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q-Rung Orthopair Fuzzy Hypergraphs with Applications

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Abstract: The concept of q-rung orthopair fuzzy sets generalizes the notions of intuitionistic fuzzy sets and Pythagorean fuzzy sets to describe complicated uncertain information more effectively. Their most dominant attribute is that the sum of the q^{th} power of the truth-membership and the q^{th} power of the falsity-membership must be equal to or less than one, so they can broaden the space of uncertain data. This set can adjust the range of indication of decision data by changing the parameter q, $q \geq 1$. In this research study, we design a new framework for handling uncertain data by means of the combinative theory of q-rung orthopair fuzzy sets and hypergraphs. We define q-rung orthopair fuzzy hypergraphs to achieve the advantages of both theories. Further, we propose certain novel concepts, including adjacent levels of q-rung orthopair fuzzy hypergraphs, (α, β)-level hypergraphs, transversals, and minimal transversals of q-rung orthopair fuzzy hypergraphs. We present a brief comparison of our proposed model with other existing theories. Moreover, we implement some interesting concepts of q-rung orthopair fuzzy hypergraphs for decision-making to prove the effectiveness of our proposed model.

Keywords: q-rung orthopair fuzzy hypergraphs; transversals of q-rung orthopair fuzzy hypergraphs; comparison analysis; decision-making; algorithms

1. Introduction

Zadeh's [1] fuzzy set (FS) has acquired greater attention by researchers in a wide range of scientific areas, including management sciences, robotics, decision theory. and many other disciplines. FSs were further generalized to intuitionistic fuzzy sets (IFSs) by Atanassov [2] in 1983. An IFS is distinguished by a truth-membership (T) and falsity-membership (F) satisfying the condition that the sum of both membership degrees should not exceed one. IF values play an important role in both theoretical and practical progress of IFSs. Applications of IFSs appear in various fields, including medical diagnosis, optimization problems, and decision-making. Recently, Liu et al. [3] introduced and explored various types of centroid transformations of IF values. Furthermore, Feng et al. [4] defined two different types of generalized IF soft subsets and various new operations for generalized IF soft sets. However, in many practical decision-making problems, the sum of truth-membership and falsity-membership may not be less than one, but the sum of their squares may be less than one. To handle such types of difficulties, Yager [5,6] introduced the novel concept of Pythagorean fuzzy sets (PFSs), which is the generalization of IFSs. Compensating the constraint that the summation of both membership degrees does not exceed one and that the sum of squares of the membership degrees should not be greater than one makes PFSs more powerful, generalizable, and effective. Naz et al. [7] proposed a novel approach to decision-making with Pythagorean fuzzy information. PFSs can deal with various real-life problems more effectively, still there are cases that cannot be handled using PFSs. Take an example: The truth-membership and falsity-membership values suggested by a decision-maker are 0.8 and 0.9, respectively. Then, the problem can never be handled by means of PFSs, as $0.8^2 + 0.9^2 = 1.45 > 1$. In order to deal with such types of cases, Yager [8] submitted the idea of q-rung orthopair fuzzy sets (q-ROFSs). A q-ROFS is represented by means of two membership degrees; one is the truth and the other is falsity, with the characteristic that the summation of the q^{th} power of truth-membership and the q^{th} power of falsity-membership should not be greater than one. Thus, q-ROFSs extend the concepts of IFSs and PFSs, so that the uncertain information can be dealt with in a widened range. After that, Liu and Wang [9] developed and applied certain simple weighted operators to aggregate q-ROFSs in decision-making. Certain applications of q-ROFSs in decision-making have been discussed in [10,11]. This set can adjust the range of indication of decision data by changing the parameter q, $q \geq 1$. Wei et al. [12] defined q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision-making.

The ambiguousness in the representation of different objects or in the relationships between them generates the essentiality of fuzzy graphs, which were originally studied and developed by Kaufmann [13] in 1977. A valuable contribution to fuzzy graphs and fuzzy hypergraphs was presented in [14]. Akram and Naz [15] discussed the energy of Pythagorean fuzzy graphs with applications. Recently, certain operations on PFGs and IFG of the three-type and n-type were discussed by Akram et al. [16]. The same authors discussed certain Pythagorean fuzzy graphs and also defined q-rung orthopair fuzzy competition graphs with applications in [17]. Kaufmann [13], in 1977, defined fuzzy hypergraphs. Fuzzy hypergraphs were redefined and generalized by Lee-Kwang and Keon-Myung [18]. Parvathi et al. [19] originated the notion of IF hypergraphs. Later on, this idea was generalized by Akram and Dudek [20]. They studied the various features of IF hypergraphs and proposed the applicability of IF hypergraphs in radio coverage networks and clustering problems. Akram and Sarwar [21] introduced the transversals and minimal transversals of m-polar fuzzy hypergraphs and investigated their certain properties. Akram and Luqman [22] studied the transversals of bipolar neutrosophic hypergraphs. They developed and implemented an algorithm to find minimal bipolar neutrosophic transversals.

In this paper, we propose certain novel concepts, including q-rung orthopair fuzzy hypergraphs, (α, β)-level hypergraphs, transversals, and minimal transversals of q-rung orthopair fuzzy hypergraphs. Further, we implement some interesting concepts of q-rung orthopair fuzzy hypergraphs for decision-making. This paper is arranged as follows: In Section 2, some basic and necessary concepts are reviewed and q-rung orthopair fuzzy hypergraphs are defined. Section 3 deals with some interesting concepts, including q-rung orthopair fuzzy transversals, minimal transversals, and locally-minimal transversals of q-rung orthopair fuzzy hypergraphs. We explain the method for finding the minimal transversal of q-rung orthopair fuzzy hypergraphs. In Section 4, we discuss how the concept of q-rung orthopair fuzzy hypergraphs can be applied to decision-making to analyze real-life phenomena. Section 5 presents a detailed comparison of q-ROFSs with other existing theories. The last section deals with the conclusions.

2. q-Rung Orthopair Fuzzy Hypergraphs

Definition 1. *A q-rung orthopair fuzzy set (q-ROFS) Q in the universe X is an object having the representation [5]:*

$$Q = \langle x, T_Q(x), F_Q(x) | x \in X \rangle,$$

where the function $T_Q : X \to [0,1]$ defines the truth-membership and $F_Q : X \to [0,1]$ defines the falsity-membership of the element $x \in X$, and for every $x \in X$,

$$0 \leq T_Q^q(x) + F_Q^q(x) \leq 1, q \geq 1.$$

Furthermore, $\pi_Q(x) = \sqrt[q]{1 - T_Q^q(x) - F_Q^q(x)}$ is called a q-ROF index or indeterminacy degree of x to the set Q.

For convenience, Liu and Wang [9] called the pair $(T^q_Q(x), F^q_Q(x))$ as a q-rung orthopair fuzzy number (q-ROFN), which is denoted as (T^q_Q, F^q_Q).

Remark 1.

- *When $q = 1$, one-ROFS is called an IFS.*
- *When $q = 2$, two-ROFS is called a PFS.*

Definition 2. *An intuitionistic fuzzy graph (IFG) on a non-empty set X is an ordered pair $G = (V, E)$ [23], where V is an IFS on X and E is an IFR on X such that:*

$$T_E(x_1x_2) \leq \min\{T_V(x_1), T_V(x_2)\} \ , \ F_E(x_1x_2) \leq \max\{F_V(x_1), F_V(x_2)\},$$

and $0 \leq T_E(x_1x_2) + F_E(x_1x_2) \leq 1$, where $T_E : X \times X \to [0,1]$ and $F_E : X \times X \to [0,1]$ represent the truth-membership and falsity-membership degrees of E, respectively. Here, V is the vertex set and E is the edge set of G.

Definition 3. *A Pythagorean fuzzy graph (PFG) on a non-empty set X is an ordered pair $\tilde{G} = (\tilde{V}, \tilde{E})$ [7], where $\tilde{V}$ is a PFS on X and $\tilde{E}$ is a PFR on X such that:*

$$T_{\tilde{E}}(x_1x_2) \leq \min\{T_{\tilde{V}}(x_1), T_{\tilde{V}}(x_2)\} \ , \ F_{\tilde{E}}(x_1x_2) \leq \max\{F_{\tilde{V}}(x_1), F_{\tilde{V}}(x_2)\},$$

and $0 \leq T^2_{\tilde{E}}(x_1x_2) + F^2_{\tilde{E}}(x_1x_2) \leq 1$, where $T_{\tilde{E}} : X \times X \to [0,1]$ and $F_{\tilde{E}} : X \times X \to [0,1]$ represent the truth-membership and falsity-membership degrees of $\tilde{E}$, respectively. Here, $\tilde{V}$ is the vertex set and $\tilde{E}$ is the edge set of $\tilde{G}$.

Definition 4. *A q-rung orthopair fuzzy relation (q-ROFR) $\mathcal{R}$ on X is defined as:*

$$\mathcal{R} = \{x_1x_2, T_{\mathcal{R}}(x_1x_2), F_{\mathcal{R}}(x_1x_2)|x_1, x_2 \in X \times X\},$$

where $T_{\mathcal{R}} : X \times X \to [0,1]$ and $F_{\mathcal{R}} : X \times X \to [0,1]$ represent the truth-membership and falsity-membership function of $\mathcal{R}$, respectively, such that $0 \leq T^q_{\mathcal{R}}(x_1x_2) + F^q_{\mathcal{R}}(x_1x_2) \leq 1$, $q \geq 1$, for all $x_1x_2 \in X \times X$.

Example 1. *Let $X = \{x_1, x_2, x_3\}$ be a non-empty set and $\mathcal{R}$ be a subset of $X \times X$ such that $\mathcal{R} = \{(x_1x_2, 0.9, 0.7)$,$(x_1x_3, 0.7, 0.9)$, $(x_2x_3, 0.6, 0.8)\}$. Note that, $0 \leq T^5_{\mathcal{R}}(x_1x_2) + F^5_{\mathcal{R}}(x_1x_2) \leq 1$, for all $x_1x_2 \in X \times X$. Hence, $\mathcal{R}$ is a five-ROFR on X.*

Definition 5. *A q-rung orthopair fuzzy graph (q-ROFG) on a non-empty set X is defined as an ordered pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a q-ROFS on X and $\mathcal{E}$ is a q-ROFR on X such that:*

$$T_{\mathcal{E}}(x_1x_2) \leq \min\{T_{\mathcal{V}}(x_1), T_{\mathcal{V}}(x_2)\} \ , \ F_{\mathcal{E}}(x_1x_2) \leq \max\{F_{\mathcal{V}}(x_1), F_{\mathcal{V}}(x_2)\},$$

and $0 \leq T^q_E(x_1x_2) + F^q_E(x_1x_2) \leq 1$, $q \geq 1$ for all $x_1, x_2 \in X$, where $T_{\mathcal{E}} : X \times X \to [0,1]$ and $F_{\mathcal{E}} : X \times X \to [0,1]$ represent the truth-membership and falsity-membership degrees of $\mathcal{E}$, respectively.

Example 2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a five-ROFG, where $\mathcal{V} = (v_1, 0.8, 0.6), (v_2, 0.5, 0.6), (v_3, 0.9, 0.6), (v_4, 0.7, 0.6)$ is a five-ROFS on X and $\mathcal{E} = \{(v_1v_2, 0.5, 0.6), (v_3v_2, 0.5, 0.6), (v_1v_4, 0.7, 0.6)\}$ is a five-ROFR on X. The corresponding five-ROFG is shown in Figure 1.*

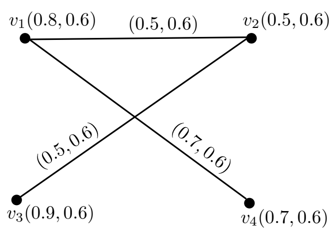

Figure 1. A five-rung orthopair fuzzy graph.

Remark 2.

