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On the Solvability of a Mixed Problem for a High-Order Partial Differential Equation with Fractional Derivatives with Respect to Time, with Laplace Operators with Spatial Variables and Nonlocal Boundary Conditions in Sobolev Classes

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Abstract: In this paper, we study the solvability of a mixed problem for a high-order partial differential equation with fractional derivatives with respect to time, and with Laplace operators with spatial variables and nonlocal boundary conditions in Sobolev classes.

Keywords: Banach space; Sobolev space; Laplace operators; nonlocal boundary conditions.

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1. Introduction

The spectral theory of operators finds numerous uses in various fields of mathematics and their applications.

An important part of the spectral theory of differential operators is the distribution of their eigenvalues. This classical question was studied for a second-order operator on a finite interval by Liouville and Sturm. Later, G.D. Birkhoff [1–3] studied the distribution of eigenvalues for an ordinary differential operator of arbitrary order on a finite interval with regular boundary conditions.

For quantum mechanics, it is especially interesting to distribute the eigenvalues of operators defined throughout the space and having a discrete spectrum. E.C. Titchmarsh [4–9] was the first to rigorously establish the formula for the distribution of the number of eigenvalues for a one-dimensional Sturm-Liouville operator on the whole axis with potential growing at infinity. He also first strictly established the distribution formula for the Schrödinger operator. B.M. Levitan [10–12] deserves much credit for the improvement of E.C. Titchmarsh's method.

In solving many mathematical physics problems, the need arises for the expansion of an arbitrary function in a Fourier series with respect to Sturm-Liouville eigenvalues. The so-called regular case of the Sturm-Liouville problem corresponding to a finite interval and a continuous coefficient of the equation has been studied for a relatively long time and is usually described in detail in the manuals on the equations of mathematical physics and integral equations.

The Sturm-Liouville problem for the so-called singular case, as well as with nonlocal boundary conditions, is much less known.

As it is known, so-called fractal media are studied in solid-state physics and, in particular, diffusion phenomena in them. In one of the models studied in [13], diffusion in a strongly porous (fractal) medium is described by an equation of the type of heat-conduction equation, but with a fractional derivative with respect to time coordinate

$$D_t^{(\alpha)} u(x,t) = \frac{\partial^2 (u(x,t)}{\partial x^2}\ , \ 0 < \alpha < 1. \tag{1}$$

The formulation of initial-boundary value problems for Equation (1), similar to the problems for parabolic differential equations, makes sense if by a regularized fractional derivative:

$$D^{(\alpha)} \varphi(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{d}{dt} \int_0^1 (t-\tau)^{-\alpha} \varphi(\tau) d\tau - t^{-\alpha} \varphi(0) \right], \ t \geq 0 \tag{2}$$

Study of the form equations

$$D_t^{(\alpha)} u = Au \tag{3}$$

where A is an elliptic operator (in [14–16]). In recent years, many authors studied fractional differential equations in [17–34].

2. Problem Formulation

In this work, we consider the equation of the form

$$D_{0t}^{\alpha} u(x,t) + (-\Delta)^{\nu} u(x,t) = f(x,t), \ \ (x,t) \in \Pi \times (0,\infty), \ l-1 < \alpha \leq l, \ l, \nu \in N \tag{4}$$

with initial conditions

$$\lim_{t \to 0} D_{0t}^{\alpha-k} u(x,t) = \varphi_k(x), \quad k = 1,2,\ldots,l \tag{5}$$

and boundary conditions

$$\begin{cases} \alpha_j \cdot (-\Delta)^i u(x_1,...,x_j,...,x_N,t) \mid_{x_j=0} + \beta_j \cdot (-\Delta)^i u(x_1,...,x_j,...,x_N,t) \mid_{x_j=\pi} = 0, \\ 1 \leq j \leq p, \\ \beta_j \cdot \frac{\partial (-\Delta)^i u(x_1,...,x_j,...,x_N,t)}{\partial x_j} \mid_{x_j=0} + \alpha_j \cdot \frac{\partial (-\Delta)^i u(x_1,...,x_j,...,x_N,t)}{\partial x_j} \mid_{x_j=\pi} = 0, \quad 1 \leq j \leq p, \\ (-\Delta)^i u(x_1,...,x_j,...,x_N,t) \mid_{x_j=0} = (-\Delta)^i u(x_1,...,x_j,...,x_N,t) \mid_{x_j=\pi}, \quad p+1 \leq j \leq q, \\ \frac{\partial (-\Delta)^i u(x_1,...,x_j,...,x_N,t)}{\partial x_j} \mid_{x_j=0} = \frac{\partial (-\Delta)^i u(x_1,...,x_j,...,x_N,t)}{\partial x_j} \mid_{x_j=\pi}, \quad p+1 \leq j \leq q, \\ (-\Delta)^i u(x_1,...,x_j,...,x_N,t) \mid_{x_j=0} = 0, \quad q+1 \leq j \leq N, \\ (-\Delta)^i u(x_1,...,x_j,...,x_N,t) \mid_{x_j=\pi} = 0, \quad q+1 \leq j \leq N, \\ 1 \leq p \leq q \leq N, \ i = 0,1,\ldots,\nu-1, \end{cases} \tag{6}$$

where $(x,t) = (x_1,\ldots,x_j,\ldots,x_N,t) \in \Pi \times (0,\infty)$, $\Pi = (0,\pi) \times \cdots \times (0,\pi)$, $\alpha_j = const$, $\beta_j = const$, and $f(x,t)$, $\varphi_k(x)$, $k = 1,2,\ldots,l$ are functions that can be expanded in terms of the system of eigenfunctions $\{v_n(x), n \in Z^N\}$ of the spectral problem:

$$(-\Delta)^{\nu} v(x) = \mu v(x), \tag{7}$$

$$\begin{cases} \alpha_j \cdot (-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)\mid_{x_j=0} + \beta_j \cdot (-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)\mid_{x_j=\pi} = 0, \\ 1 \le j \le p, \\ \beta_j \cdot \frac{\partial(-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)}{\partial x_j}\mid_{x_j=0} + \alpha_j \cdot \frac{\partial(-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)}{\partial x_j}\mid_{x_j=\pi} = 0, \quad 1 \le j \le p, \\ (-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)\mid_{x_j=0} = (-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)\mid_{x_j=\pi}, \quad p+1 \le j \le q, \\ \frac{\partial(-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)}{\partial x_j}\mid_{x_j=0} = \frac{\partial(-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)}{\partial x_j}\mid_{x_j=\pi}, \quad p+1 \le j \le q, \\ (-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)\mid_{x_j=0} = 0, \quad q+1 \le j \le N, \\ (-\Delta)^i v(x_1,\ldots,x_j,\ldots,x_N)\mid_{x_j=\pi} = 0, \quad q+1 \le j \le N, \\ 1 \le p \le q \le N,\ i = 0,1,\ldots,\nu-1. \end{cases} \tag{8}$$

Here, for $\alpha < 0$,, fractional integral D^α has the form

$$D^\alpha_{at} u(x,t) = \frac{sign(t-a)}{\Gamma(-\alpha)} \int_a^t \frac{u(x,\tau)\cdot d\tau}{|t-\tau|^{\alpha+1}},$$

$D^\alpha_{at} u(x,t) = u(x,t)$ for $\alpha = 0$, and for $l-1 < \alpha \le l$, $l \in N$, the fractional derivative has the form

$$D^\alpha_{at} u(x,t) = sign^l(t-a)\frac{d^l}{dt^l} D^{\alpha-l}_{at} u(x,t) =$$

$$= \frac{sign^{l+1}(t-a)}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_a^t \frac{u(x,\tau)\cdot d\tau}{|t-\tau|^{\alpha-l+1}}.$$

In [17], Problems (4)–(6) and, accordingly, spectral Problems (7) and (8) in the case $\nu = 1$, were considered.

3. Preliminaries

More detailed information for this section can be found in [17]. We look for eigenfunctions of spectral Problems (7) and (8) in the form of the product $v(x) = y_1(x_1)\cdot\cdots\cdot y_N(x_N)$. Then, we obtain, instead of spectral Problems (7) and (8), the following spectral problem:

$$-y''(x) = \mu y(x), \mu = \lambda^2 \tag{9}$$

$$\begin{cases} \alpha y(0) + \beta y(\pi) = 0, \\ \beta y'(0) + \alpha y'(\pi) = 0. \end{cases} \tag{10}$$

In the case of $|\alpha| = |\beta|$, i.e., with boundary conditions $y(0) = y(\pi)$, $y'(0) = y'(\pi)$ or $y(0) = -y(\pi)$, $y'(0) = -y'(\pi)$, spectral Problems (7) and (8) were investigated by many authors (see, for example, [35–41]). In order to simplify calculations, we confined ourselves to the case of $|\alpha| \ne |\beta|$, $\alpha \ne 0$, $\beta \ne 0$. It is not difficult to see that $\mu = 0$ is not an eigenvalue of Problems (9) and (10). In fact, if $\mu = 0$ is the eigenvalue, then $y'' = 0$, $y = ax + b$, $\alpha b + \beta(a\pi + b) = 0$, $\beta a + \alpha a = 0$. We obtained from here $a = 0$, $b = 0$, i.e., $y \equiv 0$. Similarly, for $\mu < 0$, Problems (9) and (10) have no nontrivial solutions.

