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Nonlocal Fractional Evolution Inclusions of Order $\alpha \in (1, 2)$

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Abstract: This paper studies the existence of mild solutions and the compactness of a set of mild solutions to a nonlocal problem of fractional evolution inclusions of order $\alpha \in (1, 2)$. The main tools of our study include the concepts of fractional calculus, multivalued analysis, the cosine family, method of measure of noncompactness, and fixed-point theorem. As an application, we apply the obtained results to a control problem.

Keywords: fractional evolution inclusions; mild solutions; condensing multivalued map

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1. Introduction

In the past several decades, there has been a significant development in the theory and applications for fractional evolution equations and inclusions; for example, see the monographs by Miller and Ross [1], Podlubny [2], Kilbas et al. [3], Zhou [4], and the recent papers [5,6]. More recently, time-fractional diffusion and wave equations have been attracting the widespread attention of many fields of science and engineering [7,8]. The interest in the study of these topics arises from the fact that fractional diffusion equations $\alpha \in (0, 1)$ or fractional wave equations $\alpha \in (1, 2)$ can capture some nonlocal aspects of phenomena or systems. Examples of these phenomena include porous media, memory effects, anomalous diffusion, viscoelastic media, and so on. The papers [9–11] cover many of these applications.

By virtue of semigroup theory and the operator theoretical method, some fractional diffusion and wave equations can be abstracted as fractional evolution equations. Bajlekova [12] exploited the concept of the fractional resolvent solution operator to investigate the associated fractional abstract Cauchy problem. A number of papers [13–17] and the references therein were inspired by this concept, and the topic of the existence of mild solutions to fractional abstract equations of order $\alpha \in (1, 2)$ was also studied. For further discussion in [18], the authors considered the controllability results for fractional evolution equations of order $\alpha \in (1, 2)$ by applying the concepts of Mainardi's Wright function (a probability density function) and strongly continuous cosine families.

The study of fractional evolution inclusions of order $\alpha \in (0, 1)$ also gained significant importance (see, e.g., [19,20]). However, the study of fractional evolution inclusions of order $\alpha \in (1, 2)$ supplemented with nonlocal conditions is yet to be initiated. We need to point out that the work spaces are of finite dimension if the strongly continuous cosine families are compact (see, e.g., [21,22]). Motivated by this fact and the above-mentioned works and relying on the known material, we aim to develop a suitable definition for mild solutions of fractional evolution equations in terms of Mainardi's Wright function. For this purpose, we consider the following nonlocal problem of fractional evolution inclusions without further assumptions regarding the compactness of the cosine families or the associated sine families.

$$\begin{cases} {}^C D_t^\alpha x(t) \in Ax(t) + F(t, x(t)), & t \in J = [0, a],\ a > 0, \\ x(0) + g(x) = x_0,\ x'(0) = x_1, \end{cases} \tag{1}$$

where ${}^C D_t^\alpha$ is a Caputo fractional derivative of order $1 < \alpha < 2$; A is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t \geq 0}$ of uniformly bounded linear operators in a Banach space X; $F : [0, a] \times X \to X$ is a multivalued map; g is a given appropriate function; and x_0, x_1 are elements of space X.

Here, we emphasize that the present work is also motivated by an inclusion of the following partial differential model:

$$\begin{cases} \partial_t^\alpha u(t,z) \in \partial_z^2 u(t,z) + F(t, z, u(t,z)), & z \in [0, \pi],\ t \in [0, a], \\ u(t, 0) = u(t, \pi) = 0, & t \in [0, a], \\ u(0, z) + g(u) = u_0(z),\ u'(0, z) = u_1(z), & z \in [0, \pi], \end{cases}$$

where ∂_t^α is a Caputo fractional partial derivative. This model includes a class of fractional wave equations that have a memory effect and are not observed in integer-order differential equations; further, this class of equations indicates the coexistence of finite wave speed and absence of a wavefront (see, e.g., [9]). It is interesting that for the case of $\alpha = 2$, the above fractional partial differential inclusion reduces to a second-order differential inclusion involving one-dimensional wave equations with nonlocal initial-boundary conditions. For the case of $\alpha = 1$ or $\alpha \in (0, 1)$ with the initial value $u_1(z)$ vanished, the model contains the classical diffusion equations or fractional diffusion equations. In addition, these types of equations can be handled by the method of semigroup theory (see, e.g., [20]) but not cosine families.

The rest of this paper is organized as follows. In Section 2, we recall some preliminary concepts related to our study. In Section 3, we establish an existence result for mild solutions of Equation (1) and discuss the compactness of the set of mild solutions. In Section 4, we show the utility of the obtained work by applying it to a control problem.

2. Preliminaries

Let X be a Banach space with the norm $\|\cdot\|$. Denote by $\mathcal{L}(X)$ the space of all bounded linear operators from X to X equipped with the norm $\|\cdot\|_{\mathcal{L}(X)}$. Let $C(J, X)$ denote the space of all continuous functions from J into X equipped with the usual sup-norm $\|x\|_C = \sup_{t \in J} \|x(t)\|$, where $J = [0, a], a > 0$. A measurable function $f : J \to X$ is Bochner integrable if $\|f\|$ is Lebesgue integrable. Let $L^p(J, X)$ $(p \geq 1)$ be the Banach space of measurable functions (defined in the sense of Bochner integral) endowed with the norm

$$\|f\|_p = \left(\int_J \|f(t)\|^p dt \right)^{\frac{1}{p}}.$$

Definition 1. *The fractional integral with the lower limit zero for a function $u : [0, \infty) \to X$ is given by*

$$I_{0+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} u(s) ds, \quad t > 0,\ \alpha \in \mathbb{R}_+,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2. *The Riemann–Liouville derivative with the lower limit zero for a function $u : [0, \infty) \to X$ is defined by*

$$ {}^{L}D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \quad t > 0,\ n-1 < \alpha < n,\ \alpha \in \mathbb{R}_+. $$

Definition 3. *The Caputo derivative with the lower limit zero for a function u is defined by*

$$ {}^{C}D^{\alpha}_{0+}u(t) = {}^{L}D^{\alpha}_{0+}\Big(u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k \Big), \quad t > 0,\ n-1 < \alpha < n, \alpha \in \mathbb{R}_+. $$

Definition 4. [23] *A family of bounded linear operators $\{C(t)\}_{t\in\mathbb{R}}$ mapping the Banach space X into itself is called a strongly continuous cosine family if and only if $C(0) = I$, $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$, and the map $t \mapsto C(t)x$ is strongly continuous for each $x \in X$.*

Let $\{S(t)\}_{t\in\mathbb{R}}$ denote the strongly continuous sine families associated with the strongly continuous cosine families $\{C(t)\}_{t\in\mathbb{R}}$, where

$$ S(t)x = \int_0^t C(s)x ds, \quad x \in X,\ t \in \mathbb{R}. \tag{2} $$

In addition, an operator A is said to be an infinitesimal generator of cosine families $\{C(t)\}_{t\in\mathbb{R}}$ if

$$ Ax = \frac{d^2}{dt^2} C(t)x \Big|_{t=0}, \quad \text{for all } x \in \mathcal{D}(A), $$

where the domain of A is given by $\mathcal{D}(A) = \{x \in X : \ C(t)x \in C^2(\mathbb{R}, X)\}$.