- *When $q = 1$, one-ROFG is called an IFG.*
- *When $q = 2$, two-ROFG is called a PFG.*

Definition 6. *The support of a q-ROFS $Q = \langle \mathrm{x}, T_Q(\mathrm{x}), F_Q(\mathrm{x}) | \mathrm{x} \in X\rangle$ is defined as $supp(Q) = \{\mathrm{x} | T_Q(\mathrm{x}) \neq 0, F_Q(\mathrm{x}) \neq 1\}$.*
The height of a q-ROFS $Q = \langle \mathrm{x}, T_Q(\mathrm{x}), F_Q(\mathrm{x}) | \mathrm{x} \in X\rangle$ is defined as $h(Q) = (\max_{\mathrm{x}\in X} T_Q(\mathrm{x}), \min_{\mathrm{x}\in X} F_Q(\mathrm{x}))$.

If $h(Q) = (1, 0)$, then q-ROFS Q is called normal.

Definition 7. *Let X be a non-empty set. A q-rung orthopair fuzzy hypergraph (q-ROFH) $\mathcal{H}$ on X is defined in the form of an ordered pair $\mathcal{H} = (\mathcal{Q}, \zeta)$, where $\mathcal{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \ldots \mathcal{Q}_n\}$ is a finite collection of non-trivial q-ROF subsets on X and ζ is a q-ROFR on q-ROFSs $\mathcal{Q}_i$ such that:*

1. $$T_\zeta(E_k) = T_\zeta(x_1, x_2, x_3, \ldots, x_m) \leq \min\{\mathcal{Q}_i(x_1), \mathcal{Q}_i(x_2), \mathcal{Q}_i(x_3), \ldots, \mathcal{Q}_i(x_m)\},$$
$$F_\zeta(E_k) = F_\zeta(x_1, x_2, x_3, \ldots, x_m) \leq \max\{\mathcal{Q}_i(x_1), \mathcal{Q}_i(x_2), \mathcal{Q}_i(x_3), \ldots, \mathcal{Q}_i(x_m)\},$$
for all $x_1, x_2, x_3, \cdots, x_m \in X$.

2. $\bigcup_i supp(\mathcal{Q}_i) = X$, *for all $\mathcal{Q}_i \in \mathcal{Q}$.*

Definition 8. *The height of a q-ROFH $\mathcal{H} = (\mathcal{Q}, \zeta)$ is defined as $h(\mathcal{H}) = \{\max(\zeta_l), \min(\zeta_m)\}$, where $\zeta_l = \max T_{\zeta_j}(x_i)$ and $\zeta_m = \min F_{\zeta_j}(x_i)$. Here, $T_{\zeta_j}(x_i)$ and $F_{\zeta_j}(x_i)$ denote the truth-membership degree and falsity-membership degree of vertex x_i to the hyperedge ζ_j, respectively.*

Definition 9. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a q-ROFH. The order of $\mathcal{H}$, which is denoted by $O(\mathcal{H})$, is defined as $O(\mathcal{H}) = \sum_{x\in X} \wedge \mathcal{Q}_i(x)$. The size of $\mathcal{H}$, which is denoted by $S(\mathcal{H})$, is defined as $S(\mathcal{H}) = \sum_{x\in X} \vee \mathcal{Q}_i(x)$.*

In a q-ROFH, adjacent vertices x_i and x_j are the vertices that are the part of the same q-ROF hyperedge. Two q-ROF hyperedges ζ_i and ζ_j are said to be adjacent hyperedges if they possess the non-empty intersection, i.e., $supp(\zeta_i) \cap supp(\zeta_i) \neq \emptyset$.

We now define the adjacent level between two q-ROF vertices and q-ROF hyperedges.

Definition 10. *The adjacent level between two vertices x_i and x_j is denoted by $\gamma(x_i, x_j)$ and is defined as $\gamma(x_i, x_j) = (\max_k \min[T_k(x_i), T_k(x_j)], \min_k \max[F_k(x_i), F_k(x_j)])$.*

The adjacent level between two hyperedges ζ_i and ζ_j is denoted by $\sigma(\zeta_i, \zeta_j)$ and is defined as $\sigma(\zeta_i, \zeta_j) = (\max_j \min[T_j(x), T_k(x)], \min_j \max[F_j(x), F_k(x)])$.

Definition 11. *A simple q-ROFH $\mathcal{H} = (\mathcal{Q}, \zeta)$ is defined as a hypergraph, which has no repeated hyperedges contained in it, i.e., if $\zeta_i, \zeta_j \in \zeta$ and $\zeta_i \subseteq \zeta_j$, then $\zeta_i = \zeta_j$.*

A q-ROFH $\mathcal{H} = (\mathcal{Q}, \zeta)$ is support simple if $\zeta_i, \zeta_j \in \zeta$, $supp(\zeta_i) = supp(\zeta_j)$ and $\zeta_i \subseteq \zeta_j$, then $\zeta_i = \zeta_j$.

A q-ROFH $\mathcal{H} = (\mathcal{Q}, \zeta)$ is strongly support simple if $\zeta_i, \zeta_j \in \zeta$ and $supp(\zeta_i) = supp(\zeta_j)$, then $\zeta_i = \zeta_j$.

Definition 12. *A q-ROFS $Q : X \to [0, 1]$ is called an elementary set if T_Q and F_Q are single-valued on the support of Q.*

A q-ROFH $\mathcal{H} = (\mathcal{Q}, \zeta)$ is elementary if all its hyperedges are elementary.

Proposition 1. *A q-ROFH $\mathcal{H} = (\mathcal{Q}, \zeta)$ is the generalization of the fuzzy hypergraph and IF hypergraph.*

An upper bound on the cardinality of hyperedges of a q-ROFH of order n can be achieved by using the following result.

Theorem 1. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a simple q-ROFH of order n. Then, $|\zeta|$ acquires no upper bound.*

Proof. Let $X = \{x_1, x_2\}$. Define $\zeta_N = \{\mathcal{Q}_j, j = 1, 2, 3, \cdots, N\}$, where:

$$T_{\mathcal{Q}_j}(x_1) = \frac{1}{1+j} \ , \ F_{\mathcal{Q}_j}(x_1) = 1 - \frac{1}{1+j},$$

and

$$T_{\mathcal{Q}_j}(x_2) = \frac{1}{1+j} \ , \ F_{\mathcal{Q}_j}(x_2) = 1 - \frac{1}{1+j}.$$

Then, $\mathcal{H}_N = (\mathcal{Q}, \zeta_N)$ is a simple q-ROFH having N hyperedges. □

Theorem 2. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be an elementary and simple q-ROFH on a non-empty set X having n elements. Then, $|\zeta| \leq 2^n - 1$. The equality holds if and only if $\{supp(\zeta_j) | \zeta_j \in \zeta, \zeta \neq 0\} = P(X) \setminus \emptyset$.*

Proof. Since $\mathcal{H}$ is elementary and simple, then at most one $\zeta_i \in \zeta$ can have each non-trivial subset of X as its support; therefore, we have $|\zeta| \leq 2^n - 1$.

To prove that the relation satisfies the equality, consider a set of mappings $\zeta = \{(T_A, F_A) | A \subseteq X\}$ such that

$$T_A(x) = \begin{cases} \frac{1}{|A|} & \text{if} \quad x \in A, \\ 0 & \text{otherwise.} \end{cases} \ , \ F_A(x) = \begin{cases} \frac{1}{|A|} & \text{if} \quad x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then, each set containing a single element has height $(1, 1)$, and the height of the set having two elements is $(0.5, 0.5)$, and so on. Hence, $\mathcal{H}$ is simple and elementary with $|\zeta| = 2^n - 1$. □

Definition 13. *The cut level set of a q-ROFS Q is defined to be a crisp set of the following form: $Q^{(\alpha,\beta)} = \{x \in X | T_Q(x) \geq \alpha \ , \ F_Q(x) \leq \beta\}$, where $\alpha, \beta \in [0, 1]$ and $0 \leq \alpha^q + \beta^q \leq 1$, $q \geq 1$.*

Definition 14. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a q-ROFH. The (α, β)-level hypergraph of $\mathcal{H}$ is defined as $\mathcal{H}^{(\alpha,\beta)} = (\mathcal{Q}^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)})$, where:*

1. $\zeta^{(\alpha,\beta)} = \{\zeta_i^{(\alpha,\beta)} : \zeta_i \in \zeta\}$ *and* $\zeta_i^{(\alpha,\beta)} = \{x \in X | T_{\zeta_i}(x) \geq \alpha \ , \ F_{\zeta_i}(x) \leq \beta\}$,
2. $\mathcal{Q}^{(\alpha,\beta)} = \bigcup\limits_{\zeta_i \in \zeta} \zeta_i^{(\alpha,\beta)}$.

Example 3. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a four-ROFH as shown in Figure 2, where $\zeta = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$. The incidence matrix of $\mathcal{H}$ is given in Table 1.*

Table 1. Incidence matrix of $\mathcal{H}$.

I	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
x_1	$(0.1, 0.2)$	$(0.1, 0.2)$	$(0.1, 0.2)$	$(0, 1)$	$(0, 1)$
x_2	$(0.2, 0.3)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0, 1)$
x_3	$(0.3, 0.4)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0.3, 0.4)$
x_4	$(0, 1)$	$(0, 1)$	$(0.4, 0.5)$	$(0, 1)$	$(0, 1)$
x_5	$(0, 1)$	$(0.5, 0.6)$	$(0, 1)$	$(0, 1)$	$(0, 1)$
x_6	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0.5, 0.4)$
x_7	$(0, 1)$	$(0, 1)$	$(0.4, 0.3)$	$(0.4, 0.3)$	$(0, 1)$
x_8	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0.6, 0.5)$	$(0, 1)$
x_9	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0.6, 0.7)$	$(0.6, 0.7)$

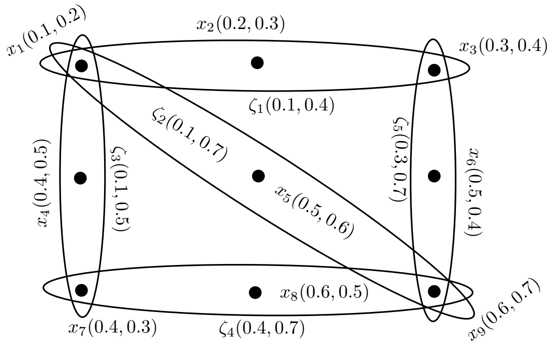

Figure 2. Four-rung orthopair fuzzy hypergraph.

By direct calculations, it can be seen that it is a four-ROFH. All the above-mentioned concepts can be well explained by considering this example. Here, $h(\mathcal{H}) = \{\max(\zeta_l), \min(\zeta_m)\} = (0.6, 0.2)$. *Since* $\mathcal{H}$ *does not contain repeated hyperedges, it is a simple four-ROFH. Furthermore,* $\mathcal{H}$ *is support simple and strongly support simple, i.e., whenever* $\zeta_i, \zeta_j \in \zeta$ *and* $supp(\zeta_i) = supp(\zeta_j)$, *then* $\zeta_i = \zeta_j$. *The adjacency level between* x_1, x_2 *and between two hyperedges* ζ_1, ζ_2 *is given as follows:*

$$\begin{aligned} \gamma(x_1, x_2) &= (\max_k \min[T_k(x_1), T_k(x_2)], \min_k \max[F_k(x_1), F_k(x_2)]), k = 1, 2, 3, 4, 5. \\ &= (0.1, 0.3), \\ \sigma(\zeta_1, \zeta_2) &= (\max \min[T_1(x), T_2(x)], \min \max[F_1(x), F_2(x)]) \\ &= (0.2, 0.6). \end{aligned}$$

For $\alpha = 0.1$, $\beta = 0.4 \in [0, 1]$, *the* $(0.1, 0.4)$*-level hypergraph of* $\mathcal{H}$ *is* $\mathcal{H}^{(0.1,0.4)} = (\mathcal{Q}^{(0.1,0.4)}, \zeta^{(0.1,0.4)})$, *where:*

$$\begin{aligned} \zeta^{(0.1,0.4)} &= \{\zeta_1^{(0.1,0.4)}, \zeta_2^{(0.1,0.4)}, \zeta_3^{(0.1,0.4)}, \zeta_4^{(0.1,0.4)}, \zeta_5^{(0.1,0.4)}\} \\ &= \{\{x_1, x_2, x_3\}, \{x_5\}, \{x_4\}, \{x_8, x_9\}, \{x_3, x_6, x_9\}\}, \\ \mathcal{Q}^{(0.1,0.4)} &= \{x_1, x_2, x_3\} \cup \{x_5\} \cup \{x_4\} \cup \{x_8, x_9\} \cup \{x_3, x_6, x_9\} \\ &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9\}. \end{aligned}$$

Note that the $(0.1, 0.4)$*-level hypergraph of* $\mathcal{H}$ *is a crisp hypergraph as shown in Figure 3.*

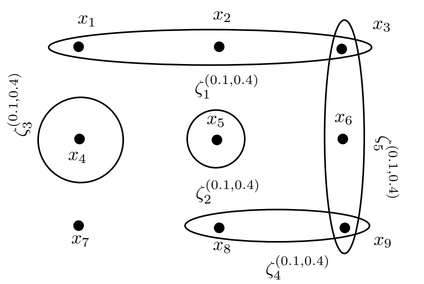

Figure 3. The $(0.1, 0.4)$-level hypergraph of $\mathcal{H}$.