For $\mu > 0$, the general solution of Problem (9) has the form

$$y(x) = Acos\lambda x + Bsin\lambda x.$$

From boundary conditions, we have:

$$\alpha y(0) + \beta y(\pi) = \alpha A + \beta(Acos\lambda\pi + Bsin\lambda\pi) = 0,$$

$$\beta y'(0) + \alpha y'(\pi) = \beta(\lambda B) + \alpha(\lambda B cos\lambda\pi - \lambda A sin\lambda\pi) = 0,$$

i.e.,

$$\begin{cases} (\alpha + \beta cos\lambda\pi)A + \beta sin\lambda\pi B = 0, \\ \alpha sin\lambda\pi A - (\beta + \alpha cos\lambda\pi)B = 0. \end{cases}$$

Hence, the nontrivial solutions of Problems (9) and (10) are only possible in the case of

$$(\alpha + \beta cos\lambda\pi)(-\beta - \alpha cos\lambda\pi) - \alpha\beta sin^2(\lambda\pi) = 0.$$

Furthermore,

$$-\alpha\beta - \alpha^2 cos\lambda\pi - \beta^2 cos\lambda\pi - \alpha\beta cos^2\lambda\pi - \alpha\beta sin^2\lambda\pi = 0,$$

i.e., $-(\alpha^2 + \beta^2)cos\lambda\pi = 2\alpha\beta$ or $cos\lambda\pi = \dfrac{-2\alpha\beta}{\alpha^2 + \beta^2}$.

Therefore, $\lambda\pi = \arccos \dfrac{-2\alpha\beta}{\alpha^2 + \beta^2}$ or

$$\lambda\pi = \pm \arccos \frac{-2\alpha\beta}{\alpha^2 + \beta^2} + 2n\pi, \quad n \in Z.$$

Further,

$$\mu_n^{\pm} = (2n + \varepsilon_n\varphi)^2 = (-2n - \varepsilon_n\varphi)^2 = \mu_{-n}^{\mp}, \ \varepsilon_n = \pm 1, \ \varphi = \frac{1}{\pi} \arccos \frac{-2\alpha\beta}{\alpha^2 + \beta^2}, \ n \in Z.$$

That's why $\mu_n^{\pm} \neq \mu_{-n}^{\pm}$ means that $\varepsilon_{-n} \neq -\varepsilon_n$, i.e., $\varepsilon_{-n} = \varepsilon_n$, $n \in Z$. Thus, the eigenvalues and eigenfunctions of Problems (9) and (10) are

$$\mu_n = \lambda_n^2 = (2n + \varepsilon_n\varphi)^2, \ \ \varphi = \frac{1}{\pi} \arccos \frac{-2\alpha\beta}{\alpha^2 + \beta^2}, \ \ \varepsilon_n = \pm 1, \varepsilon_{-n} = \varepsilon_n, \ \ n \in Z$$

and

$$y_n(x) = B_n \left(\frac{\beta + \alpha \cos \lambda_n \pi}{\alpha \sin \lambda_n \pi} \cos \lambda_n x + \sin \lambda_n x \right),$$

respectively, where

$$\frac{\beta + \alpha \cos \lambda_n \pi}{\alpha \sin \lambda_n \pi} = \frac{\beta - \frac{2\alpha^2\beta}{\alpha^2+\beta^2}}{\varepsilon_n \alpha \sqrt{1 - \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2)^2}}} = \frac{\beta(\beta^2 - \alpha^2)}{\varepsilon_n \alpha \mid \beta^2 - \alpha^2 \mid} = \varepsilon_n sign(\beta^2 - \alpha^2)\frac{\beta}{\alpha},$$

hence, $y_n(x) = B_n \left(\varepsilon_n sign(\beta^2 - \alpha^2) \frac{\beta}{\alpha} \cos \lambda_n x + sin\lambda_n x \right)$. Choosing

$$B_n = \varepsilon_n sign(\beta^2 - \alpha^2) \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1 + (2n)^{2s}}}$$

we obtain

$$y_n(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\alpha^2 + \beta^2}} \frac{1}{\sqrt{1 + (2n)^{2s}}} \left(\beta \cos \lambda_n x + \varepsilon_n sign(\beta^2 - \alpha^2) \alpha \sin \lambda_n x \right).$$

Denote $\omega_n = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\alpha^2+\beta^2}} \frac{1}{\sqrt{1+(2n)^{2s}}}$. Then,

$$y_n(x) = \omega_n \left(\beta \cos \lambda_n x + \varepsilon_n sign(\beta^2 - \alpha^2)\alpha \sin \lambda_n x\right).$$

The norm in space $W_2^s(0,\pi)$ is introduced as follows:

$$\left\|f\right\|^2_{W_2^s(0,\pi)} = \left\|f\right\|^2_{L_2(0,\pi)} + \left\|D^s f\right\|^2_{L_2(0,\pi)}.$$

Let $\varepsilon_n = \varepsilon_{-n}$. Then, system of vectors

$$z_n(x) = \omega_n \left(\beta \cos 2nx + \varepsilon_n sign(\beta^2 - \alpha^2)\alpha \sin 2nx\right)$$

forms the complete orthonormal system in $W_2^s(0,\pi)$. The following lemma holds.

Lemma 1. *Let $\{a_n\}$ be a finite system of complex numbers. Then, inequalities*

$$\left\|\sum_{-N}^{N} a_n(y_n(x) - z_n(x))\right\|_{L_2(0,\pi)} \leq \sqrt{2} \cdot \max_{x\in[0,\pi]} \left|e^{i\varphi x} - 1\right| \cdot \sqrt{\sum_{-N}^{N} \mid a_n \cdot c_n \mid^2}$$

are valid where

$$c_n = \frac{1}{\sqrt{1+(2n)^{2s}}},\ s = 1, 2, 3, ...$$

Proof. Calculating the difference of $y_n(x) - z_n(x)$, we obtain

$$y_n(x) - z_n(x) =$$

$$\omega_n[\beta(\cos \lambda_n x - \cos 2nx) + \varepsilon_n sign(\beta^2 - \alpha^2)\alpha(\sin \lambda_n x - \sin 2nx)] =$$

$$= \omega_n[(\varepsilon_n sign(\beta^2 - \alpha^2)\alpha + \beta i)\frac{e^{i\varepsilon_n \varphi x} - 1}{2i} e^{2nix} +$$

$$+(\varepsilon_n sign(\beta^2 - \alpha^2)\alpha - \beta i)\frac{1 - e^{-i\varepsilon_n \varphi x}}{2i} e^{-2nix}].$$

Then,

$$\sum_{-N}^{N} a_n(y_n - z_n) = \sum_{-N}^{N} a_n \omega_n \left[(\varepsilon_n sign(\beta^2 - \alpha^2)\alpha + \beta i)\frac{e^{i\varepsilon_n \varphi x} - 1}{2i} e^{2nix} + \right.$$

$$\left. +(\varepsilon_n sign(\beta^2 - \alpha^2)\alpha - \beta i)\frac{1 - e^{-i\varepsilon_n \varphi x}}{2i} e^{-2nix}\right].$$

Using properties of the norm, we have

$$\left\|\sum_{-N}^{N} a_n(y_n - z_n)\right\|_{L_2(0,\pi)} =$$

$$= \left\| \frac{sign(\beta^2 - \alpha^2)\alpha + \beta i}{2i}(e^{i\varphi x} - 1) \sum_{-N,\varepsilon_n=1}^{N} a_n \omega_n e^{2nix} + \right.$$

$$+\frac{-sign(\beta^2 - \alpha^2)\alpha + \beta i}{2i}(e^{-i\varphi x} - 1) \sum_{-N,\varepsilon_n=-1}^{N} a_n \omega_n e^{2nix} +$$

$$+\frac{sign(\beta^2-\alpha^2)\alpha-\beta i}{2i}(1-e^{-i\varphi x})\sum_{-N,\varepsilon_n=1}^{N}a_n\omega_n e^{-2nix}+$$

$$+\frac{-sign(\beta^2-\alpha^2)\alpha-\beta i}{2i}(1-e^{i\varphi x})\sum_{-N,\varepsilon_n=-1}^{N}a_n\omega_n e^{-2nix}\Bigg\|_{L_2(0,\pi)}=$$

$$=\Bigg\|\frac{sign(\beta^2-\alpha^2)\alpha+\beta i}{2i}(e^{i\varphi x}-1)\times$$

$$\times\left(\sum_{-N,\varepsilon_n=1}^{N}a_n\omega_n e^{2nix}+\sum_{-N,\varepsilon_n=-1}^{N}a_n\omega_n e^{-2nix}\right)+$$