A multivalued map G is called upper semicontinuous (u.s.c.) on X if, for each $x_* \in X$, the set $G(x_*)$ is a nonempty subset of X, and for every open set $B \subseteq X$ such that $G(x_*) \subset B$, there exists a neighborhood V of x_* with the property that $G(V(x_*)) \subset B$. G is convex-valued if $G(x)$ is convex for all $x \in X$. G is closed if its graph $\Gamma_G = \{(x, y) \in X \times X : \ y \in G(x)\}$ is a closed subset of the space $X \times X$. The map G is bounded if $G(B)$ is bounded in X for every bounded set $B \subseteq X$. We say that G is completely continuous if $G(B)$ is relatively compact for every bounded subset B of X. Furthermore, if G is completely continuous with nonempty values, then G is u.s.c. if and only if G has a closed graph. If there exists an element $x \in X$ such that $x \in G(x)$, then G has a fixed point.

Let B be a subset of X. Then, we define

$$ \begin{aligned} &\mathcal{P}(X) = \{B \subseteq X : B \text{ is nonempty}\}, \quad \mathcal{P}_{cv}(X) = \{B \in \mathcal{P}(X) : B \text{ is convex}\}, \\ &\mathcal{P}_{cl}(X) = \{B \in \mathcal{P}(X) : B \text{ is closed}\}, \quad \mathcal{P}_{bd}(X) = \{B \in \mathcal{P}(X) : B \text{ is bounded}\}, \\ &\mathcal{P}_{cp}(X) = \{B \in \mathcal{P}(X) : B \text{ is compact}\}, \quad \mathcal{P}_{cl,cv}(X) = \mathcal{P}_{cl}(X) \cap \mathcal{P}_{cv}(X). \end{aligned} $$

In addition, let $co(B)$ be the convex hull of a subset B, and let $\overline{co}(B)$ be the closed convex hull in X. A multivalued map $G : J \to \mathcal{P}_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $Z : J \to X$ defined by $Z(t) = d(x, G(t)) = \inf\{\|x - z\| : \ z \in G(t)\}$ is Lebesgue measurable. Let $G : J \to \mathcal{P}(X)$. A single-valued map $f : J \to X$ is called a selection of G if $f(t) \in G(t)$ for every $t \in J$.

Definition 5. *A multivalued map $F : J \times X \to \mathcal{P}(X)$ is called L^1-Carathéodory if*

(i) *the map $t \mapsto F(t, x)$ is measurable for each $x \in X$;*
(ii) *the map $u \mapsto F(t, x)$ is upper semicontinuous on X for almost all $t \in J$;*
(iii) *for each positive real number r, there exists $h_r \in L^1(J, \mathbb{R}_+)$ such that*

$$ \|F(t, x)\|_{\mathcal{P}(X)} = \sup\{\|v\| : \ v(t) \in F(t, x)\} \leq h_r(t), \ for \ \|x\| \leq r, \ for \ a.e. \ t \in J. $$

For every $\Omega \in \mathcal{P}(X)$, the Hausdorff measure of noncompactness (MNC) is defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \ \Omega \text{ has a finite } \varepsilon\text{-net}\},$$

and the Kuratowski MNC is defined by

$$\tau(\Omega) = \inf\{d > 0 : \ \Omega \ \subset \bigcup_{j=1}^{n} M_j \text{ and } \operatorname{diam}(M_j) \le d\},$$

where the diameter of M_j is given by $\operatorname{diam}(M_j) = \sup\{\|x-y\| : x, y \in M_j\}$, $j = 1, \ldots, n$. The Hausdorff and Kuratowski MNCs are connected by the relations:

$$\chi(\Omega) \le \tau(\Omega) \le 2\chi(\Omega).$$

A measure of noncompactness χ (or τ) is called: monotone if $\Omega_1, \Omega_2 \in \mathcal{P}(X)$ with $\Omega_1 \subseteq \Omega_2$ implies $\chi(\Omega_1) \le \chi(\Omega_2)$; nonsingular if $\chi(\{c\} \cup \Omega) = \chi(\Omega)$ for every $c \in X$, $\Omega \in \mathcal{P}(X)$; regular if $\chi(\Omega) = 0$ is equivalent to the relative compactness of Ω.

We now introduce the MNC ν as follows: for a bounded set $D \subset C(J, X)$, we define

$$\nu(D) = \max_{D \in \Theta(D)} \left(\sup_{t \in J} \chi(D(t)), \ \operatorname{mod}_C(D) \right),$$

where $\Theta(D)$ is the collection of all denumerable subsets of D and $\operatorname{mod}_C(D)$ is the modulus of equicontinuity of the set of functions D that have the following form

$$\operatorname{mod}_C(D) = \lim_{\delta \to 0} \sup_{x \in D} \max_{|t_2 - t_1| < \delta} \|x(t_2) - x(t_1)\|.$$

It is known that the MNC ν is monotone, nonsingular, and regular. For more details on the MNC, we refer to [24,25].

Lemma 1. ([24]). *Let $W \subset X$ be bounded. Then, for each $\varepsilon > 0$, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset W$ such that*

$$\chi(W) \le 2\chi\left(\{x_n\}_{n=1}^{\infty}\right) + \varepsilon.$$

Lemma 2. ([26]). *Let χ_C be the Hausdorff MNC on $C(J, X)$, and let $W(t) = \{x(t) : x \in W\}$. If $W \subset C(J, X)$ is bounded, then for every $t \in J$,*

$$\chi(W(t)) \le \chi_C(W).$$

Furthermore, if W is equicontinuous, then the map $t \mapsto \chi(W(t))$ is continuous on J and

$$\chi_C(W) = \sup_{t \in J} \chi(W(t)).$$

Lemma 3. ([26]). *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into X. If there exists a function $\rho(\cdot) \in L^1(J, \mathbb{R}_+)$ satisfying $\|x_n(t)\| \le \rho(t)$ for almost all $t \in J$ and for every $n \ge 1$, then the function $\psi(t) = \chi(\{x_n(t)\}_{n=1}^{\infty}) \in L^1(J, \mathbb{R}_+)$ satisfies*

$$\chi\left(\left\{\int_0^t x_n(s)ds : \ n \ge 1\right\}\right) \le 2\int_0^t \psi(s)ds.$$

Lemma 4. ([27, Lemma 4]). *Let $\{f_n\}_{n=1}^{\infty} \subset L^p(J, X)$ $(p \ge 1)$ be an integrable bounded sequence satisfying*

$$\chi(\{f_n\}_{n=1}^{\infty}) \le \gamma(t), \quad \text{a.e., } t \in J,$$

where $\gamma(\cdot) \in L^1(J,\mathbb{R}_+)$. Then, for each $\epsilon > 0$, there exists a compact $K_\epsilon \subseteq X$, a measurable set $J_\epsilon \subset J$ with measure less than ϵ, and a sequence of functions $\{g_n^\epsilon\}_{n=1}^\infty \subset L^p(J,X)$ such that $\{g_n^\epsilon(t)\}_{n=1}^\infty \subseteq K_\epsilon$, for $t \in J$, and

$$\|f_n(t) - g_n^\epsilon(t)\| < 2\gamma(t) + \epsilon, \quad for\ each\ n \geq 1\ and\ for\ every\ t \in J - J_\epsilon.$$

Lemma 5. ([28]) *Let χ be the Hausdorff MNC on X. If $\{W_n\}_{n=1}^\infty \subset X$ is a nonempty decreasing closed sequence and $\lim_{n\to\infty} \chi(W_n) = 0$, then $\bigcap_{n=1}^\infty W_n$ is nonempty and compact.*

Definition 6. *Let D be a subset of a Banach space X. A multivalued function $F : D \to \mathcal{P}(X)$ is said to be ν-condensing if $\nu(F(\Omega)) \ngeq \nu(\Omega)$ for every bounded and not relatively compact set $\Omega \subseteq D$.*

Lemma 6. ([25, Corollary 3.3.1]). *Let Ω be a convex closed subset of a Banach space X and ν be a nonsingular MNC defined on subsets of Ω. If $F : \Omega \to P_{cv,cp}(\Omega)$ is a closed ν-condensing multivalued map, then F has a fixed point.*

Lemma 7. ([25, Proposition 3.5.1]). *Let Ω be a closed subset of a Banach space X and $F : \Omega \to P_{cp}(X)$ be a closed multivalued function that is ν-condensing on every bounded subset of Ω, where ν is a monotone MNC in X. If the set of fixed points of F is bounded, then it is compact.*

Throughout this paper, we suppose that A is the infinitesimal generator of a strongly continuous cosine family of uniformly bounded linear operators $\{C(t)\}_{t\geq 0}$ in a Banach space X: that is, there exists $M \geq 1$ such that $\|C(t)\|_{\mathcal{L}(X)} \leq M$ for $t \geq 0$. In the sequel, we always set $q = \frac{\alpha}{2}$ for $\alpha \in (1,2)$.