Remark 3. *If* $\alpha \geq \mu$ *and* $\beta \leq \nu$ *and* $\mathcal{Q}$ *is a q-ROFS on X, then* $\mathcal{Q}^{(\alpha,\beta)} \subseteq \mathcal{Q}^{(\mu,\nu)}$. *Thus, we can have* $\zeta^{(\alpha,\beta)} \subseteq \zeta^{(\mu,\nu)}$, *for level hypergraphs of* $\mathcal{H}$, *i.e., if a q-ROFH has distinct hyperedges, its* (α, β)*-level hyperedges may be the same, and hence,* (α, β)*-level hypergraphs of a simple q-ROFHs may have repeated edges.*

Definition 15. *Let* $\mathcal{H} = (\mathcal{Q}, \zeta)$ *be a q-ROFH and* $\mathcal{H}^{(\alpha,\beta)}$ *be the* (α, β)*-level hypergraph of* $\mathcal{H}$. *The sequence of real numbers* $\rho_1 = (T_{\rho_1}, F_{\rho_1})$, $\rho_2 = (T_{\rho_2}, F_{\rho_2})$, $\rho_3 = (T_{\rho_3}, F_{\rho_3})$, $\cdots$, $\rho_n = (T_{\rho_n}, F_{\rho_n})$, $0 < T_{\rho_1} < T_{\rho_2} < T_{\rho_3} < \cdots < T_{\rho_n}$, $F_{\rho_1} > F_{\rho_2} > F_{\rho_3} > \cdots > F_{\rho_n} > 0$, *where* $(T_{\rho_n}, F_{\rho_n}) = h(\mathcal{H})$, *such that:*

(i) *if* $\rho_{i-1} = (T_{\rho_{i-1}}, F_{\rho_{i-1}}) < \rho = (T_\rho, F_\rho) \leq \rho_i = (T_{\rho_i}, F_{\rho_i})$, *then* $\zeta^\rho = \zeta^{\rho_i}$,
(ii) $\zeta^{\rho_i} \subseteq \zeta^{\rho_{i+1}}$,

is called the fundamental sequence of $\mathcal{H}$*, denoted by* $f_S(H)$*. The set of* ρ_i*-level hypergraphs* $\{\mathcal{H}^{\rho_1}, \mathcal{H}^{\rho_2}, \mathcal{H}^{\rho_3}, \cdots, \mathcal{H}^{\rho_n}\}$ *is called the core hypergraphs of* $\mathcal{H}$ *or simply the core set of* $\mathcal{H}$ *and is denoted by* $c(\mathcal{H})$.

Definition 16. *A q-ROFH* $\mathcal{H}_1 = (\mathcal{Q}_1, \zeta_1)$ *is called a partial hypergraph of* $\mathcal{H}_2 = (\mathcal{Q}_2, \zeta_2)$ *if* $\zeta_1 \subseteq \zeta_2$ *and is denoted as* $\mathcal{H}_1 \subseteq \mathcal{H}_2$.

Definition 17. *Let* $\mathcal{H} = (\mathcal{Q}, \zeta)$ *be a q-ROFH having fundamental sequence* $f_S(\mathcal{H}) = \{\rho_1, \rho_2, \rho_3, \cdots, \rho_n\}$*, and let* $\rho_{n+1} = 0$*; if for all hyperedges* $\zeta_k \in \zeta$, $k = 1, 2, 3, \cdots, n$ *and for all* $\rho \in (\rho_{i+1}, \rho_i]$*, we have* $\zeta_i^\rho = \zeta_i^{\rho_i}$*, then* $\mathcal{H}$ *is called sectionally elementary.*

Theorem 3. *Let* $\mathcal{H} = (\mathcal{Q}, \zeta)$ *be an elementary q-ROFH. Then, the necessary and sufficient condition for* $\mathcal{H} = (\mathcal{Q}, \zeta)$ *to be strongly support simple is that* $\mathcal{H}$ *is support simple.*

Proof. Suppose that $\mathcal{H}$ is support simple, elementary, and $supp(\zeta_i) = supp(\zeta_j)$, for $\zeta_i, \zeta_j \in \zeta$. Let $h(\zeta_i) \leq h(\zeta_j)$. Since $\mathcal{H}$ is elementary, we have $\zeta_i \leq \zeta_j$, and since $\mathcal{H}$ is support simple, we have $\zeta_i = \zeta_j$. Hence, $\mathcal{H}$ is strongly support simple. On the same lines, the converse part may be proven. □

Definition 18. *A q-ROFH* $\mathcal{H} = (\mathcal{Q}, \zeta)$ *is said to be a* $\mathcal{B} = (T_\mathcal{B}, F_\mathcal{B})$ *a tempered q-rung orthopair fuzzy hypergraph if for* $H = (X, \xi)$*, a crisp hypergraph, and a q-ROFS* $\mathcal{B} = (T_\mathcal{B}, F_\mathcal{B})$:$X \to [0,1]$ *such that* $\zeta = \{D_A = (T_{D_A}, F_{D_A}) | A \subset X\}$*, where*

$$T_{D_A}(x) = \begin{cases} \min(T_\mathcal{B}(y)) : y \in A & \text{if} \quad x \in A, \\ 0 & \text{otherwise.} \end{cases},$$

$$F_{D_A}(x) = \begin{cases} \max(F_\mathcal{B}(y)) : y \in A & \text{if} \quad x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4. *Consider a three-ROFH* $\mathcal{H} = (\mathcal{Q}, \zeta)$ *as shown in Figure 4. The incidence matrix of* $\mathcal{H} = (\mathcal{Q}, \zeta)$ *is given in Table 2.*

Define a three-ROFS $\mathcal{B} = \{(x_1, 0.6, 0.7), (x_2, 0.7, 0.6), (x_3, 0.8, 0.7), (x_4, 0.6, 0.5), (x_5, 0.7, 0.8)\}$*. By direct calculations, we have:*

$T_{D_{\{x_1,x_3,x_5\}}}(x_1) = \min\{0.6, 0.8, 0.7\} = 0.6$,
$F_{D_{\{x_1,x_3,x_5\}}}(x_1) = \max\{0.7, 0.8, 0.7\} = 0.8$,
$T_{D_{\{x_2,x_3,x_4\}}}(x_2) = \min\{0.7, 0.8, 0.6\} = 0.6$,
$F_{D_{\{x_2,x_3,x_4\}}}(x_2) = \max\{0.6, 0.5, 0.7\} = 0.7$,
$T_{D_{\{x_1,x_4\}}}(x_4) = \min\{0.6, 0.6\} = 0.6$,
$F_{D_{\{x_1,x_4\}}}(x_4) = \max\{0.7, 0.7\} = 0.7$,
$T_{D_{\{x_2,x_5\}}}(x_5) = \min\{0.7, 0.7\} = 0.7$,
$F_{D_{\{x_2,x_5\}}}(x_5) = \max\{0.6, 0.8\} = 0.8$. *Similarly, all other values can be calculated by using the same method. Thus, we have* $\zeta_1 = (T_{D_{\{x_1,x_3,x_5\}}}, F_{D_{\{x_1,x_3,x_5\}}})$, $\zeta_2 = (T_{D_{\{x_2,x_3,x_4\}}}, F_{D_{\{x_2,x_3,x_4\}}})$, $\zeta_3 = (T_{D_{\{x_1,x_4\}}}, F_{D_{\{x_1,x_4\}}})$, $\zeta_4 = (T_{D_{\{x_2,x_5\}}}, F_{D_{\{x_2,x_5\}}})$*. Hence,* $\mathcal{H}$ *is a* $\mathcal{B}$*-tempered three-ROFH.*

Table 2. Incidence matrix of $\mathcal{H}$.

I	ζ_1	ζ_2	ζ_3	ζ_4
x_1	$(0.6, 0.7)$	$(0, 1)$	$(0.6, 0.7)$	$(0, 1)$
x_2	$(0, 1)$	$(0.7, 0.6)$	$(0, 1)$	$(0.7, 0.6)$
x_3	$(0.8, 0.7)$	$(0.8, 0.7)$	$(0, 1)$	$(0, 1)$
x_4	$(0, 1)$	$(0.6, 0.5)$	$(0.6, 0.7)$	$(0, 1)$
x_5	$(0.7, 0.8)$	$(0, 1)$	$(0, 1)$	$(0.7, 0.8)$

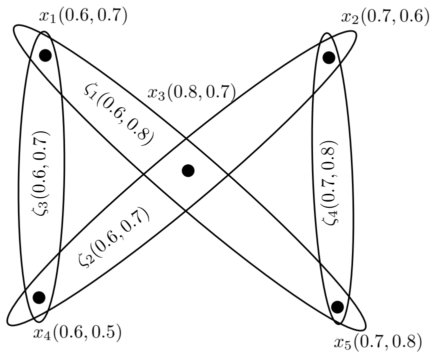

Figure 4. $\mathcal{B}$-tempered three-rung orthopair fuzzy hypergraph.

3. Transversals of q-Rung Orthopair Fuzzy Hypergraphs

Definition 19. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a q-ROFH on X. A q-ROF subset τ of X, which satisfies the condition $\tau^{h(\zeta_i)} \cap \zeta_i^{h(\zeta_i)} \neq \emptyset$, for all $\zeta_i \in \zeta$, is called a q-rung orthopair fuzzy transversal(q-ROFT) of $\mathcal{H}$.*
τ is called the minimal transversal of $\mathcal{H}$ if $\tau_1 \subset \tau$, τ_1 is not a q-ROFT. $t_r(\mathcal{H})$ denotes the collection of minimal transversals of $\mathcal{H}$.

We now discuss some results on q-ROFTs.