$$+\frac{-sign(\beta^2-\alpha^2)\alpha+\beta i}{2i}(e^{-i\varphi x}-1)\times$$

$$\times\left(\sum_{-N,\varepsilon_n=-1}^{N}a_n\omega_n e^{2nix}+\sum_{-N,\varepsilon_n=1}^{N}a_n\omega_n e^{-2nix}\right)\Bigg\|_{L_2(0,\pi)}\leq$$

$$\leq\frac{\sqrt{\alpha^2+\beta^2}}{2}\cdot\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|\times$$

$$\times\left(\left\|\sum_{-N,\varepsilon_n=1}^{N}a_n\omega_n e^{2nix}+\sum_{-N,\varepsilon_n=-1}^{N}a_n\omega_n e^{-2nix}\right\|_{L_2(0,\pi)}+\right.$$

$$\left.+\left\|\sum_{-N,\varepsilon_n=-1}^{N}a_n\omega_n e^{2nix}+\sum_{-N,\varepsilon_n=1}^{N}a_n\omega_n e^{-2nix}\right\|_{L_2(0,\pi)}\right)=$$

$$=\frac{\sqrt{\alpha^2+\beta^2}}{2}\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|\left(\sqrt{\sum_{-N,\varepsilon_n=1}^{N}|\ a_n\omega_n\ |^2+\sum_{-N,\varepsilon_n=-1}^{N}|\ a_n\omega_n\ |^2}+\right.$$

$$\left.+\sqrt{\sum_{-N,\varepsilon_n=-1}^{N}|\ a_n\omega_n\ |^2+\sum_{-N,\varepsilon_n=1}^{N}|\ a_n\omega_n\ |^2}\right)\cdot\sqrt{\pi}=$$

$$=\sqrt{\alpha^2+\beta^2}\cdot\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|\cdot\sqrt{\pi}\cdot\sqrt{\sum_{-N}^{N}|\ a_n\omega_n\ |^2}.$$

Thus, denoting $c_n=\frac{1}{\sqrt{1+(2n)^{2s}}}$, we obtain

$$\left\|\sum_{-N}^{N}a_n(y_n(x)-z_n(x))\right\|_{L_2(0,\pi)}\leq\sqrt{2}\cdot\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|\cdot\sqrt{\sum_{-N}^{N}|a_n\cdot c_n|^2}.$$

□

Lemma 2. *Let $\{a_n\}$ be a finite system of complex numbers. Then, inequalities*

$$\left\|D^s\sum_{-N}^{N}a_n(y_n(x)-z_n(x))\right\|_{L_2(0,\pi)}\leq$$

$$\leq\sqrt{2}\Big[\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|+(\varphi+1)^s-1\Big]\cdot\sqrt{\sum_{-N}^{N}|a_n\cdot c_n\cdot(2n)^s|^2}$$

are valid at $s = 1, 2, 3, \ldots$.

Proof. Denote

$$\theta = \sqrt{2} \cdot \max_{x\in[0,\pi]} |e^{i\varphi x} - 1|,$$

since

$$\sum_{-N}^{N} a_n(y_n - z_n) = \frac{sign(\beta^2 - \alpha^2)\alpha + \beta i}{2i} \cdot (e^{i\varphi x} - 1) \cdot$$
$$\cdot \Big(\sum_{-N,\varepsilon_n=1}^{N} a_n \cdot \omega_n \cdot e^{2nix} + \sum_{-N,\varepsilon_n=-1}^{N} a_n \cdot \omega_n \cdot e^{-2nix} \Big) +$$
$$+ \frac{-sign(\beta^2 - \alpha^2)\alpha + \beta i}{2i} \cdot \Big(e^{-i\varphi x} - 1 \Big) \cdot$$
$$\cdot \Big(\sum_{-N,\varepsilon_n=-1}^{N} a_n \cdot \omega_n \cdot e^{2nix} + \sum_{-N,\varepsilon_n=1}^{N} a_n \cdot \omega_n \cdot e^{-2nix} \Big),$$

using properties of the norm, we have

$$\left\| D^s \sum_{-N}^{N} a_n(y_n - z_n) \right\|_{L_2(0,\pi)} \le \frac{\sqrt{\alpha^2 + \beta^2}}{2} \cdot \left(\left\| \sum_{k=0}^{s} C_s^k \cdot D^k(e^{i\varphi x} - 1) \cdot \right.\right.$$
$$\left. \cdot D^{s-k} \left(\sum_{-N,\varepsilon_n=1}^{N} a_n \cdot \omega_n \cdot e^{2nix} + \sum_{-N,\varepsilon_n=-1}^{N} a_n \cdot \omega_n \cdot e^{-2nix} \right) \right\|_{L_2(0,\pi)} +$$
$$+ \left\| \sum_{k=0}^{s} C_s^k \cdot D^k(e^{-i\varphi x} - 1) \cdot \right.$$
$$\left.\left. \cdot D^{s-k} \left(\sum_{-N,\varepsilon_n=-1}^{N} a_n \cdot \omega_n \cdot e^{2nix} + \sum_{-N,\varepsilon_n=1}^{N} a_n \cdot \omega_n \cdot e^{-2nix} \right) \right\|_{L_2(0,\pi)} \right) \le$$
$$\le \frac{\sqrt{\alpha^2 + \beta^2}}{2} \cdot \left(\max_{x\in[0,\pi]} \left| e^{i\varphi x} - 1 \right| \times \right.$$
$$\times \left\| \sum_{-N,\varepsilon_n=1}^{N} a_n \cdot \omega_n \cdot (2n)^s e^{2nix} + \sum_{-N,\varepsilon_n=-1}^{N} a_n \cdot \omega_n \cdot (-2n)^s e^{-2nix} \right\|_{L_2(0,\pi)} +$$
$$+ \sum_{k=1}^{s} C_s^k \cdot \varphi^k \cdot \left\| \sum_{-N,\varepsilon_n=1}^{N} a_n \cdot \omega_n \cdot (2n)^{s-k} e^{2nix} + \right.$$
$$\left. + \sum_{-N,\varepsilon_n=-1}^{N} a_n \cdot \omega_n \cdot (-2n)^{s-k} e^{-2nix} \right\|_{L_2(0,\pi)} + \max_{x\in[0,\pi]} \left| e^{i\varphi x} - 1 \right| \times$$
$$\times \left\| \sum_{-N,\varepsilon_n=-1}^{N} a_n \cdot \omega_n \cdot (2n)^s \cdot e^{2nix} + \sum_{-N,\varepsilon_n=1}^{N} a_n \cdot \omega_n \cdot (-2n)^s \cdot e^{-2nix} \right\|_{L_2(0,\pi)} +$$
$$+ \sum_{k=1}^{s} C_s^k \cdot \varphi^k \cdot \left\| \sum_{-N,\varepsilon_n=-1}^{N} a_n \cdot \omega_n \cdot (2n)^{s-k} \cdot e^{2nix} + \right.$$
$$\left.\left. + \sum_{-N,\varepsilon_n=1}^{N} a_n \cdot \omega_n \cdot (-2n)^{s-k} \cdot e^{-2nix} \right\|_{L_2(0,\pi)} \right) \le \sqrt{\alpha^2 + \beta^2} \times$$

$$\times\left(\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|\sqrt{\sum_{-N}^{N}|a_n\omega_n(2n)^s|^2}+\sum_{k=1}^{s}C_s^k\varphi^k\sqrt{\sum_{-N}^{N}|a_n\omega_n(2n)^{s-k}|^2}\right)\times$$

$$\times\sqrt{\pi}=\sqrt{2}\left(\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|\cdot\sqrt{\sum_{-N}^{N}|a_n\cdot c_n\cdot(2n)^s|^2}+\right.$$

$$\left.+\sum_{k=1}^{s}C_s^k\cdot\varphi^k\cdot\sqrt{\sum_{-N}^{N}|a_n\cdot c_n\cdot(2n)^{s-k}|^2}\right)\leq$$

$$\leq\sqrt{2}\Big[\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|+(\varphi+1)^s-1\Big]\cdot\sqrt{\sum_{-N}^{N}|a_n\cdot c_n\cdot(2n)^s|^2}.$$

Thus, inequalities

$$\left\|D^s\sum_{-N}^{N}a_n(y_n(x)-z_n(x))\right\|_{L_2(0,\pi)}\leq$$

$$\leq\sqrt{2}\Big[\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|+(\varphi+1)^s-1\Big]\cdot\sqrt{\sum_{-N}^{N}|a_n\cdot c_n\cdot(2n)^s|^2}$$

hold at $s=1,2,3,\ldots$. □

Using Lemmas 1 and 2, we obtain

Lemma 3. *Let $\{a_n\}$ be a finite system of complex numbers. Then the following inequality*

$$\left\|\sum_{-N}^{N}a_n(y_n(x)-z_n(x))\right\|_{W_2^s(0,\pi)}\leq$$

$$\leq\sqrt{\theta^2+2\Big(\frac{\theta}{\sqrt{2}}+(\varphi+1)^s-1\Big)^2}\cdot\sigma(s)\cdot\sqrt{\sum_{-N}^{N}|a_n|^2}$$

is valid where $\sigma(0)=\frac{1}{\sqrt{2}}$, $\sigma(s)=1$ at $s>0$.