As argued in [18], we define a mild solution of Equation (1) as follows.

Definition 7. *A function $x \in C(J,X)$ is said to be a mild solution of Equation (1) if $x(0) + g(x) = x_0$, $x'(0) = x_1$ and there exists $f \in L^1(J,X)$ such that $f(t) \in F(t,x(t))$ on a.e. $t \in J$ and*

$$x(t) = C_q(t)(x_0 - g(x)) + K_q(t)x_1 + \int_0^t (t-s)^{q-1} P_q(t-s) f(s) ds,$$

where

$$C_q(t) = \int_0^\infty M_q(\theta) C(t^q\theta) d\theta, \quad K_q(t) = \int_0^t C_q(s) ds, \quad P_q(t) = \int_0^\infty q\theta M_q(\theta) S(t^q\theta) d\theta,$$

$$M_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}} \xi_q(\theta^{-\frac{1}{q}}),\ \xi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in (0,\infty),$$

and $M_q(\cdot)$ is the Mainardi's Wright-type function defined on $(0,\infty)$ such that

$$M_q(\theta) \geq 0\ for\ \theta \in (0,\infty)\ and\ \int_0^\infty M_q(\theta) d\theta = 1.$$

Remark 1. *In considering the case of $\alpha \in (0,1)$, we know from the references that there is a similar representation of mild solutions if the initial value $x_1 = 0$ for the case of $\alpha \in (1,2)$. However, the biggest difference is that the operator A (typically the Laplacian operator) generates a C_0-semigroup, and one can use the method of semigroup theory to obtain some well-known results for the case of $\alpha \in (0,1)$ instead of cosine families. Further, if α tends to 1, the method of semigroup theory can be also used to deal with first-order evolution problems; if α tends to 2, we can directly solve an evolution problem by using the concept of cosine families. Thus, the studied evolution problem in Equation (1) is more different from the case of $\alpha \in (0,1]$, and it is valuable to consider the existence of Equation (1).*

Remark 2. *The setting $q = \alpha/2$ for $\alpha \in (1,2)$ is derived from the constraint of the Laplace transform of Mainardi's Wright-type function and the resolvent of cosine families (see [18]). This reflects the fact that the probability density function is closely related to the mild solutions of the corresponding evolution problems.*

Lemma 8. ([18]) *The operators $C_q(t)$, $K_q(t)$, and $P_q(t)$ (appearing in Definition 7) have the following properties:*

(i) *For any $t \geq 0$, the operators $C_q(t)$, $K_q(t)$, and $P_q(t)$ are linear operators;*
(ii) *For any fixed $t \geq 0$ and for any $x \in X$, the following estimates hold:*

$$\|C_q(t)x\| \leq M\|x\|, \quad \|K_q(t)x\| \leq M\|x\|t, \quad \|P_q(t)x\| \leq \frac{M}{\Gamma(2q)}\|x\|t^q;$$

(iii) $\{C_q(t), t \geq 0\}$, $\{K_q(t), t \geq 0\}$, *and* $\{t^{q-1}P_q(t), t \geq 0\}$ *are strongly continuous.*

Lemma 9. ([29]) *Let X be a separable metric space and let $G : \Omega \to \mathcal{P}_{cl}(X)$ be a multivalued map with nonempty closed images. Then, G is measurable if and only if there exist measurable single-valued maps $g_n : \Omega \to X$ such that*

$$G(\omega) = \overline{\bigcup\{g_n(\omega),\ n \geq 1\}}, \quad for\ every\ \omega \in \Omega.$$

Lemma 10. ([30, Theorem 8.2.10]) *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measurable space, and let X, Y be two complete separable metric spaces. If $F : \Omega \to \mathcal{P}(X)$ is a measurable multivalued map with nonempty closed images and $G : \Omega \times X \mapsto Y$ is a Carathéodory map (that is, for every $x \in X$, the multivalued map $\omega \mapsto G(\omega, x)$ is measurable, and for every $\omega \in \Omega$, the multivalued map $x \mapsto G(\omega, x)$ is continuous), then for every measurable map $h : \Omega \mapsto Y$ satisfying $h(\omega) \in G(\omega, F(\omega))$ for almost all $\omega \in \Omega$, there exists a measurable selection $f(\omega) \in F(\omega)$ such that $h(\omega) = G(\omega, f(\omega))$ for almost all $\omega \in \Omega$.*

3. Main Results

We need to state the following hypotheses for the forthcoming analysis.

Hypothesis 1. *The operator A is the infinitesimal generator of a uniformly bounded cosine family $\{C(t)\}_{t \geq 0}$ in X.*

Hypothesis 2. *The multivalued map $F : J \times X \to \mathcal{P}_{cl,cv}(X)$ is an L^1-Carathéodory multivalued map satisfying the following conditions:*

(i) *For every $t \in J$, the map $F(t, \cdot) : X \to \mathcal{P}_{cl,cv}(X)$ is u.s.c.;*
(ii) *For each $x \in X$, the map $F(\cdot, x) : J \to \mathcal{P}_{cl,cv}(X)$ is measurable and the set*

$$S_{F,x} = \{f \in L^1(J, X) :\ f(t) \in F(t, x(t)) \text{ for a.e. } t \in J\}$$

is nonempty.

Hypothesis 3. *There exists a function $k_f(\cdot) \in L^1(J, \mathbb{R}_+)$ such that*

$$\|F(t, x)\| = \sup\{\|f\| :\ f \in F(t, x)\} \leq k_f(t)(1 + \|x\|), \text{ for a.a. } t \in J \text{ and all } x \in X.$$

Hypothesis 4. *There exists a function $\beta(\cdot) \in L^1(J, \mathbb{R}_+)$ such that $\chi(F(t, D)) \leq \beta(t)\chi(D)$ for every bounded subset $D \subset C(J, X)$.*

Hypothesis 5. *$g : C(J, X) \to X$ is a continuous and compact function, and there exist constants N_{g1}, N_{g2} such that $\|g(x)\| \leq N_{g1}\|x\|_C + N_{g2}$* for $x \in C(J, X)$.