Remark 4. *Although τ can be regarded as a minimal transversal of $\mathcal{H}$, it is not necessary for $\tau^{(\alpha,\beta)}$ to be the minimal transversal of $\mathcal{H}^{(\alpha,\beta)}$, for all $\alpha, \beta \in [0, 1]$. Furthermore, it is not necessary for the family of minimal q-ROFTs to form a hypergraph on X. For those q-ROFTs that satisfy the above property, we have:*

Definition 20. *A q-ROFT τ with the property that $\tau^{(\alpha,\beta)}$ is a minimal transversal of $\mathcal{H}^{(\alpha,\beta)}$, for $\alpha, \beta \in [0, 1]$, is called the locally-minimal q-ROFT of $\mathcal{H}$. The collection of locally-minimal q-ROFTs of $\mathcal{H}$ is denoted by $t_r^*(\mathcal{H})$.*

Lemma 1. *Let $f_S(\mathcal{H}) = \{\rho_1, \rho_2, \rho_3, \cdots, \rho_n\}$ be the fundamental sequence of a q-ROFH $\mathcal{H}$ and τ be the q-ROFT of $\mathcal{H}$. Then, $h(\tau) \geq h(\zeta_i)$, for each $\zeta_i \in \zeta$, and if τ is minimal, then $h(\tau) = \max\{h(\zeta_i) | \zeta_i \in \zeta\} = \rho_1$.*

Proof. Since τ is a q-ROFT of $\mathcal{H}$, then $\tau^{h(\zeta_i)} \cap \zeta_i^{h(\zeta_i)} \neq \emptyset$. Consider an arbitrary element of $supp(\tau)$, then $\zeta_i(x) > h(\zeta_i)$, and we have $h(\tau) \geq h(\zeta_i)$. If τ is a minimal transversal, then $h(\zeta_i) = \{\max T_{\zeta_i}(x), \min F_{\zeta_i}(x) | x \in X \text{ and } \zeta_i \in \zeta\} = \rho_1$. Hence, $h(\tau) = \max\{h(\zeta_i) | \zeta_i \in \zeta\} = \rho_1$. □

Theorem 4. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a q-ROFH, then the statements:*

(i) *τ is a q-ROFT of $\mathcal{H}$,*
(ii) *For all $\zeta_i \in \zeta$ and for each $\rho = \{T_\rho, F_\rho\} \in [0, 1]$ satisfying $0 < (T_\rho, F_\rho) < h(\zeta_i)$, $\tau^\rho \cap \zeta^\rho \neq \emptyset$,*
(iii) *τ^ρ is a transversal of $\mathcal{H}^\rho$, for all $\rho \in [0, 1]$, $0 < \rho < \rho_1$,*

are equivalent.

Proof. (i) $\Rightarrow$ (ii). Suppose τ is a q-ROFT of $\mathcal{H}$. For any $\rho \in [0,1]$, which satisfies $0 < (T_\rho, F_\rho) < h(\zeta_i)$, $\tau^\rho \supseteq \tau^{h(\zeta_i)}$ and $\zeta_i^\rho \supseteq \zeta_i^{h(\zeta_i)}$. Hence, $\tau^\rho \cap \zeta^\rho \supseteq \tau^{h(\zeta_i)} \cap \zeta_i^{h(\zeta_i)} \neq \emptyset$, because τ is a transversal.
(ii) $\Rightarrow$ (iii). Let $\tau^\rho \cap \zeta_i^\rho \neq \emptyset$, for all $\zeta)i \in \zeta$ and $0 < T_\rho < T_{\rho_1}$, $0 > F_\rho < F_{\rho_1}$, which implies that τ^ρ is a transversal of $\mathcal{H}^\rho$.
(iii) $\Rightarrow$ (i). This part can be proven trivially. □

Theorem 5. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a q-ROFH. For each $x \in X$ such that $\tau(x) \in f_S(\mathcal{H})$ and for all $\tau \in t_r(\mathcal{H})$, the fundamental sequence of $t_r(\mathcal{H}) \subset f_S(\mathcal{H})$.*

Proof. Let the fundamental sequence of $\mathcal{H}$ be $f_S(\mathcal{H}) = \{\rho_1, \rho_2, \rho_3, \cdots, \rho_n\}$ and $\tau \in t_r(\mathcal{H})$, for $\tau(x) \in (\rho_{i+1}, \rho_i]$. Consider a mapping ψ defined by:

$$\psi(u) = \begin{cases} \rho_i & \text{if} \quad x = u, \\ \tau(u) & \text{otherwise.} \end{cases}$$

Thus, from the definition of ψ, it follows that $\psi^{\rho_i} = \tau^{\rho_i}$, and the definition of fundamental sequence of $\mathcal{H}$ implies that $\mathcal{H}^\rho = \mathcal{H}^{\rho_i}$, for all $\rho \in (\rho_{i+1}, \rho_i]$. Since τ is a q-ROFT of $\mathcal{H}$ and $\psi^\rho = \tau^\rho$, for all $\rho \notin (\rho_{i+1}, \rho_i]$, ψ is a q-ROFT. Now, $\psi \leq \tau$, and the minimality of τ both implies that $\psi = \tau$. Thus, $\tau(x) = \psi(x) = \rho_1$ and $\tau(x) \in f_S(\mathcal{H})$. Hence, we conclude that $f_S(t_r(\mathcal{H})) \subseteq f_S(\mathcal{H})$. □

Theorem 6. *The collection of all minimal transversals $t_r(\mathcal{H})$ is sectionally elementary.*

Proof. Let the fundamental sequence of $t_r(\mathcal{H})$ be $f_s(t_r(\mathcal{H})) = \{\rho_1, \rho_2, \rho_3, \cdots, \rho_n\}$. Consider an element τ of $t_r(\mathcal{H})$ and some $\rho \in (\rho_{i+1}, \rho_i]$ such that $\tau^{\rho_i} \subset \tau^\rho$. In consideration of $[t_r(\mathcal{H})]^\rho = [t_r(\mathcal{H})]^{\rho_i}$, we have $\psi \in t_r(\mathcal{H})$ satisfying $\psi^\rho = \tau^{\rho_i}$. Then, the condition $\psi^\rho \supset \tau^{\rho_i}$ implies the existence of a q-rung orthopair fuzzy set $\mathcal{R}$ such that:

$$\mathcal{R}(x) = \begin{cases} \rho, & \text{if} \quad x \in \psi^{\rho_i} \setminus \tau^{\rho_i}, \\ \psi(x), & \text{otherwise.} \end{cases}$$

is the q-ROFT of $\mathcal{H}$. Now, $\rho < \psi$ yields a contradiction to the minimality of ψ. □

Lemma 2. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a q-ROFH. Consider an element x of $supp(\tau)$, where $\tau \in t_r(\mathcal{H})$, then there exists a q-rung orthopair fuzzy hyperedge ζ of $\mathcal{H}$ such that:*

(i) $\tau(x) = h(\zeta) = \zeta(x) > 0$,
(ii) $\tau^{h(\zeta)} \cap \zeta^{h(\zeta)} = \{x\}$.

Proof. (i) Let $\tau(x) > 0$ and Q denote the set of all q-rung orthopair fuzzy hyperedges of $\mathcal{H}$ such that for each element ζ of Q, $\zeta(x) \geq \tau(x)$. Then, this set is non-empty because $\tau^{\tau(x)}$ is a transversal of $\mathcal{H}^{\tau(x)}$ and $x \in \tau^{\tau(x)}$. Additionally, each element ζ of Q satisfies the inequality $h(\zeta) \geq \zeta(x) \geq \tau(x)$. Suppose, on the contrary, that (i) is false, then for each $\zeta \in Q$, $h(\zeta) > \tau(x)$, and we have an element $x^\zeta \neq x$, where $x^\zeta \in \zeta^{h(\zeta)} \cap \tau^{h(\zeta)}$. Here, we define a q-ROFS Q' as:

$$Q'(v) = \begin{cases} \tau(v) & \text{if} \quad x \neq v, \\ \max\{h(\zeta) | h(\zeta) < \tau(x)\} & \text{if} \quad x = v. \end{cases}$$

Note that Q' is a q-ROFT of $\mathcal{H}$ and $Q' < \tau$, which is a contradiction to the fact that τ is minimal. Hence, (i) holds for some ζ. (ii) Suppose each element of Q satisfies (i) and also has an element $x^\zeta \neq x$, where $x^\zeta \in \zeta^{h(\zeta)} \cap \tau^{h(\zeta)}$. Following the same arguments as used in (i) above completes the proof. □

Theorem 7. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be an ordered q-ROFH with $f_S(\mathcal{H}) = \{\rho_1, \rho_2, \rho_3, \cdots, \rho_n\}$, and $c(\mathcal{H}) = \{\mathcal{H}^{\rho_1}, \mathcal{H}^{\rho_2}, \mathcal{H}^{\rho_3}, \cdots, \mathcal{H}^{\rho_n}\}$. Then, $t_r^\star(\mathcal{H})$ is non-empty. Further, if τ_n is a minimal transversal of $\mathcal{H}^{\rho_n}$, then there exists $T \in t_r^\star(\mathcal{H})$ such that $supp(T) = \tau_n$.*

Proof. Let τ_n be a minimal transversal of $\mathcal{H}^{\rho_n}$; $\mathcal{H}^{\rho_{n-1}}$ is a partial hypergraph of $\mathcal{H}^{\rho_n}$ because $\mathcal{H}$ is ordered, and consequently, τ_{n-1} is a minimal transversal of $\mathcal{H}^{\rho_{n-1}}$ such that $\tau_{n-1} \subseteq \tau_n$. By continuing the same argument, we establish a nested sequence of minimal transversals $\tau_1 \subseteq \tau_2 \subseteq \tau_3 \subseteq \cdots \subseteq \tau_n$, where every τ_i is a minimal transversal of $\mathcal{H}^{\rho_i}$. Let $\eta_j = \eta_j(\tau_j, \rho_j)$ be an elementary q-ROF set having height ρ_j and support τ_j. Then, $T = \max\{\eta_j | 1 \leq j \leq n\}$ is the locally-minimal transversal of $\mathcal{H}$ having support τ_n. □

We now present a construction for finding $t_r(\mathcal{H})$ as follows.

Construction 1. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a q-ROFH having the set of core hypergraphs $c(\mathcal{H}) = \{\mathcal{H}^{\rho_1}, \mathcal{H}^{\rho_2}, \mathcal{H}^{\rho_3}, \cdots, \mathcal{H}^{\rho_n}\}$. An iterative procedure to find the minimal transversal τ of $\mathcal{H}$ is as follows:*

1. *Find a crisp minimal transversal τ_1 of $\mathcal{H}^{\rho_1}$.*
2. *Find a minimal transversal τ_2 of $\mathcal{H}^{\rho_2}$ that satisfies $\tau_1 \subseteq \tau_2$, i.e., formulate a new hypergraph $\mathcal{H}_2$ having hyperedges ζ^{ρ_2}, which is augmented having a loop at each $x \in \tau_1$. In accordance with this, we can say that $\zeta(H_2) = \zeta^{\rho_2} \cup \{\{x\} | x \in \tau_1\}$. Let τ_2 be an arbitrary minimal transversal of $\mathcal{H}_2$.*
3. *By continuing the same procedure repeatedly, we have a sequence of minimal transversals $\tau_1 \subseteq \tau_2 \subseteq \tau_3 \subseteq \cdots \subseteq \tau_j$ such that τ_j is the minimal transversal of $\mathcal{H}^{\rho_j}$ with the property $\tau_{j-1} \subseteq \tau_j$.*
4. *Consider an elementary q-rung orthopair fuzzy set μ_j having the support τ_j and $h(\mu_j) = \rho_j$, $1 \leq j \leq n$. Then, $\tau = \bigcup\limits_{j=1}^{n} \{\mu_j | 1 \leq j \leq n\}$ is a minimal q-ROFT of $\mathcal{H}$.*

Example 5. *Consider a five-ROFH $\mathcal{H} = (\mathcal{Q}, \zeta)$, as shown in Figure 5, where $\zeta = \{\zeta_1, \zeta_2, \zeta_3\}$. Incidence matrix of $\mathcal{H} = (\mathcal{Q}, \zeta)$ is given in Table 3. By routine calculations, we have $h(\zeta_1) = (0.8, 0.6)$, $h(\zeta_2) = (0.8, 0.5)$ and $h(\zeta_3) = (0.8, 0.5)$. Consider a q-rung orthopair fuzzy subset τ_1 of X such that $\tau_1 = \{(x_1, 0.8, 0.6), (x_2, 0.7, 0.9), (x_3, 0.8, 0.5)\}$. Note that, $\zeta_1^{h(\zeta_1)} = \{x_1\}$, $\zeta_2^{h(\zeta_2)} = \{x_3\}$, and $\zeta_3^{h(\zeta_3)} = \{x_3\}$. Furthermore, $\tau_1^{(0.8,0.6)} = \{x_1\}$, $\tau_2^{(0.8,0.5)} = \{x_3\}$, and $\tau_3^{(0.8,0.5)} = \{x_3\}$. It can be seen that $\tau_1^{h(\zeta_i)} \cap \zeta_i^{h(\zeta_i)} \neq \emptyset$, for all $\zeta_i \in \zeta$. Thus, τ_1 is a five-ROFT of $\mathcal{H}$. Similarly, $\tau_2 = \{(x_1, 0.8, 0.6), (x_3, 0.8, 0.5)\}$, $\tau_3 = \{(x_1, 0.8, 0.6), (x_3, 0.8, 0.5), (x_4, 0.6, 0.8)\}$, $\tau_4 = \{(x_1, 0.8, 0.6), (x_3, 0.8, 0.5), (x_5, 0.7, 0.5),\}$ are other transversals of $\mathcal{H}$. The minimal transversal is τ_2, i.e., whenever $\tau \subseteq \tau_2$, τ is not a five-ROFT.*

Let $\alpha = 0.8$, $\beta = 0.5$, then $\zeta_1^{(0.8,0.5)} = \{\emptyset\}$, $\zeta_2^{(0.8,0.5)} = \{x_3\}$, $\zeta_3^{(0.8,0.5)} = \{x_3\}$ show that $\tau_2^{(0.8,0.5)}$ is not a minimal transversal of $\mathcal{H}^{(0.8,0.5)}$.