Lemma 4. *Let $\alpha\neq 0$, $\beta\neq 0$, $|\alpha|\neq|\beta|$ be real numbers, and*

$$\rho=\sqrt{\theta^2+2\left(\frac{\theta}{\sqrt{2}}+(\varphi+1)^s-1\right)^2}\cdot\sigma(s)<1$$

where $\sigma(0)=\frac{1}{\sqrt{2}}$, $\sigma(s)=1$ at $s>0$, $\theta=\sqrt{2}\cdot\max_{x\in[0,\pi]}\left|e^{i\varphi x}-1\right|$, $\lambda_n=2n+\varepsilon_n\cdot\varphi$, $\varphi=\frac{1}{\pi}\arccos\frac{-2\alpha\beta}{\alpha^2+\beta^2}$, $\varepsilon_n=\varepsilon_{-n}=\pm1$ at $n\in Z$.

Then, eigenfunction system

$$y_n(x)=\sqrt{\frac{2}{\pi}}\cdot\frac{\beta\cos\lambda_n x+\varepsilon_n\cdot sign(\beta^2-\alpha^2)\cdot\alpha\sin\lambda_n x}{\sqrt{\alpha^2+\beta^2}\cdot\sqrt{1+(2n)^{2s}}},\ \ n\in Z,$$

of spectral Problems (9) and (10) forms the Riesz basis in the space $W_2^s(0,\pi)$.

Proof. Vector system

$$z_n(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\beta \cos 2nx + \varepsilon_n \cdot sign(\beta^2 - \alpha^2) \cdot \alpha \sin 2nx}{\sqrt{\alpha^2 + \beta^2} \cdot \sqrt{1 + (2n)^{2s}}}, n \in Z$$

forms the complete orthonormal system in Hilbert space $W_2^s(0, \pi)$,, and vector system

$$y_n(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\beta \cos \lambda_n x + \varepsilon_n \cdot sign(\beta^2 - \alpha^2) \cdot \alpha \sin \lambda_n x}{\sqrt{\alpha^2 + \beta^2} \cdot \sqrt{1 + (2n)^{2s}}}, n \in Z$$

by virtue of Lemma 3 satisfying the theorem conditions by R. Paley and N. Wiener (see p. 224, [39]). This theorem implies that system of vectors $\{y_n(x)\}_{n \in Z}$ forms the Riesz basis in space $W_2^s(0, \pi)$. □

Lemma 5. *Operator*

$$Ly = -y''$$

with domain

$$D(L) = \{y(x) : y(x) \in C^2(0, \pi) \cap C^1[0, \pi], y'' \in L_2(0, \pi),$$
$$\alpha y(0) + \beta y(\pi) = 0, \beta y'(0) + \alpha y'(0) = 0\}$$

is a symmetric operator in class $L_2(0, \pi)$.

Proof. Indeed, since functions f and $\overline{g}$ belong to domain $D(L)$, we have $Lf \in L_2(0, \pi)$, $L\overline{g} = \overline{Lg} \in L_2(0, \pi)$, and the second Green formula

$$\int_G (Lu \cdot v - u \cdot Lv)dx = -\int_{\partial G} \left(\frac{\partial u}{\partial n} \cdot v - u \cdot \frac{\partial v}{\partial n} \right) ds$$

at $u = f$ and $v = \overline{g}$ takes the form

$$\int_0^\pi (Lf \cdot \overline{g} - f \cdot \overline{Lg})dx = -\left(f'(x)\overline{g(x)} - f(x)\overline{g'(x)} \right) \Big|_0^\pi.$$

Further, functions f and $\overline{g}$ satisfy the boundary conditions:

$$\alpha f(0) + \beta f(\pi) = 0,\ \beta f'(0) + \alpha f'(\pi) = 0,\ \alpha\overline{g(0)} + \beta\overline{g(\pi)} = 0,\ \beta\overline{g'(0)} + \alpha\overline{g'(\pi)} = 0.$$

By assumption, $\alpha \neq 0$, $\beta \neq 0$. Therefore,

$$f(0) \cdot \overline{g(\pi)} - f(\pi) \cdot \overline{g(0)} = 0$$

and

$$f'(0) \cdot \overline{g'(\pi)} - f'(\pi) \cdot \overline{g'(0)} = 0,$$

i.e., $f(0) \cdot \overline{g(\pi)} = f(\pi) \cdot \overline{g(0)}$ and $f'(0) \cdot \overline{g'(\pi)} = f'(\pi) \cdot \overline{g'(0)}$. For here, we obtain

$$\frac{f(\pi)}{f(0)} = \frac{\overline{g(\pi)}}{\overline{g(0)}} = k_0 = -\frac{\alpha}{\beta}$$

and

$$\frac{f'(\pi)}{f'(0)} = \frac{\overline{g'(\pi)}}{\overline{g'(0)}} = k_1 = -\frac{\beta}{\alpha},\ k_0 \cdot k_1 = 1.$$

So, $f(\pi) = k_0 f(0), \overline{g(\pi)} = k_0 \overline{g(0)}$ è $f'(\pi) = k_1 f'(0), \overline{g'(\pi)} = k_1 \overline{g'(0)}$. Thus,

$$\int_0^\pi (Lf \cdot \overline{g} - f \cdot \overline{Lg})\, dx = -\left(f'(x) \cdot \overline{g(x)} - f(x) \cdot \overline{g'(x)}\right)\Big|_0^\pi =$$

$$= -\left(f'(\pi) \cdot \overline{g(\pi)} - f(\pi) \cdot \overline{g'(\pi)}\right) + \left(f'(0) \cdot \overline{g(0)} - f(0) \cdot \overline{g'(0)}\right) =$$

$$= -\left(f'(0) \cdot \overline{g(0)} - f(0) \cdot \overline{g'(0)}\right) + \left(f'(0) \cdot \overline{g(0)} - f(0) \cdot \overline{g'(0)}\right) = 0.$$

Thereby, $(Lf, g) = (f, Lg), \forall f, g \in D(L)$. □

Theorem 1. *Let $\alpha \neq 0$, $\beta \neq 0$, $|\alpha| \neq |\beta|$ be real number, and*

$$\rho = \sqrt{\theta^2 + 2(\frac{\theta}{\sqrt{2}} + (\varphi + 1)^s - 1)^2} \cdot \sigma(s) < 1$$

where $\sigma(0) = \frac{1}{\sqrt{2}}$, $\sigma(s) = 1$ at $s > 0$, $\theta = \sqrt{2} \cdot \max_{x \in [0,\pi]} \left|e^{i\varphi x} - 1\right|$, $\lambda_n = 2n + \varepsilon_n \cdot \varphi$, $\varphi = \frac{1}{\pi} \arccos \frac{-2\alpha\beta}{\alpha^2 + \beta^2}$, $\varepsilon_n = \varepsilon_{-n} = \pm 1$ at $n \in Z$. Then the system of eigenfunctions

$$\overline{y}_n(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\beta \cos \lambda_n x + \varepsilon_n \cdot sign(\beta^2 - \alpha^2) \cdot \alpha \sin \lambda_n x}{\sqrt{\alpha^2 + \beta^2} \cdot \sqrt{1 + \mid \lambda_n \mid^{2s}}}, \;\; n \in Z,$$

of spectral Problems (9) and (10) form the complete orthonormal system in Sobolev classes $W_2^s(0, \pi)$.

Proof. Symmetry of operator L implies that eigenfunctions $\{\overline{y}_n(x)\}_{n \in Z}$ of operator L, corresponding to the different eigenvalues, are orthogonal in classes $L_2(0, \pi)$.

System of functions $\{D^\alpha \overline{y}_n(x)\}_{n \in Z}$ is also the system of eigenfunctions of a similar operator corresponding to different eigenvalues, which implies that functions of system $\{D^\alpha \overline{y}_n(x)\}_{n \in Z}$ are orthogonal in classes $L_2(0, \pi)$.

As a result, we see that system of eigenfunctions $\{\overline{y}_n(x)\}_{n \in Z}$ of operator L, corresponding to different eigenvalues, are orthogonal in the Sobolev classes $W_2^s(0, \pi)$. It is known that, if a sequence of vectors $\{\psi_n(x)\}_{n \in Z}$ forms the Riesz basis in a Hilbert space H, then system of vectors

$$\{\widehat{\psi}_n(x)\}_{n \in Z} \left(\widehat{\psi}_n(x) = \frac{\psi_n(x)}{\|\psi_n(x)\|}, n \in Z\right)$$

also forms the Riesz basis in H (see p. 374, [42]).