Remark 3. *If X is a finite dimension Banach space, then for each $x \in C(J,X)$, $S_{F,x} \neq \emptyset$ (see, e.g., Lasota and Opial [31]). If X is an infinite dimension Banach space and $x \in C(J,X)$, it follows from Hu and Papageorgiou [32] that $S_{F,x} \neq \emptyset$ if and only if the function $\varsigma : J \to \mathbb{R}_+$ given by $\varsigma(t) := \inf\{\|\nu\| : \nu \in F(t,x)\}$ belongs to $L^1(J,\mathbb{R}_+)$.*

Lemma 11. ([31]). *Let X be a Banach space, let $F : J \times X \to \mathcal{P}_{cp,cv}(X)$ be a L^1-Carathéodory multivalued map with $S_{F,x} \neq \emptyset$ (see (H2)), and let Ψ be a linear continuous operator from $L^1(J,X)$ to $C(J,X)$. Then,*

$$\Psi \circ S_F : C(J,X) \to \mathcal{P}_{cp,cv}(C(J,X)), \quad x \mapsto (\Psi \circ S_F)(x) := \Psi(S_{F,x}),$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Theorem 1. *Assume that* (H1)–(H5) *are satisfied. Then, Equation (1) has at least one mild solution provided that $\|k_f\|_1 < (1 - MN_{g1})M^{-1}a^{1-2q}\Gamma(2q)$ and $\|\beta\|_1 < (8M)^{-1}a^{1-2q}\Gamma(2q)$.*

Proof. By (H2), we can define a multivalued map $\mathscr{P} : C(J,X) \to \mathcal{P}(C(J,X))$ as follows: for $x \in C(J,X)$, $\mathscr{P}(x)$ is the set of all functions $y \in \mathscr{P}(x)$ satisfying

$$y(t) = C_q(t)(x_0 - g(x)) + K_q(t)x_1 + \int_0^t (t-s)^{q-1} P_q(t-s) f(s) ds, \quad t \in J,$$

where $f \in S_{F,x}$. It will be verified in several steps, claims and parts that the operator $\mathscr{P}$ has fixed points that correspond to mild solutions of Equation (1).

Step 1. $\mathscr{P}$ maps a bounded closed convex set into a bounded closed convex set.

By the hypothesis of function $k_f(\cdot)$ in (H3), there exists $r > 0$ such that

$$M\|x_0\| + MN_{g1}r + MN_{g2} + Ma\|x_1\| + \frac{Ma^{2q-1}}{\Gamma(2q)}\|k_f\|_1 + \frac{Ma^{2q-1}}{\Gamma(2q)}\|k_f\|_1 r \leq r. \tag{3}$$

Furthermore, we introduce $W_0 = \{x \in C(J,X) : \|x\|_C \leq r\}$ and observe that W_0 is a nonempty bounded closed and convex subset of $C(J,X)$. Let $x \in W_0$ and $y \in \mathscr{P}(x)$, then, there exists $f \in S_{F,x}$ such that for each $t \in J$ and for any $x \in W_0$, we have

$$y(t) = C_q(t)(x_0 - g(x)) + K_q(t)x_1 + \int_0^t (t-s)^{q-1} P_q(t-s) f(s) ds.$$

By (H3) and (H4), we have

$$\begin{aligned}
\|y(t)\| \leq & \|C_q(t)\|_{\mathcal{L}(X)}\|x_0 - g(x)\| + \|K_q(t)\|_{\mathcal{L}(X)}\|x_1\| + \int_0^t (t-s)^{q-1}\|P_q(t-s)f(s)\| ds \\
\leq & M\|x_0\| + M\|g(x)\| + Mt\|x_1\| + \frac{M}{\Gamma(2q)}\int_0^t (t-s)^{2q-1} k_f(s)(1 + |x(s)|) ds \\
\leq & M\|x_0\| + MN_{g1}\|x\|_C + MN_{g2} + Mt\|x_1\| + \frac{Mt^{2q-1}}{\Gamma(2q)}\|k_f\|_1 + \frac{Mt^{2q-1}}{\Gamma(2q)}\|k_f\|_1\|x\|_C \\
\leq & M\|x_0\| + MN_{g1}r + MN_{g2} + Ma\|x_1\| + \frac{Ma^{2q-1}}{\Gamma(2q)}\|k_f\|_1 + \frac{Ma^{2q-1}}{\Gamma(2q)}\|k_f\|_1 r \\
\leq & r.
\end{aligned}$$

Therefore, $\|y\|_C \leq r$, which implies that $\mathscr{P}(W_0) \subseteq W_0$.

Define $W_1 = \overline{co}\mathscr{P}(W_0)$. Clearly, $W_1 \subset C(J, X)$ is a nonempty bounded closed and convex set. Repeating the arguments employed in the previous step, for any $x \in W_1$, $y \in \mathscr{P}(x)$, it follows that there exists $f \in S_{F,x}$ such that for each $t \in J$ and for any $x \in W_1$,

$$y(t) = C_q(t)(x_0 - g(x)) + K_q(t)x_1 + \int_0^t (t-s)^{q-1} P_q(t-s) f(s) ds.$$

By (H3) and (H4), together with Lemma 8 (ii), we have

$$\begin{aligned} \|y(t)\| \leq & \|C_q(t)(x_0 - g(x))\| + \|K_q(t)x_1\| + \int_0^t (t-s)^{q-1} \|P_q(t-s) f(s)\| ds \\ \leq & M\|x_0\| + MN_{g1}r + MN_{g2} + Ma\|x_1\| + \frac{Ma^{2q-1}}{\Gamma(2q)}\|k_f\|_1 + \frac{Ma^{2q-1}}{\Gamma(2q)}\|k_f\|_1 r \leq r, \end{aligned}$$

which implies that $\mathscr{P}(W_1) \subseteq W_1$ and $W_1 \subset W_0$.

Next, for every $n \geq 1$, we define $W_{n+1} = \overline{co}\mathscr{P}(W_n)$. From the above proof, it is easy to see that W_n is a nonempty bounded closed and convex subset of $C(J, X)$. Furthermore, $W_2 = \overline{co}\mathscr{P}(W_1) \subset W_1$. By induction, we know that the sequence $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty bounded closed and convex subsets of $C(J, X)$. Furthermore, we set $W = \bigcap_{n=1}^{\infty} W_n$ and note that W is bounded closed and convex since W_n is bounded closed and convex for every $n \geq 1$.

Now, we establish that $\mathscr{P}(W) \subseteq W$. Indeed, $\mathscr{P}(W) \subseteq \mathscr{P}(W_n) \subseteq \overline{co}\mathscr{P}(W_n) = W_{n+1}$ for every $n \geq 1$. Therefore, $\mathscr{P}(W) \subseteq \bigcap_{n=2}^{\infty} W_n$. On the other hand, $W_n \subset W_1$ for every $n \geq 1$. Hence,

$$\mathscr{P}(W) \subseteq \bigcap_{n=2}^{\infty} W_n = \bigcap_{n=1}^{\infty} W_n = W.$$

Step 2. The multivalued map $\mathscr{P}$ is ν-condensing.

Let $\mathbb{B} \subseteq W$ be such that

$$\nu(\mathbb{B}) \leq \nu(\mathscr{P}(\mathbb{B})). \tag{4}$$

We show below that $\mathbb{B}$ is a relatively compact set; that is, $\nu(\mathbb{B}) = 0$.

Let $\sigma(\mathbb{B}) = \sup_{t \in J} \chi(\mathbb{B}(t))$, and let $\nu(\mathscr{P}(\mathbb{B}))$ be achieved on a sequence $\{y_n\}_{n=1}^{\infty} \subset \mathscr{P}(\mathbb{B})$; that is,

$$\nu(\{y_n\}_{n=1}^{\infty}) = \max\Big(\sigma(\{y_n\}_{n=1}^{\infty}),\ \mathrm{mod}_C(\{y_n\}_{n=1}^{\infty})\Big).$$

Then,

$$y_n(t) = C_q(t)(x_0 - g(x_n)) + K_q(t)x_1 + \int_0^t (t-s)^{q-1} P_q(t-s) f_n(s) ds, \quad t \in J,$$

where $\{x_n\}_{n=1}^{\infty} \subset \mathbb{B}$ and $f_n \in Sel_{F,x_n}$ for every $n \geq 1$.