Table 3. Incidence matrix of $\mathcal{H}$.

I	ζ_1	ζ_2	ζ_3
x_1	$(0.8, 0.6)$	$(0.8, 0.6)$	$(0, 1)$
x_2	$(0.7, 0.9)$	$(0, 1)$	$(0.7, 0.9)$
x_3	$(0, 1)$	$(0.8, 0.5)$	$(0.8, 0.5)$
x_4	$(0.6, 0.8)$	$(0.6, 0.8)$	$(0, 1)$
x_5	$(0, 1)$	$(0, 1)$	$(0.7, 0.5)$

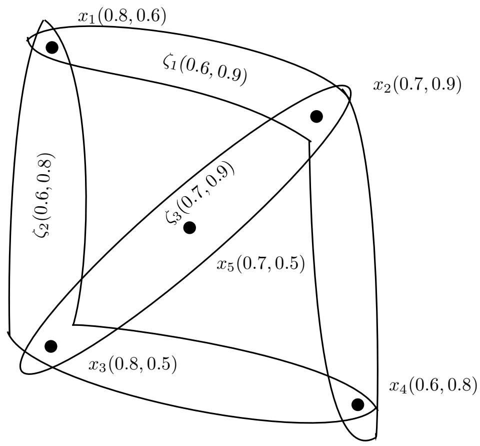

Figure 5. Five-rung orthopair fuzzy hypergraph (ROFH).

Theorem 8. *Let $\mathcal{H} = (\mathcal{Q}, \zeta)$ be a q-ROFH and $x \in X$. Then, there exists an element τ of $t_r(\mathcal{H})$ such that $x \in supp(\tau)$ if and only if there is a hyperedge $\zeta_1 \in \zeta$, which satisfies:*

(i) $\zeta_1(x) = h(\zeta')$,
(ii) *For every $\xi \in \zeta$ with $h(\xi) > h(\zeta_1)$, $\xi^{h(\zeta_i)} \not\subset \zeta_1^{h(\zeta_1)}$,*
(iii) *The $h(\zeta_1)$ level cut of ζ_1 is not a proper subset of any other hyperedge of $\mathcal{H}^{h(\zeta_1)}$.*

Proof.

(i) Let us suppose that $\tau(x) > 0$ and τ is an element of $t_r(\mathcal{H})$, then the first condition directly follows from Lemma 2.
(ii) To prove the second condition, suppose that for every ζ_1 that satisfies the first condition, there is $\xi \in \zeta$ such that $h(\xi) > h(\zeta_1)$ and $\xi^{h(\xi)} \subseteq \zeta_1^{h(\zeta_1)}$. Then, there exists an element $v \neq x$, where $v \in \xi^{h(\xi)} \cap \tau^{h(\xi)} \subseteq \zeta_1^{h(\zeta_1)} \cap \tau^{h(\zeta_1)}$, which is a contradiction.
(iii) To prove that the $h(\zeta_1)$ level cut of ζ_1 is not a proper subset of any other hyperedge of $\mathcal{H}^{h(\zeta_1)}$, suppose that for every ζ_1 that satisfies the above two conditions, there is $\xi \in \zeta$ with $\emptyset \subset \xi^{h(\xi)} \subset \zeta_1^{h(\zeta_1)}$, as $\xi^{h(\xi)} \neq \emptyset$, and from second condition, we have $h(\xi) = \zeta_1(x) = \tau(x)$. If $h(\xi) = \zeta_1(x)$, our supposition accommodates $\xi' \in \zeta$ such that $\emptyset \subset \xi'^{h(\zeta_1)} \subset \xi^{h(\zeta_1)} \subset \zeta_1^{h(\zeta_1)}$. This recursive procedure must end after a finite number of steps, so assume that $\xi(x) < h(\xi)$, which implies the existence of an element $v \neq x$, where $v \in \xi^{h(\zeta_1)} \cap \tau^{h(\zeta_1)} \subseteq \zeta_1^{h(\zeta_1)} \cap \tau^{h(\zeta_1)}$, which is again a contradiction.

The sufficient condition is proven by using the construction given in Algorithm 1. By using the first condition, we have $h(\zeta_1) = \rho_1$, $\rho_1 \in f_S(\mathcal{H})$, and from the other two conditions, we have $y_\xi \in \xi^{h(\xi)} \setminus \zeta_1^{h(\zeta_1)}$ such that $\xi \neq \zeta_1$ and $h(\xi) \geq h(\zeta_1)$. Then, $Q \cap \zeta_1^{h(\zeta_1)}$, where Q is the collection of all such vertices. An initial sequence of transversals is constructed in a way that $\tau_j \subseteq Q$, for $1 \leq j \leq n$ and $\tau_i \subseteq Q \cup \{x\}$. Continuing the construction 1 will give a minimal q-ROFT with $\tau(x) = \zeta_1(x) = h(\zeta_1)$. □

Definition 21. *Let Q be a q-ROFS and $\alpha, \beta \in [0, 1]$. The lower truncation of Q at level α, β is a q-ROFS $Q_{\langle\alpha,\beta\rangle}$ given by:*

$$Q_{\langle\alpha,\beta\rangle}(x) = \begin{cases} Q(x) & \text{if} \quad x \in Q^{(\alpha,\beta)}, \\ (0,1) & \text{otherwise.} \end{cases}$$

The upper truncation of Q at level α, β is a q-ROFS $Q^{\langle\alpha,\beta\rangle}$ given by:

$$Q^{\langle\alpha,\beta\rangle}(x) = \begin{cases} (\alpha,\beta) & \text{if} \quad x \in Q^{(\alpha,\beta)}, \\ Q(x) & \text{otherwise.} \end{cases}$$

Definition 22. *Let $\mathcal{E}$ be a collection of q-ROFSs of X and:*

$$\mathcal{E}^{\langle\alpha,\beta\rangle} = \{q^{\langle\alpha,\beta\rangle} | q \in \mathcal{E}\} \quad , \quad \mathcal{E}_{\langle\alpha,\beta\rangle} = \{q_{\langle\alpha,\beta\rangle} | q \in \mathcal{E}\}.$$

Then, the upper and lower truncations of a q-ROFH $\mathcal{H} = (\mathcal{Q}, \zeta)$ at α, β level are a pair of q-ROFHs, $\mathcal{H}^{\langle\alpha,\beta\rangle}$ and $\mathcal{H}_{\langle\alpha,\beta\rangle}$, defined by
$\mathcal{H}^{\langle\alpha,\beta\rangle} = (X, \mathcal{E}^{\langle\alpha,\beta\rangle})$ and $\mathcal{H}_{\langle\alpha,\beta\rangle} = (X, \mathcal{E}_{\langle\alpha,\beta\rangle})$.

Definition 23. *Let Q be a q-ROFS on X, then each $(\mu,\nu) \in (0, h(Q))$ for which:*

$$Q^{(\alpha,\beta)} \nsubseteq Q^{(\mu,\nu)} \ \ , \ \ (\mu,\nu) < (\alpha,\beta) \leq h(Q),$$

is called the transition level of Q.

Definition 24. *Let Q be a non-trivial q-ROFS of X. Then:*

(i) *the sequence $\mathcal{S}(Q) = \{t_1^Q, t_2^Q, t_3^Q, \cdots, t_n^Q\}$ is called the basic sequence determined by Q, where:*

- $t_1^Q > t_2^Q > t_3^Q > \cdots > t_n^Q > 0$,
- $t_1^Q = h(Q)$,
- $\{t_2^Q, t_3^Q, \cdots, t_n^Q\}$ *is the set of transition levels of Q.*

(ii) *The set of cuts of Q, $\mathcal{C}(Q)$, is defined as $\mathcal{C}(Q) = \{Q^t | t \in \mathcal{S}(Q)\}$.*
(iii) *The join $\max\{\eta(Q^t, t) | t \in \mathcal{S}(Q)\}$ of basic elementary q-ROFSs $E(Q) = \{\eta(Q^t, t) | t \in \mathcal{S}(Q)\}$ is called the basic elementary join of Q.*

Lemma 3. *Let $\mathcal{H}$ be a q-ROFH with $f_S(\mathcal{H}) = \{\rho_1, \rho_2, \rho_3, \cdots, \rho_n\}$. Then:*

(i) *if $t = (\mu,\nu)$ is a transition level of $\tau \in t_r(\mathcal{H})$, then there is an $\epsilon > 0$ such that, $\forall \ (\alpha,\beta) \in (t, t+\epsilon]$, $\tau^{(\mu,\nu)}$ is a minimal $\mathcal{H}^{(\mu,\nu)}$-transversal extension of $\tau^{(\alpha,\beta)}$, i.e., if $\tau^{(\alpha,\beta)} \subseteq \tau' \subseteq \tau^{(\mu,\nu)}$, then τ' is not a transversal of $\mathcal{H}^{(\mu,\nu)}$.*
(ii) *$t_r(\mathcal{H})$ is sectionally elementary.*
(iii) *$f_S(t_r(\mathcal{H}))$ is properly contained in $f_S(\mathcal{H})$.*
(iv) *$\tau^{(\alpha,\beta)}$ is a minimal transversal of $\mathcal{H}^{(\alpha,\beta)}$, for each $\tau \in t_r(\mathcal{H})$ and $\rho_2 < (\alpha,\beta) \leq \rho_1$.*

Proof.

(i) Let $\tilde{t} = (\mu,\nu)$ be a transition level of $\tau \in t_r(\mathcal{H})$. Then, by definition, we have $\tau^{(\alpha,\beta)} \nsubseteq \tau^{(\mu,\nu)}$, $(\mu,\nu) < (\alpha,\beta) \leq h(\mathcal{H})$, for all α, β. Since τ possesses a finite support, this implies the existence of an $\epsilon > 0$ such that $\tau^{(\alpha,\beta)}$ is constant on $(\tilde{t}, \tilde{t}+\epsilon]$. Assume that there is a transversal T of $\mathcal{H}^{(\mu,\nu)}$ such that $\tau^{(\alpha',\beta')} \subseteq T \subseteq \tau^{(\mu,\nu)}$, for $\alpha', \beta' \in (\tilde{t}, \tilde{t}+\epsilon]$. We claim that this supposition is false. To demonstrate the existence of this claim, we suppose that the assumption is true and consider the collection of basic elementary q-ROFSs $E(\tau) = \{\eta(\tau^t, t) | t \in S(\tau)\}$ of τ. Note that a nested sequence of X is formed by $c(\tau) \cup T$, where $c(\tau)$ is used to denote the basic cuts of τ. Since $\mathcal{H} = (\mathcal{Q}, \zeta)$ is defined on a finite set X and $\mathcal{Q}$ is a finite collection of q-ROFSs of X, then each $\rho \in (0, h(\mathcal{H}))$ corresponds to a number $\epsilon_\rho > 0$ such that:

- $\mathcal{H}^{(\alpha,\beta)}$ is constant on $(\rho, \rho + \epsilon_\rho]$,
- $\mathcal{H}^{(\alpha,\beta)}$ is constant on $(\rho - \epsilon_\rho, \rho]$.

