dt, \tag{28}$$

$$\left(I_{-}^{\vartheta,0,\eta}f\right)(x)=\left(K_{\eta,\vartheta}^{-}f\right)(x)=\frac{x^{\eta}}{\Gamma(\vartheta)}\int_x^{\infty}(t-x)^{\vartheta-1}t^{-\vartheta-\eta}f(t)dt. \tag{29}$$

Following the above relations, we have

$$\left(I_{\eta,\vartheta}^{+}f\right)(x)=\sum_{s=0}^{[w/u]}\sum_{n\geq 1}\frac{2n\,(-w)_{us}\,A_{w,s}\;y^s\,z^n}{(p^2+n^2)^{\tau+1}\,s!\,c^R}x^{T}$$

$$\times\frac{\Gamma(T+\eta+1)}{\Gamma(T+\vartheta+\eta+1)}{}_{k+1}F_k\left[\begin{array}{c}\rho+vs-\delta n,\Delta(k,T+\eta+1);\\ \\ \Delta(k,T+\vartheta+\eta+1);\end{array}\; -\frac{x^k}{c^k}\right] \tag{30}$$

and

$$\left(K_{\eta,\vartheta}^{-}f\right)(x)=\sum_{s=0}^{[w/u]}\sum_{n\geq 1}\frac{2n\,(-w)_{us}\,A_{w,s}\;y^s\,z^n}{(p^2+n^2)^{\tau+1}\,s!\,c^R}x^{T'}$$

$$\times\frac{\Gamma(\eta-T')}{\Gamma(\vartheta+\eta-T')}{}_{k+1}F_k\left[\begin{array}{c}\rho+vs-\delta n,\Delta(k,T'-\vartheta-\eta+1);\\ \\ \Delta(k,T'-\eta+1);\end{array}\; -\frac{x^k}{c^k}\right]. \tag{31}$$

4. Conclusions

The concept of fractional calculus has been singled out as an outstanding mathematical tool for modelling of relevant systems in various fields of science and engineering. Particularly, the image formulas involving various special functions plays an important role for evaluating the integrals and for providing solution of fractional differential and integral equations. Here, certain generalized fractional integrals of Saigo's type connected with the product of Srivastava's polynomials and the

Mathieu series have been investigated. Analogous results associated with Riemann–Liouville and Erdélyi–Kober fractional integral operators, which have been depicted in corollaries, are also discussed. The results presented in this article are easily converted in terms of a similar type of new interesting integrals with different arguments after some suitable parametric replacements. Further, we conclude with the remark that suitably assigning values to the bounded sequence $A_{w,s}$ $(w, s \geq 0)$, the image formulas given in Theorem 1 and 2 being of general nature, will lead to several integrals involving product of variety orthogonal polynomials and generalized Mathieu series.

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