Since g is compact, the set $\{g(x_n) :\ n \geq 1\}$ is relatively compact and $C_q(t)$, $K_q(t)$ are strongly continuous for $t \geq 0$. Hence, for every $t \in J$, we have

$$\nu\big(\{C_q(t)(x_0 - g(x_n)) + K_q(t)x_1,\ n \geq 1\}\big) = 0.$$

Therefore, it is enough to estimate that

$$\nu\left(\left\{\int_0^t (t-s)^{q-1} P_q(t-s) f_n(s) ds,\ n \geq 1\right\}\right) = 0.$$

Claim I. $\sigma(\{y_n\}_{n=1}^{\infty}) = 0$.

For any $t \in J$, using (H4), Lemma 3, and Lemma 8 (ii), we have

$$\begin{aligned}\sigma(\{y_n\}_{n=1}^{\infty}) =& \sup_{t\in J}\chi(\{y_n(t)\}_{n=1}^{\infty}) \leq 2\sup_{t\in J}\int_0^t (t-s)^{q-1}\chi\left(\{P_q(t-s)f_n(s)\}_{n=1}^{\infty}\right)ds\\ \leq& \sup_{t\in J}\frac{2M}{\Gamma(2q)}\int_0^t (t-s)^{2q-1}\beta(s)\chi\left(\{x_n(s)\}_{n=1}^{\infty}\right)ds\\ \leq& \sup_{t\in J}\frac{2M}{\Gamma(2q)}\int_0^t (t-s)^{2q-1}\beta(s)ds\sigma(\{x_n\}_{n=1}^{\infty})\\ \leq& \frac{2Ma^{2q-1}}{\Gamma(2q)}\int_0^a \beta(s)ds\sigma(\{x_n\}_{n=1}^{\infty}) < \frac{1}{4}\sigma(\{x_n\}_{n=1}^{\infty}).\end{aligned}$$

On the other hand, Equation (4) implies that $\sigma(\{y_n\}_{n=1}^{\infty}) \geq \sigma(\{x_n\}_{n=1}^{\infty})$. In consequence, we have $\sigma(\{y_n\}_{n=1}^{\infty}) = 0$.

Claim II. $\mathrm{mod}_C(\{y_n\}_{n=1}^{\infty}) = 0$; that is, the set $\mathbb{B}$ is equicontinuous.

Let

$$\widetilde{y}_n(\cdot) = \int_0^{\cdot} (\cdot - s)^{q-1}P_q(t-s)f_n(s)ds.$$

Therefore, it remains to be verified that $\mathrm{mod}_C(\{\widetilde{y}_n\}_{n=1}^{\infty}) = 0$. Then, for any $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}\|\widetilde{y}_n(t_2) - \widetilde{y}_n(t_1)\| \leq& \int_{t_1}^{t_2} \|(t_2-s)^{q-1}P_q(t_2-s)f_n(s)\|ds\\ &+ \int_0^{t_1} \|((t_2-s)^{q-1}P_q(t_2-s) - (t_1-s)^{q-1}P_q(t_2-s))f_n(s)\|ds\\ =& I_1 + I_2.\end{aligned}$$

According to Lemma 8 (ii), we get

$$\begin{aligned}I_1 \leq& \frac{M}{\Gamma(2q)}\int_{t_1}^{t_2}(t_2-s)^{2q-1}k_f(s)(1+\|x_n(s)\|)ds\\ \leq& \frac{M}{\Gamma(2q)}(t_2-t_1)^{2q-1}\int_{t_1}^{t_2}k_f(s)ds(1+\|x_n\|_C) \to 0, \quad \text{as } t_2 \to t_1.\end{aligned}$$

Let $T_q(t) = t^{q-1}P_q(t)$ for $t \in J$. Then, we know from Lemma 8 (iii) that $T_q(t)$ is a strongly continuous operator. For I_2, taking $\varepsilon > 0$ to be small enough, we obtain

$$\begin{aligned}I_2 \leq& \int_0^{t_1-\varepsilon} \|(T_q(t_2-s) - T_q(t_1-s))f_n(s)\|ds + \int_{t_1-\varepsilon}^{t_1} \|(T_q(t_2-s) - T_q(t_1-s))f_n(s)\|ds\\ \leq& \int_0^{t_1} k_f(s)(1+\|x_n(s)\|)ds \sup_{s\in[0,t_1-\varepsilon]}\|T_q(t_2-s) - T_q(t_1-s)\|_{\mathcal{L}(X)}\\ &+ \Big(\frac{M\varepsilon^{2q-1}}{\Gamma(2q)} + \frac{M(t_2-t_1+\varepsilon)^{2q-1}}{\Gamma(2q)}\Big)\int_{t_1-\varepsilon}^{t_1} k_f(s)(1+\|x_n(s)\|)ds\\ \leq& \|k_f\|_1(1+\|x_n\|_C)\sup_{s\in[0,t_1-\varepsilon]}\|T_q(t_2-s) - T_q(t_1-s)\|_{\mathcal{L}(X)}\\ &+ \Big(\frac{M\varepsilon^{2q-1}}{\Gamma(2q)} + \frac{M(t_2-t_1+\varepsilon)^{2q-1}}{\Gamma(2q)}\Big)(1+\|x_n\|_C)\int_{t_1-\varepsilon}^{t_1} k_f(s)ds\\ \to& 0, \quad \text{as } t_2 \to t_1,\ \varepsilon \to 0.\end{aligned}$$

Consequently, we have

$$\operatorname{mod}_C\left(\left\{\int_0^t (t-s)^{q-1}P_q(t-s)f_n(s)ds,\ n\geq 1\right\}\right)=0.$$

As a conclusion, it follows that $\operatorname{mod}_C(\{y_n\}_{n=1}^{\infty})=0$. Hence, the multivalued map $\mathscr{P}$ is ν-condensing.

Step 3. The multimap $\mathscr{P}(x)$ is convex and compact for each $x\in W$.

Part I. $\mathscr{P}(x)$ has convex values for each $x\in W$.

In fact, if y_1, y_2 belong to $\mathscr{P}(x)$ for each $x\in W$, then there exist $f_1, f_2\in S_{F,x}$ such that for each $t\in J$, we have

$$y_i(t)=C_q(t)(x_0-g(x))+K_q(t)x_1+\int_0^t (t-s)^{q-1}P_q(t-s)f_i(s)ds,\quad i=1,2.$$

Let $\theta\in[0,1]$. Then, for each $t\in J$, we get

$$\begin{aligned}(\theta y_1+(1-\theta)y_2)(t)=&C_q(t)(x_0-g(x))+K_q(t)x_1\\&+\int_0^t (t-s)^{q-1}P_q(t-s)(\theta f_1+(1-\theta)f_2)(s)ds.\end{aligned}$$

As F has convex values by the definition of $S_{F,x}$, we deduce that $\theta f_1(s)+(1-\theta)f_2(s)\in S_{F,x}$. Thus, $\theta y_1+(1-\theta)y_2\in\mathscr{P}(x)$.