By virtue of Lemma 4, system of eigenvectors $\{y_n(x)\}_{n \in Z}$ forms the Riesz basis in space $W_2^s(0, \pi)$. The orthogonality of this system implies that $\{\overline{y}_n(x)\}_{n \in Z}$ is a complete orthonormal system in the Sobolev classes $W_2^s(0, \pi)$. □

Theorem 1 and the Sobolev embedding theorem imply the following corollaries.

Corollary 1. *Let $\alpha \neq 0$, $\beta \neq 0$, $|\alpha| \neq |\beta|$ be real numbers, and*

$$\rho = \sqrt{\theta^2 + 2\left(\frac{\theta}{\sqrt{2}} + \varphi\right)^2} < 1$$

where $\theta = \sqrt{2} \cdot \max\limits_{x \in [0,\pi]} \left| e^{i\varphi x} - 1 \right|$, $\lambda_n = 2n + \varepsilon_n \cdot \varphi$, $\varphi = \frac{1}{\pi} \arccos \frac{-2\alpha\beta}{\alpha^2+\beta^2}$, $\varepsilon_n = \varepsilon_{-n} = \pm 1$ *at* $n \in Z$. *Then, the Fourier series for function* $f(x) \in W_2^1(0,\pi) \cap C[0,\pi]$ *in orthonormal eigenfunctions*

$$\overline{y}_n(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\beta \cos \lambda_n x + \varepsilon_n \cdot sign(\beta^2 - \alpha^2) \cdot \alpha \sin \lambda_n x}{\sqrt{\alpha^2 + \beta^2} \cdot \sqrt{1 + |\lambda_n|^2}}, \ n \in Z$$

of spectral Problems (9) and (10) uniformly converges on segment $[0, \pi]$ *to function* $f(x)$.

Corollary 2. *Let* $\alpha \neq 0$, $\beta \neq 0$, $|\alpha| \neq |\beta|$ *be real numbers, and*

$$\rho = \sqrt{\theta^2 + 2\left(\frac{\theta}{\sqrt{2}} + (\varphi + 1)^s - 1\right)^2} < 1$$

where $s > k$, $\theta = \sqrt{2} \cdot \max\limits_{x \in [0,\pi]} \left| e^{i\varphi x} - 1 \right|$, $\lambda_n = 2n + \varepsilon_n \cdot \varphi$, $\varphi = \dfrac{1}{\pi} \arccos \dfrac{-2\alpha\beta}{\alpha^2 + \beta^2}$, $\varepsilon_n = \varepsilon_{-n} = \pm 1$ *at* $n \in Z$. *Then the Fourier series for function* $f(x) \in W_2^s(0,\pi) \cap C^k[0,\pi]$ *in orthonormal eigenfunctions*

$$\overline{y}_n(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\beta \cos \lambda_n x + \varepsilon_n \cdot sign(\beta^2 - \alpha^2) \cdot \alpha \sin \lambda_n x}{\sqrt{\alpha^2 + \beta^2} \cdot \sqrt{1 + |\lambda_n|^2}}, \ n \in Z,$$

of spectral Problems (9) and (10) converges in the norm of space $C^k[0,\pi]$ *to function* $f(x)$.

The scalar product in space $W_2^{s_1,s_2}((0,\pi) \times (0,\pi))$ is introduced in the following way:

$$(f(x,y), g(x,y))_{W_2^{s_1,s_2}((0,\pi)\times(0,\pi))} = (f(x,y), g(x,y))_{L_2((0,\pi)\times(0,\pi))} +$$

$$+ (D_x^{s_1} f(x,y), D_x^{s_1} g(x,y))_{L_2((0,\pi)\times(0,\pi))} + (D_y^{s_2} f(x,y), D_y^{s_2} g(x,y))_{L_2((0,\pi)\times(0,\pi))} +$$
$$+ (D_{x,y}^{s_1,s_2} f(x,y), D_{x,y}^{s_1,s_2} g(x,y))_{L_2((0,\pi)\times(0,\pi))}.$$

Respectively, the norm in this space is introduced as follows:

$$\left\| f(x,y) \right\|^2_{W_2^{s_1,s_2}((0,\pi)\times(0,\pi))} =$$

$$= \left\| f(x,y) \right\|^2_{L_2((0,\pi)\times(0,\pi))} + \left\| D_x^{s_1} f(x,y) \right\|^2_{L_2((0,\pi)\times(0,\pi))} +$$

$$+ \left\| D_y^{s_2} f(x,y) \right\|^2_{L_2((0,\pi)\times(0,\pi))} + \left\| D_{x,y}^{s_1,s_2} f(x,y) \right\|^2_{L_2((0,\pi)\times(0,\pi))}.$$

Lemma 6. *If* $\{\psi_m^{(1)}(x)\}$ *and* $\{\psi_n^{(2)}(y)\}$ *are complete orthonormal systems in* $W_2^{s_1}(0,\pi)$ *and* $W_2^{s_2}(0,\pi)$,, *respectively, then the system of all products*

$$f_{mn}(x,y) = \psi_m^{(1)}(x) \cdot \psi_n^{(2)}(y)$$

is a complete orthonormal system in $W_2^{s_1,s_2}((0,\pi) \times (0,\pi))$, *where* $s_1, s_2 = 1, 2, 3, ...$ *and* $x, y \in (0, \pi)$

Proof. By virtue of the Fubini theorem,

$$\left\| f_{mn}(x,y) \right\|^2_{W_2^{s_1,s_2}((0,\pi)\times(0,\pi))} = \left\| \psi_m^{(1)}(x) \right\|^2_{L_2(0,\pi)} \cdot \left\| \psi_n^{(2)}(y) \right\|^2_{L_2(0,\pi)} +$$

$$+\left\|D_x^{s_1}\psi_m^{(1)}(x)\right\|_{L_2(0,\pi)}^2\cdot\left\|\psi_n^{(2)}(y)\right\|_{L_2(0,\pi)}^2+\left\|\psi_m^{(1)}(x)\right\|_{L_2(0,\pi)}^2\cdot\left\|D_y^{s_2}\psi_n^{(2)}(y)\right\|_{L_2(0,\pi)}^2+$$

$$+\left\|D_x^{s_1}\psi_m^{(1)}(x)\right\|_{L_2(0,\pi)}^2\cdot\left\|D_y^{s_2}\psi_n^{(2)}(y)\right\|_{L_2(0,\pi)}^2=$$

$$=\left(\left\|\psi_m^{(1)}(x)\right\|_{L_2(0,\pi)}^2+\left\|D_x^{s_1}\psi_m^{(1)}(x)\right\|_{L_2(0,\pi)}^2\right)\cdot\left\|\psi_n^{(2)}(y)\right\|_{L_2(0,\pi)}^2+$$

$$+\left(\left\|\psi_m^{(1)}(x)\right\|_{L_2(0,\pi)}^2+\left\|D_x^{s_1}\psi_m^{(1)}(x)\right\|_{L_2(0,\pi)}^2\right)\cdot\left\|D_y^{s_2}\psi_n^{(2)}(y)\right\|_{L_2(0,\pi)}^2=$$

$$=\left(\left\|\psi_m^{(1)}(x)\right\|_{L_2(0,\pi)}^2+\left\|D_x^{s_1}\psi_m^{(1)}(x)\right\|_{L_2(0,\pi)}^2\right)\cdot$$

$$\cdot\left(\left\|\psi_n^{(2)}(y)\right\|_{L_2(0,\pi)}^2+\left\|D_y^{s_2}\psi_n^{(2)}(y)\right\|_{L_2(0,\pi)}^2\right)=1.$$

If $m\neq m_1$ or $n\neq n_1$, by the same theorem

$$(f_{mn}(x,y),f_{m_1n_1}(x,y))_{W_2^{s_1,s_2}((0,\pi)\times(0,\pi))}=$$

$$=(f_{mn}(x,y),f_{m_1n_1}(x,y))_{L_2((0,\pi)\times(0,\pi))}+$$

$$+(D_x^{s_1}f_{mn}(x,y),D_x^{s_1}f_{m_1n_1}(x,y))_{L_2((0,\pi)\times(0,\pi))}+$$

$$+(D_y^{s_2}f_{mn}(x,y),D_y^{s_2}f_{m_1n_1}(x,y))_{L_2((0,\pi)\times(0,\pi))}+$$

$$+(D_{x,y}^{s_1,s_2}f_{mn}(x,y),D_{x,y}^{s_1,s_2}f_{m_1n_1}(x,y))_{L_2((0,\pi)\times(0,\pi))}=$$

$$=(\psi_m^{(1)}(x),\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)}\cdot(\psi_n^{(2)}(y),\psi_{n_1}^{(2)}(y))_{L_2(0,\pi)}+$$

$$+(D_x^{s_1}\psi_m^{(1)}(x),D_x^{s_1}\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)}\cdot(\psi_n^{(2)}(y),\psi_{n_1}^{(2)}(y))_{L_2(0,\pi)}+$$

$$+(\psi_m^{(1)}(x),\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)}\cdot(D_y^{s_2}\psi_n^{(2)}(y),D_y^{s_2}\psi_{n_1}^{(2)}(y))_{L_2(0,\pi)}+$$