It follows from these considerations that level cuts of $\tau^{\star(\alpha,\beta)}$ of the join $\tau^{\star}$= max$\{\max\{E(\tau) \setminus \eta(\tau^{\tilde{t}},\tilde{t}), \eta(\tau^{\tilde{t}},\tilde{t}-\epsilon_{\tilde{t}}), \eta(T,\tilde{t})\}\}$ persuade:

$$\tilde{\tau}^{(\alpha,\beta)} = \begin{cases} T & \text{if} \quad (\alpha,\beta) \in (\tilde{t}-\epsilon_{\tilde{t}},\tilde{t}), \\ \tau^{(\alpha,\beta)} & \text{if} \quad (\alpha,\beta) \in (0,h(\mathcal{H})) \setminus (\tilde{t},\tilde{t}-\epsilon_{\tilde{t}})]. \end{cases}$$

This relation is derived because of the supposition that $\epsilon_{\tilde{t}}$ is so small that the open interval $(\tilde{t}-\epsilon_{\tilde{t}},\tilde{t})$ does not contain any other transition level of τ.
Since it is assumed that T is a transversal of $\mathcal{H}^{\tilde{t}}$, T is a transversal of $\mathcal{H}^{(\alpha,\beta)}$, for all $(\alpha,\beta) \in (\tilde{t}-\epsilon_{\tilde{t}},\tilde{t})$, and $\mathcal{H}^{(\alpha,\beta)}$ is constant on $(\tilde{t}-\epsilon_{\tilde{t}},\tilde{t})$. Note that $\tau^{(\alpha,\beta)}$ is a transversal of $\mathcal{H}^{(\alpha,\beta)}$, for all $(\alpha,\beta) \in (0,h(\mathcal{H})]$; therefore, it follows that $\tilde{\tau}$ is a q-ROF transversal of $\mathcal{H}$, as $\tilde{\tau} < \tau$ implies that $\tau \notin t_r(\mathcal{H})$, which leads to a contradiction. Hence, the supposition is false, and the claim is satisfied.

(ii) Let $\tau \in t_r(\mathcal{H})$, then $\tau^{(\alpha,\beta)}$ is a transversal of $\mathcal{H}^{(\alpha,\beta)}$ for $0 < (\alpha,\beta) < h(\mathcal{H})$. Suppose that a transition level t of τ corresponds to an interval $(t,t+\epsilon]$, $\epsilon > 0$, on which $\tau^{(\alpha,\beta)}$ is constant. Then, for $(\alpha',\beta') \in (t,t+\epsilon]$, $\tau^{(\alpha',\beta')}$ is not a transversal of $\mathcal{H}^t$, which implies that $\tau^{(\alpha',\beta')} \notin (t_r(\mathcal{H}))^t$, where $t_r(\mathcal{H}))^t$ denotes the t-cut of $t_r(\mathcal{H})$. However, the definition of the fundamental sequence of $t_r(\mathcal{H})$ implies that $t \in f_S(t_r(\mathcal{H}))$.

(iii) To prove (iii), we suppose that if t = (μ,ν) is a transition level of some $\tau \in t_r(\mathcal{H})$, then t belongs to $f_S(\mathcal{H})$. On the contrary, suppose that the transition level t of some $\tau \in t_r(\mathcal{H})$ does not belong to $f_S(\mathcal{H})$. Then, for some $\rho_j \in f_S(\mathcal{H})$, we have $\rho_{j+1} < t < \rho_j$, where ρ_{n+1} = 0, as $\mathcal{H}^{(\alpha,\beta)}$ = $\mathcal{H}^{\rho_j}$, for all $(\alpha,\beta) \in (\rho_{j+1},\rho_j]$, and it follows that τ^t is a transversal of $\mathcal{H}^t$ = $\mathcal{H}^{\rho_j}$. Furthermore, there exists an $\epsilon > 0$, such that $\tau^{(\alpha,\beta)}$ is constant on $(t,t+\epsilon]$. Without loss of generality, we assume that $t+\epsilon \leq \rho_j$ and $(\alpha',\beta') \in (t,t+\epsilon]$. Since t is a transition level of τ, then $\tau^{(\alpha',\beta')} \subsetneq \tau^t$, and $\tau^{(\alpha',\beta')}$ is not a transversal of $\mathcal{H}^t$ (from i), which is not possible, as $\mathcal{H}^{(\alpha',\beta')}$ = $\mathcal{H}^{\rho_j}$ = $\mathcal{H}^t$; this proves our claim. Along with this result and the fact that $h(\tau)$ = $\rho_1 \in f_S(\mathcal{H})$, it follows that $f_S(t_r(\mathcal{H})) \subseteq f_S(\mathcal{H})$, for all $\tau \in t_r(\mathcal{H})$.

(iv) First, we will show that τ^{ρ_1} is a minimal transversal of $\mathcal{H}^{\rho_1}$. Suppose on the contrary that there is a minimal transversal T of $\mathcal{H}^{\rho_1}$ such that $T \subseteq \tau^{\rho_1}$. Let $\tilde{\tau}$ = max$\{\tau^{\rho_2},\eta_1\}$, where η_1 is the basic elementary q-ROFS having support T and height ρ_1. τ^{ρ_2} is considered as the upper truncation of τ at level ρ_2. It is obvious that $\tilde{\tau}$ is a transversal of $\mathcal{H}$ with $\tilde{\tau} < \tau$, which is a contradiction to the fact that τ is minimal. From the (ii) and (iii) parts, it follows that $\tau^{(\alpha,\beta)} \in t_r(\mathcal{H})^{(\alpha,\beta)}$, for $\rho_2 < (\alpha,\beta) < \rho_1$.

□

Theorem 9. *At least one minimal q-ROFT is contained in every q-ROFT of a q-ROFH $\mathcal{H}$.*

Proof. Let $f_S(\mathcal{H})$ = $\{\rho_1,\rho_2,\rho_3,\cdots,\rho_n\}$ be the fundamental sequence of $\mathcal{H}$ and suppose that ξ is a transversal of $\mathcal{H}$, which is not minimal. Let τ be a minimal transversal of $\mathcal{H}$, $\tau \leq \xi$, which is constructed in such a way $\{q_i \in Q(X) | i = 0,1,2,\cdots,n\}$ satisfying τ = $q_n \leq \cdots \leq q_1 \leq q_0 \leq \xi$, where $Q(X)$ is the collection of q-ROFSs on X. It can be noted that $h(\xi) \geq h(\mathcal{H})$ = ρ_1 and $\xi^{(\alpha,\beta)}$ is a transversal of $\mathcal{H}^{(\alpha,\beta)}$, for $0 < (\alpha,\beta) \leq \rho_1$. Therefore, the reduction process is started as q_0 = $\xi^{\langle\rho_1\rangle}$, where $\xi^{\langle\rho_1\rangle}$ represents the upper truncation level of ξ at ρ_1. Since the top level cut ξ^{ρ_1} of ρ_0 comprises a crisp minimal transversal T_1 of $\mathcal{H}^{\rho_1}$, we have q_1 = max$\{\xi^{\langle\rho_2\rangle},\lambda^{T_1}\}$, where λ^{T_1} is an elementary q-ROFS having height ρ_1 and support T_1. Note that $q_1 \leq q_2 \leq \xi$. The same procedure will determine all the other remaining members. For instance, we have q_2 = max$\{\xi^{\langle\rho_3\rangle},\lambda^{T_1},\lambda^{T_2}\}$, where λ^{T_2} is an elementary q-ROFS having height ρ_2 and support T_2, such that:

$$T_2 = \begin{cases} T_1 & \text{if } \mathrm{T}_1 \text{ is a transversal of } \mathcal{H}^{æ_2}, \\ B_2, & \text{otherwise,} \end{cases}$$

where B_2 is the minimal transversal extension of T_1, i.e., if $T_1 \subseteq B \subseteq B_2$, then B_2 is not considered as a transversal of $\mathcal{H}^{\rho_2}$, and B_2 is contained in the ρ-level of ξ because ξ^{ρ_2} contains a transversal of $\mathcal{H}^{\rho_2}$. Further, as $T_2 \subseteq \xi^{\rho_2}$, it is obvious that $q_2 \leq q_1$. When this process is finished, we certainly have $q_n = \tau$, a q-ROF transversal of $\mathcal{H}$ and included in ξ. We now claim that τ is a minimal transversal of $\mathcal{H}$, i.e., $\tau \in t_r(\mathcal{H})$. On the contrary, suppose that τ_1 is a transversal of $\mathcal{H}$ such that $\tau_1 < \tau$. Then, we have:

(i) $\tau_1^{(\alpha,\beta)} \subseteq \tau^{(\alpha,\beta)}$ for all $\alpha, \beta \in (0, h(\mathcal{H})]$,
(ii) $\tau_1^{(\alpha',\beta')} \subseteq \tau^{(\alpha',\beta')}$ for some $\alpha', \beta' \in (0, h(\mathcal{H})]$.

However, no such α', β' exist. To prove this, let $\alpha, \beta \in (\rho_2, \rho_1]$, then as $\tau_1^{(\alpha,\beta)} \subseteq \tau^{(\alpha,\beta)}$, $\tau_1^{(\alpha,\beta)}$ is a transversal of $\mathcal{H}^{(\alpha,\beta)} = \mathcal{H}^{\rho_1}$ and $\tau^{(\alpha,\beta)} \in t_r(\mathcal{H}^{\rho_1})$, which implies that $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$ on $(\rho_2, \rho_1]$. Moreover, suppose that $\alpha, \beta \in (\rho_3, \rho_2]$, then by using $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$, we have $\tau_1^{(\alpha,\beta)} \supseteq \tau^{\rho_1}$ on $(\rho_3, \rho_2]$, and if $T_2 = T_1 = \tau^{\rho_1}$, then by the previous arguments, $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$ on $(\rho_3, \rho_2]$. Furthermore, if $T_1 \subseteq T_2$ and $T_1 \subseteq \tau_1^{(\alpha,\beta)} \subsetneq T_2$, then $\tau_1^{(\alpha,\beta)}$ is not a transversal of $\mathcal{H}^{(\alpha,\beta)} = \mathcal{H}^{\rho_2}$, which is a contradiction to the fact that τ_1 is a transversal of $\mathcal{H}$. Hence, we have $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$ on $(\rho_3, \rho_2]$. In general, we have $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$ on $(0, h(\mathcal{H})]$, which completes the proof. □

4. Applications to Decision-Making

Decision-making is considered as an abstract technique, which results in the selection of an opinion or a strategy among a couple of elective potential results. Every decision-making procedure delivers a final decision, which may or may not be appropriate for our problem. We have to make hundreds of decisions everyday; some are easy, but others may be complicated, confusing, and miscellaneous. This leads to the process of decision-making. Decision-making is the foremost way to choose the most desirable alternative. It is essential in real-life problems, when there are many possible choices. Thus, decision-makers evaluate numerous merits and demerits of every choice and try to select the most fitting alternative.

4.1. Selection of the Most Desirable Appliance

Here, we consider a decision-making problem of selecting the most appropriate product from different brands or organizations. Suppose that a person wants to purchase a product, which is available of many brands. Let he/she consider the following nine organizations or brands $O = \{O_1, O_2, O_3, \cdots, O_9\}$, from which a product can be chosen to purchase. We will discuss how the (α, β)-level cuts can be applied to q-ROFH to make a good decision. The method adopted in this application is given in Algorithm 1.

A six-ROFH model depicting the problem is shown in Figure 6.