Part II. $\mathscr{P}$ has compact values. In view of the foregoing facts, it is enough to show that W is nonempty and compact in $C(J,X)$: that is, by Lemma 5, we need to show that

$$\lim_{n\to\infty}\nu(W_n)=0. \tag{5}$$

As in Step 2, we can show that $\operatorname{mod}_C(W_n)=0$; that is, W_n is equicontinuous. Hence, it remains to be shown that $\sigma(W_n)=0$. By Lemma 1, for each $\varepsilon>0$, there exists a sequence $\{y_k\}_{k=1}^{\infty}$ in $\mathscr{P}(W_{n-1})$ such that

$$\sigma(W_n)=\sigma(\mathscr{P}(W_n))\leq 2\sigma(\{y_k\}_{k=1}^{\infty})+\varepsilon.$$

Therefore, by Lemma 2 and the nonsingularity of σ, it follows that

$$\sigma(W_n)\leq 2\sigma(\{y_k\}_{k=1}^{\infty})+\varepsilon=2\sup_{t\in J}\chi(\{y_k(t)\}_{k=1}^{\infty})+\varepsilon. \tag{6}$$

Since $y_k\in\mathscr{P}(W_{n-1})$ $(k\geq 1)$, there exists $x_k\in W_{n-1}$ such that $y_k\in\mathscr{P}(x_k)$. Hence, from the compactness of g and the strong continuity of $C_q(t)$ and $K_q(t)$ for $t\in J$, there exists $f_k\in S_{F,x_k}$ such that for every $t\in J$,

$$\begin{aligned}\chi(\{y_k(t)\}_{k=1}^{\infty})\leq&\chi(\{C_q(t)(x_0-g(\{x_k\}_{k=1}^{\infty}))+K_q(t)x_1\})\\&+\chi\left(\left\{\int_0^t (t-s)^{q-1}P_q(t-s)f_k(s)ds:\ k\geq 1\right\}\right)\\=&\chi\left(\left\{\int_0^t (t-s)^{q-1}P_q(t-s)f_k(s)ds:\ k\geq 1\right\}\right).\end{aligned}$$

By (H5) and Lemma 1, for a.e. $t\in J$, we have

$$\chi(\{f_k(t)\}_{k=1}^{\infty})\leq\chi(F(t,\{x_k(t)\}_{k=1}^{\infty}))\leq\beta(t)\chi(\{x_k(t)\}_{k=1}^{\infty})\leq\beta(t)\sigma(W_{n-1}):=\gamma(t).$$

On the other hand, by (H3), for almost all $t\in J$, $\|f_k(t)\|\leq k_f(t)(1+r)$ for every $k\geq 1$. Hence, $f_k\in L^1(J,X)$, $k\geq 1$. Note that $\gamma(\cdot)\in L^1(J,\mathbb{R}_+)$ from (H4). It follows from Lemma 4 that there exists a compact $K_\epsilon \subset X$, a measurable set $J_\epsilon \subset J$ with measure less than ϵ, and a sequence of functions $\{g_k^\epsilon\} \subset L^1(J, X)$ such that $\{g_k^\epsilon(s)\}_{k=1}^\infty \subseteq K_\epsilon$ for all $s \in J$, and

$$\|f_k(s) - g_k^\epsilon(s)\| < 2\gamma(s) + \epsilon, \quad \text{for every } k \geq 1 \text{ and every } s \in J'_\epsilon = J - J_\epsilon.$$

Then, using Minkowski's inequality and the property of the MNC, we obtain

$$\begin{aligned}
&\chi\left(\left\{\int_{J'_\epsilon}(t-s)^{q-1}P_q(t-s)(f_k(s) - g_k^\epsilon(s))ds : \, k \geq 1\right\}\right) \\
\leq &\frac{2M}{\Gamma(2q)}\int_{J'_\epsilon}(t-s)^{2q-1}\chi(\{(f_k(s) - g_k^\epsilon(s)) : \, k \geq 1\})ds \\
\leq &\frac{2M}{\Gamma(2q)}\int_{J'_\epsilon}(t-s)^{2q-1}\sup_{k\geq 1}\|f_k(s) - g_k^\epsilon(s)\|ds \\
\leq &\frac{2Ma^{2q-1}}{\Gamma(2q)}\int_{J'_\epsilon}(2\gamma(s) + \epsilon)ds \\
\leq &\frac{4Ma^{2q-1}}{\Gamma(2q)}\|\gamma\|_1 + \frac{2Ma^{2q-1}}{\Gamma(2q)}\epsilon \\
\leq &\frac{4Ma^{2q-1}}{\Gamma(2q)}\sigma(W_{n-1})\|\beta\|_1 + \frac{2Ma^{2q-1}}{\Gamma(2q)}\epsilon,
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
\chi\left(\left\{\int_{J_\epsilon}(t-s)^{q-1}P_q(t-s)f_k(s)ds : \, k \geq 1\right\}\right) \leq &\frac{2M}{\Gamma(2q)}\int_{J_\epsilon}(t-s)^{2q-1}\chi\left(\{f_k(s)\}_{k=1}^\infty\right)ds \\
\leq &\frac{2M}{\Gamma(2q)}\int_{J_\epsilon}(t-s)^{2q-1}\sup_{k\geq 1}\|f_k(s)\|ds \\
\leq &\frac{Ma^{2q-1}}{\Gamma(2q)}(1+r)\int_{J_\epsilon}k_f(s)ds.
\end{aligned} \tag{8}$$

Using Equations (7) and (8), we have

$$\begin{aligned}
&\chi\left(\left\{\int_0^t(t-s)^{q-1}P_q(t-s)f_k(s)ds : \, k \geq 1\right\}\right) \leq \chi\left(\left\{\int_{J'_\epsilon}(t-s)^{q-1}P_q(t-s)f_k(s)ds : \, k \geq 1\right\}\right) \\
&\quad + \chi\left(\left\{\int_{J_\epsilon}(t-s)^{q-1}P_q(t-s)f_k(s)ds : \, k \geq 1\right\}\right) \\
\leq &\chi\left(\left\{\int_{J'_\epsilon}(t-s)^{q-1}P_q(t-s)(f_k(s) - g_k^\epsilon(s))ds : \, k \geq 1\right\}\right) \\
&\quad + \chi\left(\left\{\int_{J'_\epsilon}(t-s)^{q-1}P_q(t-s)g_k^\epsilon(s)ds : \, k \geq 1\right\}\right) \\
&\quad + \chi\left(\left\{\int_{J_\epsilon}(t-s)^{q-1}P_q(t-s)f_k(s)ds : \, k \geq 1\right\}\right) \\
\leq &\frac{4Ma^{2q-1}}{\Gamma(2q)}\sigma(W_{n-1})\|\beta\|_1 + \frac{2Ma^{2q-1}}{\Gamma(2q)}\epsilon + \frac{Ma^{2q-1}}{\Gamma(2q)}(1+r)\int_{J_\epsilon}k_f(s)ds.
\end{aligned}$$

As ϵ is arbitrary, for all $t \in J$, we get

$$\chi\left(\left\{\int_0^t(t-s)^{q-1}P_q(t-s)f_k(s)ds\right\}\right) \leq \frac{4Ma^{2q-1}}{\Gamma(2q)}\|\beta\|_1\sigma(W_{n-1}).$$

Therefore, for each $t \in J$, we have

$$\chi(\{y_k(t)\}_{k=1}^{\infty}) \leq \frac{4Ma^{2q-1}}{\Gamma(2q)}\|\beta\|_1\sigma(W_{n-1}).$$

By the above inequality, together with Equation (6) and the arbitrary nature of ε, we can deduce that

$$\sigma(W_n) \leq \frac{8Ma^{2q-1}}{\Gamma(2q)}\|\beta\|_1\sigma(W_{n-1}).$$

Then, by induction, we find that

$$0 \leq \sigma(W_n) \leq \left(\frac{8Ma^{2q-1}}{\Gamma(2q)}\|\beta\|_1\right)^n \sigma(W_0), \quad \text{for all } n \geq 1.$$

Since this inequality is true for every $n \geq 1$, passing on to the limit $n \to \infty$ and by (H4), we obtain Equation (5). Hence, $W = \bigcap_{n=1}^{\infty} W_n$ is a nonempty compact set of X, and $\mathscr{P}$ has compact values in W.

Step 4. The values of $\mathscr{P}$ are closed.