$$+(D_x^{s_1}\psi_m^{(1)}(x),D_x^{s_1}\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)}\cdot(D_y^{s_2}\psi_n^{(2)}(y),D_y^{s_2}\psi_{n_1}^{(2)}(y))_{L_2(0,\pi)}=$$

$$=((\psi_m^{(1)}(x),\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)}+(D_x^{s_1}\psi_m^{(1)}(x),D_x^{s_1}\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)})\cdot$$

$$\cdot(\psi_n^{(2)}(y),\psi_{n_1}^{(2)}(y))_{L_2(0,\pi)}+$$

$$+((\psi_m^{(1)}(x),\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)}+(D_x^{s_1}\psi_m^{(1)}(x),D_x^{s_1}\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)})\cdot$$

$$\cdot(D_y^{s_2}\psi_n^{(2)}(y),D_y^{s_2}\psi_{n_1}^{(2)}(y))_{L_2(0,\pi)}=$$

$$=((\psi_m^{(1)}(x),\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)}+(D_x^{s_1}\psi_m^{(1)}(x),D_x^{s_1}\psi_{m_1}^{(1)}(x))_{L_2(0,\pi)})\cdot$$

$$\cdot((\psi_n^{(2)}(y),\psi_{n_1}^{(2)}(y))_{L_2(0,\pi)}+(D_y^{s_2}\psi_n^{(2)}(y),D_y^{s_2}\psi_{n_1}^{(2)}(y))_{L_2(0,\pi)})=0$$

since scalar product $(f_{mn}(x,y),f_{m_1n_1}(x,y))_{W_2^{s_1,s_2}((0,\pi)\times(0,\pi))}$ of two variables exist on $\Pi=(0,\pi)\times(0,\pi)$. Let us prove the completeness of system $\{f_{mn}(x,y)\}$. Assume that there exists a function $f(x,y)$ in $W_2^{s_1,s_2}((0,\pi)\times(0,\pi))$ that is orthogonal to all functions $f_{mn}(x,y)$. Set

$$F_m(y)=(f(x,y),\psi_m^{(1)}(x))_{W_2^{s_1}(0,\pi)}.$$

It is easy to see, that function $F_m(y)$ belongs to class $W_2^{s_2}(0,\pi)$. That's why for any n,m again applying the Fubini theorem, we obtain

$$(F_m(y), \psi_n^{(2)}(y))_{W_2^{s_2}(0,\pi)} = (f(x,y), f_{mn}(x,y))_{W_2^{s_1,s_2}((0,\pi)\times(0,\pi))} = 0.$$

By completeness of system $\psi_n^{(2)}(y)$, for almost all y

$$F_m(y) = 0.$$

But then, for almost every y,, equalities

$$(f(x,y), \psi_m^{(1)}(x))_{W_2^{s_1}(0,\pi)} = 0$$

hold for all m. Completeness of system $\psi_m^{(1)}(x)$ implies that, for almost all y, the set of those x, for which

$$f(x,y) \neq 0,$$

has the measure zero. By virtue of the Fubini theorem, this means that, on $\Pi = (0,\pi)\times(0,\pi)$, function $f(x,y)$ is zero almost everywhere. □

The scalar product in space $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$ is introduced in the following way:

$$(f(x), g(x))_{W_2^{s_1,s_2,\ldots,s_N}(\Pi)} = (f(x), g(x))_{L_2(\Pi)} +$$
$$+ \sum_{j_1=1}^{N} (D_{x_{j_1}}^{s_{j_1}} f(x), D_{x_{j_1}}^{s_{j_1}} g(x))_{L_2(\Pi)} +$$
$$+ \sum_{1\le j_1<j_2\le N} (D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}} f(x), D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}} g(x))_{L_2(\Pi)} + \cdots +$$
$$+ \sum_{1\le j_1<j_2<\cdots<j_N\le N} (D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}} \ldots D_{x_{j_N}}^{s_{j_N}} f(x), D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}} \ldots D_{x_{j_N}}^{s_{j_N}} g(x))_{L_2(\Pi)}.$$

Respectively, the norm in this space is introduced as follows:

$$\left\|f(x)\right\|^2_{W_2^{s_1,s_2,\ldots,s_N}(\Pi)} = \left\|f(x)\right\|^2_{L_2(\Pi)} + \sum_{j_1=1}^{N} \left\|D_{x_{j_1}}^{s_{j_1}} f(x)\right\|^2_{L_2(\Pi)} +$$
$$+ \sum_{1\le j_1<j_2\le N} \left\|D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}} f(x)\right\|^2_{L_2(\Pi)} +$$
$$+ \cdots + \sum_{1\le j_1<j_2<\cdots<j_N\le N} \left\|D_{x_{j_1}}^{s_{j_1}} D_{x_{j_2}}^{s_{j_2}} \ldots D_{x_{j_N}}^{s_{j_N}} f(x)\right\|^2_{L_2(\Pi)}.$$

Using the method of mathematical induction and Lemma 6, we obtain the following:

Lemma 7. *If* $\{\psi_{m_1}^{(1)}(x_1)\}, \ldots, \{\psi_{m_N}^{(N)}(x_N)\}$ *are complete orthonormal systems in spaces* $W_2^{s_1}(0,\pi)$, $\ldots$, $W_2^{s_N}(0,\pi)$, *respectively, then system of all products*

$$f_m(x) = f_{m_1\ldots m_N}(x_1,\ldots,x_N) = \psi_{m_1}^{(1)}(x_1)\cdot\cdots\cdot\psi_{m_N}^{(N)}(x_N)$$

is a complete orthonormal system in $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$.

Let us apply Lemma 7 to our orthonormal systems. In space $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$ of functions of N variables $f(x) = f(x_1, \ldots, x_N)$ all products

$$v_{m_1\ldots m_N}(x_1,\ldots,x_N) = \overline{y}_{m_1}^{(1)}(x_1)\cdot\cdots\cdot\overline{y}_{m_N}^{(N)}(x_N)$$

form the complete orthonormal system. Here,

$$\overline{y}_{m_j}^{(j)}(x_j) = \sqrt{\frac{2}{\pi}}\cdot\frac{\beta_j\cos\lambda_{m_j}x_j + \varepsilon_{m_j}\cdot sign(\beta_j^2-\alpha_j^2)\cdot\alpha_j\sin\lambda_{m_j}x_j}{\sqrt{\alpha_j^2+\beta_j^2}\cdot\sqrt{1+\mid\lambda_{m_j}\mid^{2s_j}}},\ m_j\in Z$$

at $1 \le j \le p$,

$$\overline{y}_{m_j}^{(j)}(x_j) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{1+\mid 2m_j\mid^{2s_j}}}\exp(i2m_jx_j),\ m_j\in Z$$

at $p+1 \le j \le q$,

$$\overline{y}_{m_j}^{(j)}(x_j) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{1+\mid m_j\mid^{2s_j}}}\sin(m_jx_j),\ m_j\in N$$

at $q+1 \le j \le N$.

Thus, the following statement is valid:

Theorem 2. *Let $\alpha_j \neq 0$, $\beta_j \neq 0$, $|\alpha_j| \neq |\beta_j|$ be real numbers at every $1 \le j \le p$, and*

$$\rho = \max_{1\le j\le p}\sqrt{\theta_j^2 + 2(\frac{\theta_j}{\sqrt{2}} + (\varphi_j+1)^{s_j} - 1)^2}\cdot\sigma(s_j) < 1$$

where $\sigma(0) = \frac{1}{\sqrt{2}}$, $\sigma(s_j) = 1$, at $s_j > 0$, $\theta_j = \sqrt{2}\cdot\max_{x\in[0,\pi]}|e^{i\varphi_j x} - 1|$, $\lambda_{m_j} = 2m_j + \varepsilon_{m_j}\cdot\varphi_j$, $\varphi_j = \frac{1}{\pi}\arccos\frac{-2\alpha_j\beta_j}{\alpha_j^2+\beta_j^2}$, $\varepsilon_{m_j} = \varepsilon_{-m_j} = \pm 1$ at $m_j \in Z$. Then, system of eigenfunctions

$$\{v_{m_1\ldots m_N}(x_1,\ldots,x_N)\}_{(m_1,\ldots,m_p)\in Z^p,(m_{p+1},\ldots,m_q)\in Z^{q-p},(m_{q+1},\ldots,m_N)\in N^{N-q}} =$$

$$= \left\{\prod_{j=1}^{p}\sqrt{\frac{2}{\pi}}\frac{\beta_j\cos\lambda_{m_j}x_j + \varepsilon_{m_j}sign(\beta_j^2-\alpha_j^2)\cdot\alpha_j\sin\lambda_{m_j}x_j}{\sqrt{\alpha_j^2+\beta_j^2}\cdot\sqrt{1+\mid\lambda_{m_j}\mid^{2s_j}}}\right\}_{(m_1,\ldots,m_p)\in Z^p}\times$$

$$\times\left\{\prod_{j=p+1}^{q}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{1+\mid 2m_j\mid^{2s_j}}}exp(i2m_jx_j)\right\}_{(m_{p+1},\ldots,m_q)\in Z^{q-p}}\times$$

$$\times\left\{\prod_{j=q+1}^{N}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{1+\mid m_j\mid^{2s_j}}}\sin(m_jx_j)\right\}_{(m_{q+1},\ldots,m_N)\in N^{N-q}}$$

of spectral Problems (7) and (8) forms the complete orthonormal system in Sobolev classes $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$.