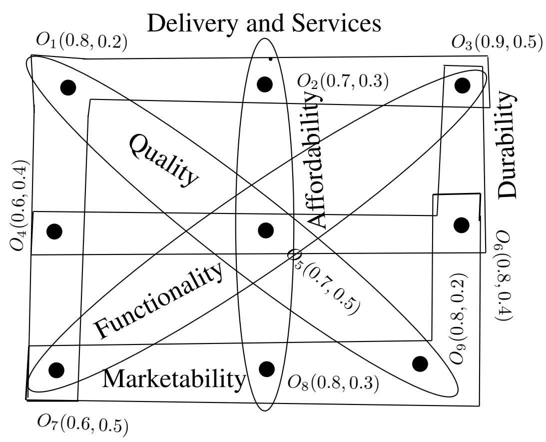

Figure 6. Six-rung orthopair fuzzy model to select the most appropriate appliance.

Algorithm 1: A decision-making method—applying the (α, β)-level cuts to q-ROFH

1. Input the truth-membership and falsity-membership degrees of all q-ROF vertices $O_1, O_2, O_3, \cdots, O_k$ such that $0 \leq T^q(O_j) + F^q(O_j) \leq 1, q \geq 1, j = 1, 2, \cdots, k$.
2. Calculate the truth-membership and falsity-membership degrees of q-ROF hyperedges using the formula

$$\begin{aligned} T_\zeta(E_l) &= T_\zeta(O_1, O_2, O_3, \ldots, O_k) \\ &\leq \min\{\mathcal{Q}_i(O_1), \mathcal{Q}_i(O_2), \mathcal{Q}_i(O_3), \ldots, \mathcal{Q}_i(O_k)\}, \\ F_\zeta(E_l) &= F_\zeta(O_1, O_2, O_3, \ldots, O_k) \\ &\leq \max\{\mathcal{Q}_i(O_1), \mathcal{Q}_i(O_2), \mathcal{Q}_i(O_3), \ldots, \mathcal{Q}_i(O_k)\}, \end{aligned}$$

   for all $O_1, O_2, O_3, \cdots, O_k$ representing the organizations as vertices.
3. Calculate the $(\alpha, \beta)-$levels of q-rung orthopair fuzzy hyperedges by using:

$$\zeta_i^{(\alpha,\beta)} = \{O_j \in O | T_{\zeta_i}(O_j) \geq \alpha \ , \ F_{\zeta_i}(O_j) \leq \beta\},$$

   for $i = 1, 2, 3, \cdots, l$, $j = 1, 2, 3 \cdots, k$ and $\alpha, \beta \in [0, 1]$.
4. Find out the crisp sets describing the most suitable organization according to the customer's satisfaction levels.

The truth-membership degrees and falsity-membership degrees of vertices (which represent the organizations) depict how much that organization fulfills the costumer's requirements and up to what percentage the product is not suitable. The hyperedges of our graph represent the characteristics of those organizations, which are (as vertices) contained in that hyperedge. This is shown in Table 4.

Table 4. Incidence matrix.

I	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
O_1	$(0.8, 0.2)$	$(0, 1)$	$(0.8, 0.2)$	$(0, 1)$	$(0, 1)$	$(0, 1)$
O_2	$(0.7, 0.3)$	$(0, 1)$	$(0, 1)$	$(0.7, 0.3)$	$(0, 1)$	$(0, 1)$
O_3	$(0.9, 0.5)$	$(0.9, 0.5)$	$(0, 1)$	$(0, 1)$	$(0.9, 0.5)$	$(0, 1)$
O_4	$(0.6, 0.4)$	$(0.6, 0.4)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0, 1)$
O_5	$(0, 1)$	$(0.7, 0.5)$	$(0.7, 0.5)$	$(0.7, 0.5)$	$(0.7, 0.5)$	$(0, 1)$
O_6	$(0, 1)$	$(0.8, 0.4)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0.8, 0.4)$
O_7	$(0.6, 0.5)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0.6, 0.5)$	$(0.6, 0.5)$
O_8	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0.8, 0.3)$	$(0, 1)$	$(0.8, 0.3)$
O_9	$(0, 1)$	$(0, 1)$	$(0.8, 0.2)$	$(0, 1)$	$(0, 1)$	$(0.8, 0.2)$

The attributes, which we have considered as hyperedges $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$ to describe the characteristics of different organizations, are delivery and service, durability, affordability, quality, functionality, and marketability Note that, if ζ_2 is considered as durability , then the membership degrees $(0.9, 0.5)$ of O_3 describe that the product manufactured by organization O_3 is 90% durable and 50% lacking in the requirements of the customer. Similarly, O_4 is 60% durable and 40% lacking the condition. In the same way, we can describe the characteristics of all products manufactured by different organizations. Now, to select the most appropriate product, we will find out the (α, β)-level cuts of all hyperedges. We choose the values of α and β in such a manner that they will be fixed according to the customer's demand. Let α = 0.7 and β = 0.4; this means that the customer will consider that product, which will satisfy 70% or more of the characteristics mentioned above and will have a deficiency less than or equal to 40%. The (α, β)-levels of all hyperedges are given as follows:

$$\zeta_1^{(0.7,0.4)} = \{O_1, O_2\}, \quad \zeta_2^{(0.7,0.4)} = \{O_6\}, \quad \zeta_3^{(0.7,0.4)} = \{O_1, O_5, O_9\},$$
$$\zeta_4^{(0.7,0.4)} = \{O_2, O_8\}, \quad \zeta_5^{(0.7,0.4)} = \{\emptyset\}, \quad \zeta_6^{(0.7,0.4)} = \{O_6, O_8, O_9\}.$$

Note that the $\zeta_1^{(0.7,0.4)}$ level set represents that O_1 and O_2 are the organizations that provide the best delivery services among all other organizations, and the $\zeta_2^{(0.7,0.4)}$ level set represents that O_6 is the organization whose products are more durable as compared to all other organizations. Similarly, $\zeta_4^{(0.7,0.4)}$ indicates that the products proposed by the O_2 and O_8 organizations, are more affordable in comparison to the others. Thus, if a customer wants some specific specialty product, for example he/she wants to purchase a product with good marketability, then the organizations O_6, O_8, and O_9 are more suitable. Similarly, if the satisfaction and dissatisfaction level of a customer are taken as $\alpha = 0.8$ and $\beta = 0.3$, respectively, then $(0.8, 0.3)$-level cuts are given as:

$$\zeta_1^{(0.8,0.3)} = \{O_1\}, \quad \zeta_2^{(0.8,0.3)} = \{\emptyset\}, \quad \zeta_3^{(0.8,0.3)} = \{O_1, O_9\},$$
$$\zeta_4^{(0.8,0.3)} = \{O_8\}, \quad \zeta_5^{(0.8,0.3)} = \{\emptyset\}, \quad \zeta_6^{(0.8,0.3)} = \{O_8, O_9\}.$$

Here, $\zeta_4^{(0.8,0.3)} = \{O_8\}$ indicates that the products proposed by organization O_8 satisfy the customer's requirement by 80%, which is affordability and so on. For $\alpha = 0.7$ and $\beta = 0.3$, we have:

$$\zeta_1^{(0.7,0.3)} = \{O_1, O_2\}, \quad \zeta_2^{(0.7,0.3)} = \{\emptyset\}, \quad \zeta_3^{(0.7,0.3)} = \{O_1, O_9\},$$
$$\zeta_4^{(0.7,0.3)} = \{O_2, O_8\}, \quad \zeta_5^{(0.7,0.3)} = \{\emptyset\}, \quad \zeta_6^{(0.7,0.3)} = \{O_8, O_9\}.$$

Hence, by considering different (α, β)-levels corresponding to the satisfaction and dissatisfaction levels of customers, we can conclude which organization fulfills the actual demands of a customer. The flowchart describing the procedure of above application is given in Figure 7.

4.2. Adaptation of the Most Alluring Residential Scheme

The essential factors for any purchase of property is the budget and location for a purchaser in particular. However, it is a complicated procedure to select a residential area for buying a house. In addition to scrutinizing the further details such as the pricing, loan options, payments, and developer's credentials, a customer must examine closely some other facilities that should be possessed by every housing community. Now, to adopt a favorable housing scheme, an obvious initial step is to compare different societies. After analyzing the characteristics of different societies, one will be able to make a wise decision. We will investigate the problem of adopting the most alluring residential scheme using seven-ROFH. The method adopted in our application is explained through Algorithm 2.

Let the set of vertices of seven-ROFH be taken as representative of those attributes' characteristics that one considers to make a comparison between different housing societies. The hyperedges of seven-ROFH represent some housing schemes that will be compared. The portrayal of our problem is illustrated in Figure 8.

The description of the hyperedges $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7\}$ and vertices $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ of the above hypergraph is given in Tables 5 and 6, respectively.

Note that each hyperedge represents a distinct housing scheme. and the vertices contained in hyperedges are those attributes that will be provided by the societies represented through hyperedges. This means that the Senate Avenue housing society provides 80% of the basic facilities of life, such as water, gas, and electricity, and 20% is not provided. Similarly, the same society accommodates its residents with easy accessibilityby 90% and only 10% lacking. In the same way, taking into account the truth-membership and falsity-membership degrees of all other attributes, we can identify the characteristics of all societies.

Algorithm 2: The investigation of the most alluring residential scheme using seven-ROFH

1. Input the truth-membership and falsity-membership degrees of all q-ROF vertices $x_1, x_2, x_3, \cdots, x_j$ such that $0 \leq T^q(x_i) + F^q(x_i) \leq 1$, $q \geq 1$, $1 \leq i \leq j$.
2. Calculate the truth-membership and falsity-membership degrees of q-ROF hyperedges using the formula:

$$\begin{aligned} T_\zeta(E_k) &= T_\zeta(x_1, x_2, x_3, \ldots, x_j) \\ &\leq \min\{\mathcal{Q}_i(x_1), \mathcal{Q}_i(x_2), \mathcal{Q}_i(x_3), \ldots, \mathcal{Q}_i(x_j)\}, \\ F_\zeta(E_k) &= F_\zeta(x_1, x_2, x_3, \ldots, x_j) \\ &\leq \max\{\mathcal{Q}_i(x_1), \mathcal{Q}_i(x_2), \mathcal{Q}_i(x_3), \ldots, \mathcal{Q}_i(x_j)\}, \end{aligned}$$

for all $x_1, x_2, \cdots, x_j$ representing the attributes of housing societies.
3. Calculate the heights of all q-rung orthopair fuzzy hyperedges by using:

$$h(\zeta_j) = (\max T_{\zeta_j}(x_i), \min F_{\zeta_j}(x_i)),$$

$j = 1, 2, \cdots, k$ and $i = 1, 2, \cdots, j$.
4. Input the different q-ROFSs.
5. Determine the q-ROFTs using the formula:

$$\tau^{h(\zeta_i)} \cap \zeta_i^{h(\zeta_i)} \neq \emptyset, \quad \text{for all } \zeta_i \in \zeta.$$

6. Find the most alluring residential area having maximum truth-membership and minimum falsity-membership degrees as obtained in Step 3.
7. Find the more advantageous schemes, satisfying the relation of minimal transversals and that will contain the attributes of all other societies.

Table 5. Description of hyperedges.

Set of Hyperedges	Corresponding Housing Scheme	Provision of Facilities	Lack of Facilities
ζ_1	Senate Avenue	70%	20%
ζ_2	Soan Gardens	50%	50%
ζ_3	CBRTown	60%	70%
ζ_4	OPFHousing Scheme	80%	50%
ζ_5	Paradise City	60%	70%
ζ_6	RP Corporation	80%	50%
ζ_7	Tele Gardens Housing Scheme	70%	50%

Table 6. Description of attributes.