Let $x_n, x_* \in W$ with $x_n \to x_*$ as $n \to \infty$, $y_n \in \mathscr{P}(x_n)$, and $y_n \to y_*$ as $n \to \infty$. We show that $y_* \in \mathscr{P}(x_*)$. Indeed, $y_n \in \mathscr{P}(x_n)$ means that there exists $f_n \in S_{F,x_n}$ such that

$$y_n(t) = C_q(t)(x_0 - g(x)) + K_q(t)x_1 + \int_0^t (t-s)^{q-1}P_q(t-s)f_n(s)ds.$$

Next, we must show that there exists $f_* \in S_{F,x_*}$ such that

$$y_*(t) = C_q(t)(x_0 - g(x)) + K_q(t)x_1 + \int_0^t (t-s)^{q-1}P_q(t-s)f_*(s)ds.$$

Since $x_n \to x_*$ and $y_n \in \mathscr{P}(x_n)$, we deduce that

$$\|(y_n(t) - C_q(t)x_0 + C_q(t)g(x_n) - K_q(t)x_1) - (y_*(t) - C_q(t)x_0 + C_q(t)g(x_*) - K_q(t)x_1)\| \to 0,$$

as $n \to \infty$.

Now, we consider the linear continuous operator

$$\mathscr{F} : L^1(J, X) \to C(J, X), \quad f \mapsto (\mathscr{F}f)(t) = \int_0^t (t-s)^{q-1}P_q(t-s)f(s)ds.$$

From Step 3 and Lemma 11, it follows that $\mathscr{F} \circ S_F$ is a closed graph operator. Furthermore, in view of the definition of $\mathscr{F}$, we have

$$(y_n(t) - C_q(t)x_0 + C_q(t)g(x_n) - K_q(t)x_1) \in \mathscr{F}(S_{F,x_n}).$$

In view of the fact that $x_n \to x_*$ as $n \to \infty$, the repeated application of Lemma 11 yields

$$y_*(t) - C_q(t)x_0 + C_q(t)g(x_*) - K_q(t)x_1 = \int_0^t (t-s)^{q-1}P_q(t-s)f(s)ds$$

for some $f \in S_{F,x_*}$. Thus, $\mathscr{P}$ is a closed multivalued map.

Therefore, as an implication of Steps 1–5, we deduce that $\mathscr{P} : W \to \mathcal{P}(W)$ is closed and ν-condensing with nonempty convex compact values. Thus, all the hypotheses of Lemma 6 are satisfied. Hence, there exists at least one fixed point $x \in W$ such that $x \in \mathscr{P}(x)$, which corresponds to a mild solution of Equation (1). $\square$

Theorem 2. *Suppose that all the assumptions of Theorem 1 are satisfied. Then, the set of mild solutions of Equation (1) is compact in $C(J, X)$.*

Proof. Note that the set of mild solutions is nonempty by Theorem 1. Indeed, letting $r > 0$, defined by Equation (3), we can get a mild solution in W_0. Now, we show that an arbitrary number of mild solutions of Equation (1) belongs to W_0. Let x be a mild solution of Equation (1). Then,

$$x(t) = C_q(t)(x_0 - g(x)) + K_q(t)x_1 + \int_0^t (t-s)^{q-1} P_q(t-s) f(s) ds,$$

where $f \in S_{F,x} = \{f \in L^1(J, X) : \ f(t) \in F(t, x(t)), \text{ for a.e. } t \in J\}$. Using an argument similar to the one used in Step 1 of the proof of Theorem 1, we have

$$\begin{aligned}
\|x\|_C =& \sup_{t \in J} \|x(t)\| \\
\leq& \sup_{t \in J} \|C_q(t)(x_0 - g(x))\| + \sup_{t \in J} \|K_q(t)x_1\| + \sup_{t \in J} \int_0^t (t-s)^{q-1} \|P_q(t-s) f(s)\| ds \\
\leq& M\|x_0\| + MN_{g1}r + MN_{g2} + Ma\|x_1\| + \frac{Ma^{2q-1}}{\Gamma(2q)} \|k_f\|_1 + \frac{Ma^{2q-1}}{\Gamma(2q)} \|k_f\|_1 r \leq r.
\end{aligned}$$

This shows that the mild solutions of Equation (1) are bounded. Thus, the conclusion follows from Lemma 7. The proof is completed. □

4. An Application

Let $\Omega \subset \mathbb{R}^N$ $(N = 1, 2, 3)$ be an open bounded set and $X = U = L^2(\Omega)$. Let us consider the following fractional partial differential equations with the constrained control u and a finite multi-point discrete mean condition:

$$\begin{cases}
\partial_t^{\alpha} y(t,z) = \Delta y(t,z) + G(t,z,y(t,z),u(t,z)), \quad t \in [0,1], \ z \in \Omega, \ u \in U, \\
y(t,z) = 0, \quad t \in [0,1], z \in \partial\Omega \\
y(0,z) - \sum_{i=0}^{n} \int_{\Omega} m(\xi,z) y(t_i,\xi) d\xi = 0, \ y'(0,z) = 0, \ z \in \Omega,
\end{cases} \tag{9}$$

where ∂_t^{α} is the Caputo fractional partial derivative of order $\alpha \in (1, 2)$, $0 \leq t_0 < t_1 < \cdots < t_n \leq 1$, $m(\xi, z) : \Omega \times \Omega \to X$ is an L^2-Lebesgue integrable function, and $G : [0, 1] \times \Omega \times X \times U \to X$ is a single-valued continuous measurable function.

We define $x(t) = y(t, \cdot)$, that is, $x(t)(z) = y(t, z)$, $t \in J$, $z \in \Omega$, here $J = [0, 1]$. The set of the constraint functions $U : J \to \mathcal{P}_{cl,cv}(X)$ is a measurable multivalued map. If $u \in U$, then it means that $u(t) \in U(t, x(t))$, for a.e. $t \in J$. The function $f : J \times X \times U$ is given by $f(t, x(t), u(t))(z) = G(t, z, y(t, z), u(t, z))$. Equation (9) is solved if we show that there exists a control function u such that Equation (9) admits a mild solution. Let the multivalued map be given by

$$F(t, x(t)) = \{f(t, x(t), u(t)), \quad u \in U\}. \tag{10}$$

Then, the set of mild solutions of the control problem in Equation (9), with the right-hand side given by Equation (10), coincides with the set of mild solutions of Equation (1).

Let A be the Laplace operator with Dirichlet boundary conditions defined by $A = \Delta$ with

$$\mathcal{D}(A) = \{v \in L^2(\Omega) : \ v \in H_0^1(\Omega) \cap H^2(\Omega)\}.$$

Let $\{-\lambda_k, \phi_k\}_{k=1}^{\infty}$ be the eigensystem of the operator A. Then, $0 < \lambda_1 \leq \lambda_2 \leq \cdots, \lambda_k \to \infty$ as $k \to \infty$, and $\{\phi_k\}_{k=1}^{\infty}$ forms an orthonormal basis of X. Furthermore,

$$Ax = -\sum_{k=1}^{\infty} \lambda_k (x, \phi_k)\phi_k, \quad x \in \mathcal{D}(A),$$

where $(\cdot, \cdot)$ is the inner product in X. It is known that the operator A generates a strongly continuous uniformly bounded cosine family (see, e.g., [9]), which, in this case, is defined by

$$C(t)x = \sum_{k=1}^{\infty} \cos(\sqrt{\lambda_k}t)(x, \phi_k)\phi_k, \quad x \in X,$$

and then $\|C(t)\|_{\mathcal{L}(X)} \leq 1$ for every $t \geq 0$. Hence, (H1) holds.