Corollary 3. *Let $\alpha_j \neq 0$, $\beta_j \neq 0$, $|\alpha_j| \neq |\beta_j|$ be real numbers at every $1 \le j \le p$, and*

$$\rho = \max_{1\le j\le p}\sqrt{\theta_j^2 + 2\left(\frac{\theta_j}{\sqrt{2}} + (\varphi_j+1)^{s_j} - 1\right)^2}\cdot\sigma(s_j) < 1$$

where $\sigma(0) = \frac{1}{\sqrt{2}}$, $\sigma(s_j) = 1$ *at* $s_j > 0$, $\theta_j = \sqrt{2} \cdot \max_{x\in[0,\pi]} |e^{i\varphi_j x} - 1|$, $\lambda_{m_j} = 2m_j + \varepsilon_{m_j} \cdot \varphi_j$, $\varphi_j = \frac{1}{\pi} \arccos \frac{-2\alpha_j\beta_j}{\alpha_j^2+\beta_j^2}$, $\varepsilon_{m_j} = \varepsilon_{-m_j} = \pm 1$ *at* $m_j \in Z$, $s_j > k + \frac{N}{2}$, $k \geq 0$, $k \in Z$. *Then, the Fourier series for function* $f(x) \in W_2^{s_1,s_2,\ldots,s_N}(\Pi) \cap C^k(\Pi)$ *in orthonormal eigenfunctions*

$$\{v_{m_1\ldots m_N}(x_1,\ldots,x_N)\}_{(m_1,\ldots,m_p)\in Z^p,(m_{p+1},\ldots,m_q)\in Z^{q-p},(m_{q+1},\ldots,m_N)\in N^{N-q}} =$$

$$= \left\{ \prod_{j=1}^{p} \sqrt{\frac{2}{\pi}} \frac{\beta_j \cos \lambda_{m_j} x_j + \varepsilon_{m_j} sign(\beta_j^2 - \alpha_j^2) \cdot \alpha_j \sin \lambda_{m_j} x_j}{\sqrt{\alpha_j^2+\beta_j^2} \cdot \sqrt{1+ \mid \lambda_{m_j} \mid^{2s_j}}} \right\}_{(m_1,\ldots,m_p)\in Z^p} \times$$

$$\times \left\{ \prod_{j=p+1}^{q} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{1+ \mid 2m_j \mid^{2s_j}}} \exp(i2m_jx_j) \right\}_{(m_{p+1},\ldots,m_q)\in Z^{q-p}} \times$$

$$\times \left\{ \prod_{j=q+1}^{N} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1+ \mid m_j \mid^{2s_j}}} \sin(m_jx_j) \right\}_{(m_{q+1},\ldots,m_N)\in N^{N-q}}$$

of spectral Problems (7) and (8) converges in the norm of space $C^k(\Pi)$ *to function* $f(x)$.

The proof of Corollary 3 is carried out using Theorem 2 and the Sobolev embedding theorem. The following are true:

4. Main Results

In this section, we give the most general case of the works done in [17].

Theorem 3. *Let* $\alpha_j \neq 0$, $\beta_j \neq 0$, $|\alpha_j| \neq |\beta_j|$ *be real numbers at every* $1 \leq j \leq p$, *and*

$$\rho = \max_{1\leq j\leq p} \sqrt{\theta_j^2 + 2(\frac{\theta_j}{\sqrt{2}} + (\varphi_j+1)^{s_j} - 1)^2} \cdot \sigma(s_j) < 1$$

where $\sigma(0) = \frac{1}{\sqrt{2}}$, $\sigma(s_j) = 1$ *at* $s_j > 0$, $\theta_j = \sqrt{2} \cdot \max_{x\in[0,\pi]} |e^{i\varphi_j x} - 1|$, $\lambda_{m_j} = 2m_j + \varepsilon_{m_j} \cdot \varphi_j$, $\varphi_j = \frac{1}{\pi} \arccos \frac{-2\alpha_j\beta_j}{\alpha_j^2+\beta_j^2}$, $\varepsilon_{m_j} = \varepsilon_{-m_j} = \pm 1$ *at* $m_j \in Z$, $s_j > k + \frac{N}{2}$, $k \geq 0$, $k \in Z$ *and* $\varphi_j(x) \in W_2^{s_1+j-\frac{N}{2},s_2+j-\frac{N}{2},\ldots,s_N+j-\frac{N}{2}}(\Pi)$, $f(x,t) \in W_2^{s_1,s_2,\ldots,s_N,s_{N+1}}(\Pi \times (0,+\infty))$. *Then, the solution of problems (4)–(6) exists, it is unique, and is represented in the form of series*

$$u(x,t) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_q=-\infty}^{\infty} \sum_{m_{q+1}=1}^{\infty} \cdots \sum_{m_N=1}^{\infty} \sum_{j=1}^{n} \varphi_{j,(m_1\ldots m_N)} t^{\alpha-j} E_{\alpha,\alpha-j+1}(-\mu_{m_1\ldots m_N} \cdot t^\alpha) +$$
$$+ \int_0^t (t-\tau)^{\alpha-1} \cdot E_{\alpha,\alpha}[-\mu_{m_1\ldots m_N}(t-\tau)^\alpha] f_{m_1\ldots m_N}(\tau) d\tau \cdot v_{m_1\ldots m_N}(x_1,\ldots,x_N) \quad (11)$$

where coefficients are determined in the following way :

$$E_{\alpha,\alpha-j+1}(-\mu_{m_1\ldots m_N} \cdot t^\alpha) = \sum_{i=0}^{\infty} \frac{(-\mu_{m_1\ldots m_N} \cdot t^\alpha)^i}{\Gamma(\alpha i + \alpha - j + 1)},$$

$$E_{\alpha,\alpha}\Big(-\mu_{m_1\ldots m_N}\cdot(t-\tau)^\alpha\Big)=\sum_{i=1}^{\infty}\frac{(-\mu_{m_1\ldots m_N})^{i-1}\cdot(t-\tau)^{\alpha(i-1)}}{\Gamma(\alpha\cdot i)},$$

$$f(x,t)=\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_q=-\infty}^{\infty}\sum_{m_{q+1}=1}^{\infty}\cdots\sum_{m_N=1}^{\infty}f_{m_1\ldots m_N}(t)\cdot v_{m_1\ldots m_N}(x_1,\ldots,x_N),$$

$$\varphi_j(x)=\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_q=-\infty}^{\infty}\sum_{m_{q+1}=1}^{\infty}\cdots\sum_{m_N=1}^{\infty}\varphi_{j,(m_1\ldots m_N)}\cdot v_{m_1\ldots m_N}(x_1,\ldots,x_N),$$

$$j=1,2,\ldots,n,\ \ \mu_{m_1\ldots m_N}=\lambda_{m_1}^2+\cdots+\lambda_{m_N}^2.$$

Proof. Since system of eigenfunctions

$$\{v_{m_1\ldots m_N}(x_1,\ldots,x_N)\}_{(m_1,\ldots,m_p)\in Z^p,(m_{p+1},\ldots,m_q)\in Z^{q-p},(m_{q+1},\ldots,m_N)\in N^{N-q}}$$

of spectral Problems (7) and (8) forms the complete orthonormal system in Sobolev classes $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$, any function from class $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$ can be represented as a convergent Fourier series in this system. For any $t>0$, expand solution $u(x,t)$ of Problems (4)–(6) into the Fourier series in eigenfunctions

$$\{v_{m_1\ldots m_N}(x_1,\ldots,x_N)\}_{(m_1,\ldots,m_p)\in Z^p,(m_{p+1},\ldots,m_q)\in Z^{q-p},(m_{q+1},\ldots,m_N)\in N^{N-q}}$$

of spectral Problems (4) and (5):

$$u(x,t)\ \ =\ \ \sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_q=-\infty}^{\infty}\sum_{m_{q+1}=1}^{\infty}\cdots\sum_{m_N=1}^{\infty}T_{m_1\ldots m_N}(t)\cdot v_{m_1\ldots m_N}(x),\tag{12}$$

$$T_{m_1\ldots m_N}(t)=(u(x,t),v_{m_1\ldots m_N}(x)).$$

By virtue of Problems (4) and (5), unknown functions $T_{m_1\ldots m_N}(t)$ must satisfy equation

$$D_{0t}^{\alpha}T_{m_1\ldots m_N}(t)+\mu_{m_1\ldots m_N}T_{m_1\ldots m_N}(t)=f_{m_1\ldots m_N}(t),\ \ l-1<\alpha\le l,\ l\in N\tag{13}$$

with initial conditions

$$\lim_{t\to 0}D_{0t}^{\alpha-k}T_{m_1\ldots m_N}(t)=\varphi_{k,m_1\ldots m_N},\ \ k=1,2,\ldots,l,\ \ \mu_{m_1\ldots m_N}=\lambda_{m_1}^2+\cdots+\lambda_{m_N}^2.\tag{14}$$