Set of Attributes	Depicting the Facility	Provision Level of the Corresponding Facility	Lack of the Corresponding Facility
x_1	Basic amenities of life	0.8	0.2
x_2	Easily Accessible	0.9	0.1
x_3	Land ownership	0.7	0.2
x_4	Power back-up	0.6	0.3
x_5	Eco-friendly construction	0.9	0.4
x_6	Social infrastructure	0.8	0.5
x_7	Drainage system	0.5	0.6
x_8	Security	0.6	0.7
x_9	Regular sanitation	0.8	0.5
x_{10}	Parking area	0.9	0.3

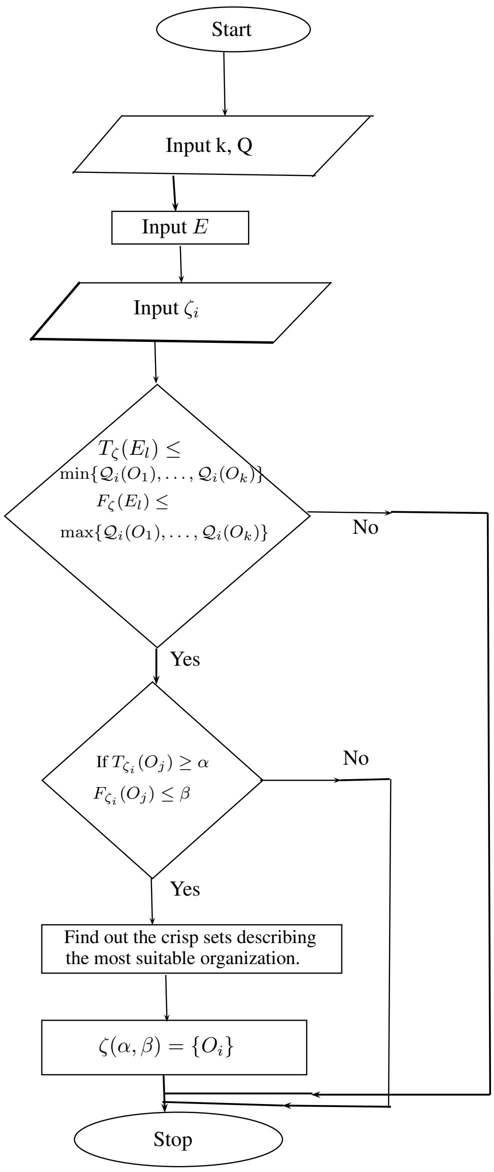

Figure 7. The flowchart of application.

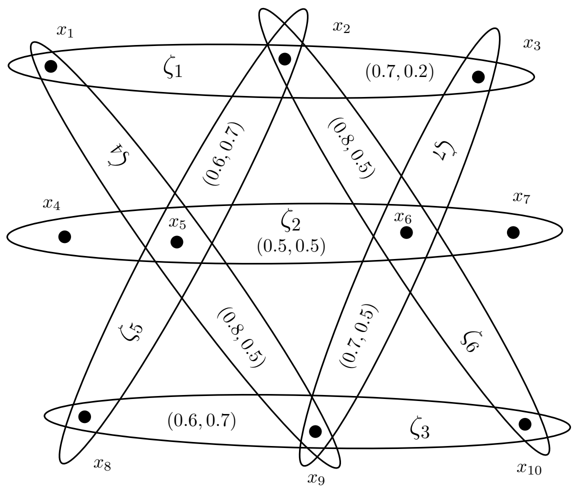

Figure 8. Seven-ROFH model for the housing schemes under consideration and their attributes.

In order to determine the overall comfort of each society, we will calculate the heights of all hyperedges, and the society having the maximum truth-membership and minimum falsity-membership will be considered as the most comfortable society in which to live. The calculated heights of all schemes are given in Table 7.

It can be noted from Table 7 that there are three societies that have the maximum membership and minimum non-membership degrees, i.e., Senate Avenue, Paradise City, and RP Corporation are those housing societies that will provide 90% facilities to their habitants, and only 10% amenities will be lacking. Thus, it is more beneficial and substantial to select one of these three housing schemes.

Table 7. Heights of hyperedges.

Heights of Hyperedges	$(\max(\zeta_l), \min(\zeta_m))$
h(Senate Avenue)	$(0.9, 0.1)$
h(Soan Gardens)	$(0.9, 0.3)$
h(CBR Town)	$(0.9, 0.3)$
h(OPF Housing Scheme)	$(0.9, 0.2)$
h(Paradise City)	$(0.9, 0.1)$
h(RP Corporation)	$(0.9, 0.1)$
h(Tele Gardens Housing Scheme)	$(0.8, 0.2)$

The same problem can be extrapolated to a more extended idea of someone wanting to build a new housing scheme, which will provide the facilities of all the above societies. The concept of seven-ROFHs can be utilized to extrapolate such housing scheme. Consider a seven-rung orthopair fuzzy set of vertices given as follows:

$$\tau_1 = \{(x_1, 0.8, 0.2), (x_2, 0.9, 0.1), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_{10}, 0.9, 0.3)\}.$$

By applying the definition of seven-ROFT, it can be seen that:

$$\begin{array}{rlrl}
\zeta_1^{(0.9,0.1)} \cap \tau_1^{(0.9,0.1)} = \{x_2\}, & & \zeta_2^{(0.9,0.3)} \cap \tau_1^{(0.9,0.3)} & = \{x_5\}, \\
\zeta_3^{(0.9,0.3)} \cap \tau_1^{(0.9,0.3)} = \{x_{10}\}, & & \zeta_4^{(0.9,0.2)} \cap \tau_1^{(0.9,0.2)} & = \{x_5\}, \\
\zeta_5^{(0.9,0.1)} \cap \tau_1^{(0.9,0.1)} = \{x_2\}, & & \zeta_6^{(0.9,0.1)} \cap \tau_1^{(0.9,0.1)} & = \{x_2\}, \\
\zeta_7^{(0.8,0.2)} \cap \tau_1^{(0.8,0.2)} = \{x_6\}. & & &
\end{array}$$

That is the q-rung orthopair fuzzy subset τ_1 satisfies the condition of the transversal, and the housing society that will be represented through this hyperedge will contain at least one attribute of each scheme mentioned above. Similarly, some other societies can be figured out by following the same method. Hence, some other seven-rung orthopair fuzzy subsets are:

$$\begin{aligned}
\tau_2 &= \{(x_1, 0.8, 0.2), (x_2, 0.9, 0.1), (x_3, 0.7, 0.2), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_{10}, 0.9, 0.3)\}, \\
\tau_3 &= \{(x_2, 0.9, 0.1), (x_4, 0.6, 0.3), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_{10}, 0.9, 0.3)\}, \\
\tau_4 &= \{(x_2, 0.9, 0.1), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_{10}, 0.9, 0.3)\}, \\
\tau_5 &= \{(x_2, 0.9, 0.1), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_7, 0.5, 0.5), (x_8, 0.6, 0.7), (x_{10}, 0.9, 0.3)\}.
\end{aligned}$$

The graphical description of these schemes is displayed in Figure 9 with the dashed lines.

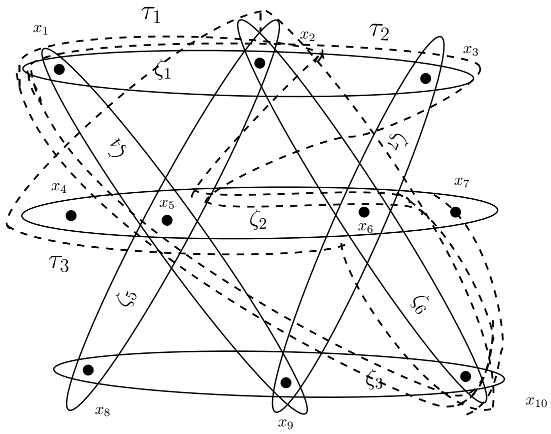

Figure 9. Seven-ROFTs.

Thus, the schemes shown through dashed lines will contain the attributes of all other societies and may be more advantageous to their dwellers.

5. Comparison Analysis of the Proposed Model with IF and PF Models

Orthopair fuzzy sets are defined as those fuzzy sets in which the membership degrees of an element are taken as the pair of values in the unit interval $[0,1]$, given as $(T(x), F(x))$. $T(x)$ indicates support for membership (truth-membership), and $F(x)$ indicates support against membership (falsity-membership) to the fuzzy set. IFSs and PFSs are examples of orthopair fuzzy sets. Atanassov's [2] IFS has been studied widely by various researchers, but the range of applicability of IFS is limited because of its constraint that the sum of truth-membership and falsity-membership must be equal to or less than one. Under this condition, IFSs cannot express some decision evaluation information effectively; because a decision-maker may provide information for a particular attribute such that the sum of the degrees of truth-membership and the degrees of falsity-membership become greater than one. In order to solve such types of problems, PFSs were defined by Yager [5], whose prominent characteristic is that the square sum of the truth-membership degree and the falsity-membership degree is less than or equal to one. Thus, a PFS can solve a number of practical problems that cannot be handled using IFS and is a generalization of IFS. Due to the more complicated information in society and the development of theories, q-ROFSs were proposed by Yager [8]. A q-ROFS is characterized in such a way that the sum of the q^{th} power of the truth-membership degree and the q^{th} power of the degrees of falsity-membership is restricted to less than or equal to one. Note that IFSs and PFSs are particular cases of q-ROFSs. The flexibility and the effectiveness of a q-ROF model can be proven as follows. Suppose that (x, y) is an IF grade, where $x \in [0,1]$, $y \in [0,1]$, and $0 \leq x + y \leq 1$, since $x^q \leq x$, $y^q \leq y$, $q \geq 1$, so we have $0 \leq x^q + y^q \leq 1$. Thus, every IF grade is also a PF grade, as well as a q-ROF grade. However, there are q-ROF grades that are not IF nor PF grades. For example, $(0.9, 0.8)$, here $(0.9)^5 + (0.8)^5 \leq 1$, but $0.9 + 0.8 = 1.7 > 1$ and $(0.9)^2 + (0.8)^2 = 1.45 > 1$. This implies that the class of q-ROFSs extends the classes of IFSs and PFSs. It is worth noting that as the parameter q increases, the space of acceptable orthopairs also increases, and thus, the bounding constraint is satisfied by more orthopairs. Thus, a wider range of uncertain information can be expressed by using q-ROFSs. We can adjust the value of the parameter q to determine the expressed information range; thus, q-ROFSs are more effective and more practical for the uncertain environment. Based on these advantages of q-ROFSs, we proposed q-ROFHs to combine the benefits of both theories. A wider range of uncertain information can be expressed using the methods proposed in this paper, and they are closer to real decision-making. Our proposed models are more general as compared to the IF and PF models, as when $q = 1$, the model reduces to the IF model, and when $q = 2$, it reduces to the PF model. Hence, our approach is more flexible and generalized, and different values of q can be chosen by decision-makers according to the different attitudes.

6. Conclusions

A q-ROF model is an extension of the IF and PF models. This model deals with real-life phenomena more precisely and efficiently. Since q-ROFSs are based on a parameter q, as the parameter q increases, the space of acceptable orthopairs also increases, and thus, the bounding constraint is satisfied by more orthopairs. Thus, q-ROFSs can express the vague information more widely and can determine a larger range for the boundary. However, the IFSs and PFSs are all good ways to deal with fuzzy information, but q-ROFSs are more general as compared to these classical models, because when $q = 1$, the model reduces to the IF model, and when $q = 2$, it reduces to the PF model. In this research article, we have applied the more generalized and powerful concept of q-ROFSs to the most productive theory of hypergraphs. After a concise review of q-ROF theory and crisp hypergraphs, we have described the novel concept of q-ROFHs and some of their properties, including height, size, elementary and sectionally elementary, and $\mathcal{B}$-tempered and transversals. We have illustrated some interesting applications of q-ROFHs in decision-making to explain the flexibility of the model when the given data possess uncertain behavior and compared our proposed models to other existing theories. We aim to broaden our study to (1) q-rung orthopair fuzzy directed hypergraphs, (2) interval valued bipolar neutrosophic hypergraphs, (3) fuzzy rough soft directed hypergraphs, and (4) fuzzy rough neutrosophic hypergraphs.

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