Taking $\alpha = \frac{3}{2}$, we have $q = \frac{3}{4}$. Let $g : C(J, X) \to X$ be given by $g(x)(z) = \sum_{i=0}^{n} K_g x(t_i)(z)$ with $K_g v(z) = \int_{\Omega} m(\xi, z)v(\xi)d\xi$ for $v \in X, z \in \Omega$ (noting that $K_g : X \to X$ is completely continuous). Thus, the assumption in (H5) holds true. With the choice of operator A, Equation (9) can be reformulated in X as the following nonlocal control problem:

$$\begin{cases} {}^C D_t^{\alpha} x(t) = Ax(t) + f(t, x(t), u(t)), \quad t \in J, \ u \in U, \\ x(0) = g(x), \ x'(0) = 0. \end{cases} \tag{11}$$

Next, the results obtained in Section 4 apply to the following problem of fractional evolution inclusions:

$$\begin{cases} {}^C D_t^{\alpha} x(t) \in Ax(t) + F(t, x(t)), \quad t \in J, \\ x(0) = g(x), \ x'(0) = 0. \end{cases} \tag{12}$$

Theorem 3. *Assume that the following conditions hold:*

Hypothesis 6. *$U : J \to \mathcal{P}_{cl,cv}(X)$ is a measurable multivalued map.*

Hypothesis 7. *The function $f : J \times X \times X \to X$ is L^1-Carathéodory, linear in the third argument, and there exists a function $k_f(\cdot) \in L^1(J, \mathbb{R}_+)$ satisfying $\|k_f\|_1 < \sqrt{\pi}(1 - n\|m\|)/2$ such that $\|f(t, x, y)\| \leq k_f(t)(1 + \|x\|)$ for almost all $t \in J$ and all $x \in X$.*

Hypothesis 8. *There exists a function $\beta(\cdot) \in L^1(J, \mathbb{R}_+)$ satisfying $\|\beta\|_1 < \sqrt{\pi}/16$ such that*

$$\chi(f(t, D, U(t, D))) \leq \beta(t)\chi(D),$$

for every bounded subset $D \subset C(J, X)$.

Then, the control problem in Equation (9) has at least one mild solution. In addition, the set of mild solutions is compact.

Proof. From (H6) and (H7), the map $t \mapsto F(t, \cdot)$ is obviously a measurable multivalued map, and then $F(\cdot, \cdot) \in \mathcal{P}_{cv,cl}(X)$. Now, we show that the selection set of F is not empty. Since U is a measurable multivalued map, it follows by Lemma 9 that there exists a sequence of measurable selections $\{u_n\}_{n=1}^{\infty} \subset U$ such that

$$U(t) = \overline{\bigcup\{u_n(t), \ n \geq 1\}} \quad \text{for every } t \in J.$$

Let $v_n(t) = f(t, x(t), u_n(t))$ for $n \geq 1$ and $t \in J$. In view of the continuity of f, v_n is thus measurable. Hence, $\overline{\{v_n(t), \ n \geq 1\}} \subseteq F(t, x(t))$. Conversely, if $f(t, x(t), u(t)) \in F(t, x(t))$ for any $u \in U$, then there exists a subsequence in U which will be still defined by $\{u_n\}_{n=1}^{\infty}$ such that $u_n \to u$ as $n \to \infty$. It follows from the continuity of f that $f(t, x(t), u_n(t)) \to f(t, x(t), u(t))$ as $n \to \infty$. Hence, $f(t, x(t), u(t)) \in \overline{\{v_n(t),\ n \geq 1\}}$. This means that

$$F(t, x(t)) = \overline{\bigcup\{v_n(t),\ n \geq 1\}},$$

Consequently, from Lemma 9, $F(\cdot, x)$ is measurable.

Next, we show that the map $x \mapsto F(\cdot, x)$ is an u.s.c. multivalued map by means of contradiction. Firstly, we suppose that F is not u.s.c. at some point $x_0 \in \Omega$. Then, there exists an open neighborhood $W \subseteq X$ such that $F(t, x_0) \subset W$, and for every open neighborhood $V \subseteq \Omega$ of x_0, there exists $x_1 \in V$ such that $F(t, x_1) \not\subset W$. Let

$$V_n = \left\{x \in \Omega,\ \|x - x_0\| < \frac{1}{n},\ n = 1, 2, \ldots\right\}.$$

Clearly, V_n is a open neighborhood of x_0. Then, for each $n \geq 1$, there exist $x_n \in V_n$, $v_n \in F(t, x_n)$, and $u_n \in U$ such that $v_n = f(t, x_n, u_n)$ and $v_n \notin W$. Moreover, as $\{u_n\}_{n=1}^{\infty} \subset U$, we set $u_n \to u$ as $n \to \infty$ for some $u \in U$. By the continuity of f, owing to $x_n \to x_0$ as $n \to \infty$, we have $v_n \to v$ as $n \to \infty$, where $v = f(t, x_0, u)$, which implies that $v \in F(t, x_0) \subset W$. This contradicts that $v_n \notin W$ for each $n \geq 1$. Thus, our supposition is false.

In addition, according to the condition in (H7), we find that F is an L^1-Carathéodory multivalued map. Hence, (H2) and (H3) are satisfied. On the other hand, the hypothesis (H8) corresponds to (H4). Thus, all of the hypotheses of Theorem 1 are satisfied. Hence, Equation (12) has at least one mild solution. Furthermore, the set of mild solutions of Equation (12) is compact by Theorem 2.

Finally, we show that the mild solutions of Equation (12) do coincide with the mild solutions of the control problem in Equation (11). Let x be a solution of Equation (12). Then, there exists a single-valued selection

$$\phi \in S_{F,x} = \{\phi \in L^1(J, X),\ \phi(t) \in F(t, x(t)),\ \text{a.e. } t \in J\}, \tag{13}$$

such that

$${}^{C}D_t^{\alpha} x(t) = Ax(t) + \phi(t),\ \text{a.e. } t \in J,\ \text{and } x(0) = g(x),\ x'(0) = 0.$$

Now, we introduce a map $\Psi(t, u) = f(t, x(t), u(t))$ and note that it is Carathéodory. Moreover, let the equality in Equation (10) be satisfied. Then, for a.a. $t \in J$ and for every $\phi(t) \in \{f(t, x(t), u(t)), u \in U\} := \Psi(t, U(t))$, we deduce by Lemma 10 that there exists a measurable selection $u(t) \in U(t)$ such that $\phi(t) = \Psi(t, u(t)) = f(t, x(t), u(t))$ for a.a. $t \in J$. Thus, the mild solution satisfies the control problem in Equation (11).

On the other hand, let x satisfy the control problem in Equation (11). Then, x is obviously a mild solution of Equation (12), and the proof is completed. □

5. Conclusions

In the current paper, we study a class of fractional evolution inclusions with nonlocal initial conditions. We obtain the sufficient conditions for ensuring the existence of mild solutions and the compactness for set of mild solutions. We can see that the probability density function is closely related to the mild solutions of the corresponding evolution inclusion problems, which enrich the knowledge of the fractional calculus. Moreover, an illustrative example is provided to demonstrate the applicability of the proposed problem.

On the other hand, many evolution inclusion problems are focused on a finite interval. This is because the solutions of some physical models may blow up, or we can gain a clearer understanding of the state of a physical system in finite time. If the time goes to infinity, it urges us to extend the concept of mild solutions such as Equation (1) in $[0, \infty)$ and, furthermore, to find the existence of global mild solutions. However, the technique for an infinite interval is more complex, and this topic may be a future work. In addition, our future works also include the topological properties of solution sets (including R_δ, acyclicity, connectedness, compactness, and contractibility) for fractional evolution inclusions of order $\alpha \in (1,2)$.

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