The solution of Cauchy Problems (13) and (14) has the form

$$T_{m_1\ldots m_N}(t)\ \ =\ \ \sum_{j=1}^{n}\varphi_{j,(m_1\ldots m_N)}t^{\alpha-j}E_{\alpha,\alpha-j+1}(-\mu_{m_1\ldots m_N}\cdot t^\alpha)+$$

$$+\int_0^t(t-\tau)^{\alpha-1}\cdot E_{\alpha,\alpha}[-\mu_{m_1\ldots m_N}(t-\tau)^\alpha]f_{m_1\ldots m_N}(\tau)d\tau\tag{15}$$

where coefficients are determined as follows:

$$E_{\alpha,\alpha-j+1}(-\mu_{m_1\ldots m_N}\cdot t^\alpha)=\sum_{i=0}^{\infty}\frac{(-\mu_{m_1\ldots m_N}\cdot t^\alpha)^i}{\Gamma(\alpha i+\alpha-j+1)},$$

$$E_{\alpha,\alpha}\Big(-\mu_{m_1\ldots m_N}\cdot(t-\tau)^{\alpha}\Big)=\sum_{i=1}^{\infty}\frac{(-\mu_{m_1\ldots m_N})^{i-1}\cdot(t-\tau)^{\alpha(i-1)}}{\Gamma(\alpha\cdot i)},$$

$$f(x,t)=\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_q=-\infty}^{\infty}\sum_{m_{q+1}=1}^{\infty}\cdots\sum_{m_N=1}^{\infty}f_{m_1\ldots m_N}(t)\cdot v_{m_1\ldots m_N}(x_1,\ldots,x_N),$$

$$\varphi_j(x)=\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_q=-\infty}^{\infty}\sum_{m_{q+1}=1}^{\infty}\cdots\sum_{m_N=1}^{\infty}\varphi_{j,(m_1\ldots m_N)}\cdot v_{m_1\ldots m_N}(x_1,\ldots,x_N),j=1,2,\ldots,n.$$

After substituting Problem (15) into Problem (12), we obtain the unique solution of Problems (4)–(6) in the form of Series (8).

Let $\nu>1$. Consider mixed Problems (4)–(6). If we look for a solution $u(x,t)$ to Problems (4)–(6) in the form of Fourier series expansion

$$u(x,t)\;=\;\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_q=-\infty}^{\infty}\sum_{m_{q+1}=1}^{\infty}\cdots\sum_{m_N=1}^{\infty}T_{m_1\ldots m_N}(t)\cdot v_{m_1\ldots m_N}(x),$$

where are $T_{m_1\ldots m_N}(t)=(u(x,t),v_{m_1\ldots m_N}(x))$ are the coefficients of the series, $\{v_{m_1\ldots m_N}\}$ is the system of eigenfunctions of spectral Problems (7) and (8).

Differential operator $(-\Delta)^{\nu}$, generated by a differential expression $l^{(\nu)}(v(x))=(-\Delta)^{\nu}v(x)$ with domain definition

$$D\Big((-\Delta)^{\nu}\Big)=\{v(x):v(x)\in C^{2\nu}(\Pi)\cap C^{2\nu-1}(\overline{\Pi}),l^{(\nu)}(v(x))\in L_2(\Pi)\}$$

satisfies Condition (8).

Similarly, as Lemma 5, it can be shown that operator $(-\Delta)^{\nu}$, is a symmetric and positive operator in space $L_2(\Pi)$. The eigenvalues of Problems (7) and (8) $\mu_{m_1\ldots m_N}\geq 0$, and each $\mu_{\mu_{m_1\ldots m_N}}=\Big(\lambda^2_{m_1}+\cdots\lambda^2_{m_1}\Big)^{\nu}$ corresponds to an eigenvalue of Problems (9) and (10), and the eigenfunctions $\{v_{m_1\ldots m_N}(x)\}$ of Problems (7) and (8) and eigenfunctions $\{y_{m_1\ldots m_N}(x)\}$ of Problems (9) and (10) coincide, i.e.,

$$v_{m_1\ldots m_N}(x)\equiv y_{m_1\ldots m_N}(x).$$

□

Therefore, the following theorem is valid:

Theorem 4. *Let $\alpha_j\neq 0$, $\beta_j\neq 0$, $|\alpha_j|\neq|\beta_j|$ be real numbers at every $1\leq j\leq p$, and*

$$\rho=\max_{1\leq j\leq p}\sqrt{\theta_j^2+2(\frac{\theta_j}{\sqrt{2}}+(\varphi_j+1)^{s_j}-1)^2}\cdot\sigma(s_j)<1$$

where $\sigma(0)=\frac{1}{\sqrt{2}}$, $\sigma(s_j)=1$ at $s_j>0$, $\theta_j=\sqrt{2}\cdot\max_{x\in[0,\pi]}\left|e^{i\varphi_j x}-1\right|$, $\lambda_{m_j}=2m_j+\varepsilon_{m_j}\cdot\varphi_j$, $\varphi_j=\frac{1}{\pi}\arccos\frac{-2\alpha_j\beta_j}{\alpha_j^2+\beta_j^2}$, $\varepsilon_{m_j}=\varepsilon_{-m_j}=\pm1$ at $m_j\in Z$, $s_j>(k+\frac{N}{2})\nu$, $k\geq 0$, $k\in Z$ and

$\varphi_j(x) \in W_2^{(s_1+j-\frac{N}{2})\nu,(s_2+j-\frac{N}{2})\nu,\ldots,(s_N+j-\frac{N}{2})\nu}(\Pi)$, $f(x,t) \in W_2^{s_1,s_2,\ldots,s_N,s_{N+1}}(\Pi \times (0,+\infty))$. Then the solution of Problems (4)–(6) exists, it is unique, and is represented in the form of series

$$u(x,t) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_q=-\infty}^{\infty} \sum_{m_{q+1}=1}^{\infty} \cdots \sum_{m_N=1}^{\infty} \sum_{j=1}^{n} \varphi_{j,(m_1\ldots m_N)} t^{\alpha-j} E_{\alpha,\alpha-j+1}(-\mu_{m_1\ldots m_N} \cdot t^{\alpha}) +$$
$$+ \int_0^t (t-\tau)^{\alpha-1} \cdot E_{\alpha,\alpha}[-\mu_{m_1\ldots m_N}(t-\tau)^{\alpha}] f_{m_1\ldots m_N}(\tau) d\tau \cdot v_{m_1\ldots m_N}(x_1,\ldots,x_N)$$

where coefficients are determined in the following way:

$$E_{\alpha,\alpha-j+1}(-\mu_{m_1\ldots m_N} \cdot t^{\alpha}) = \sum_{i=0}^{\infty} \frac{(-\mu_{m_1\ldots m_N} \cdot t^{\alpha})^i}{\Gamma(\alpha i + \alpha - j + 1)},$$

$$E_{\alpha,\alpha}\Big(-\mu_{m_1\ldots m_N} \cdot (t-\tau)^{\alpha}\Big) = \sum_{i=1}^{\infty} \frac{(-\mu_{m_1\ldots m_N})^{i-1} \cdot (t-\tau)^{\alpha(i-1)}}{\Gamma(\alpha \cdot i)},$$

$$f(x,t) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_q=-\infty}^{\infty} \sum_{m_{q+1}=1}^{\infty} \cdots \sum_{m_N=1}^{\infty} f_{m_1\ldots m_N}(t) \cdot v_{m_1\ldots m_N}(x_1,\ldots,x_N),$$

$$\varphi_j(x) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_q=-\infty}^{\infty} \sum_{m_{q+1}=1}^{\infty} \cdots \sum_{m_N=1}^{\infty} \varphi_{j,(m_1\ldots m_N)} \cdot v_{m_1\ldots m_N}(x_1,\ldots,x_N),\ j = 1,2,\ldots,n,$$

$$\mu_{m_1\ldots m_N} = \Big(\lambda_{m_1}^2 + \cdots + \lambda_{m_N}^2\Big)^{\nu}.$$

5. Conclusions

In this paper, we considered questions on the unique solvability of a mixed problem for a partial differential equation of high order with fractional Riemann-Liouville derivatives with respect to time, and with Laplace operators with spatial variables and with nonlocal boundary conditions in Sobolev classes. The solution was found in the form of a series of expansions in eigenfunctions of the Laplace operator with nonlocal boundary conditions. Initial and boundary problems with fractional Riemann-Liouville derivatives with respect to time have many applications [13]. In connection to this, we chose the fractional Riemann-Liouville derivative, although we could consider other types of fractional derivatives.

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