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Article

On a Coupled System of Fractional Differential Equations with Four Point Integral Boundary Conditions

Nazim I Mahmudov *, Sameer Bawaneh and Areen Al-Khateeb

Eastern Mediterranean University, Gazimagusa, T.R. North Cyprus, 99628 Mersin 10, Turkey; sameer.b@outlook.com (S.B.); khteb1987@live.com (A.A.-K.)

* Correspondence: nazim.mahmudov@emu.edu.tr

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Abstract: The study is on the existence of the solution for a coupled system of fractional differential equations with integral boundary conditions. The first result will address the existence and uniqueness of solutions for the proposed problem and it is based on the contraction mapping principle. Secondly, by using Leray–Schauder's alternative we manage to prove the existence of solutions. Finally, the conclusion is confirmed and supported by examples.

Keywords: fractional calculus; Caputo derivative; fractional differential equations

1. Introduction

Fractional calculus and more specifically coupled fractional differential equations are amongst the strongest tools of modern mathematics as they play a key role in developing differential models for high complexity systems. Examples include the quantum evolution of complex systems [1], dynamical systems of distributed order [2], chuashirku [3], Duffing system [4], Lorentz system [5], anomalous diffusion [6,7], nonlocal thermoelasticity systems [8,9], secure communication and control processing [10], synchronization of coupled chaotic systems of fractional order [11–14], etc. In terms of developing high complexity models, applications of coupled fractional differential equations can be significantly extended by dealing with various types of integral boundary conditions. Integral boundary conditions are in fact essential for obtaining reliable models in many practical problems, such as regularization of parabolic inverse problems [15] and flow analysis in computational fluid dynamics [16].

Some of the latest studies on integral and nonlocal boundary value problems for coupled fractional differential equations are presented in [17–25].

In [26], the following coupled system of fractional differential equations was studied:

$$\begin{cases} D^{\alpha}x\left(t\right)=f\left(t,x\left(t\right),y\left(t\right)D^{\gamma}y\left(t\right)\right), \quad t\in\left[0,T\right], \\ \qquad 1<\alpha\leq 2, \quad 0<\gamma<1, \\ D^{\beta}y\left(t\right)=g\left(t,x\left(t\right),D^{\delta}x\left(t\right),y\left(t\right)\right), \quad t\in\left[0,T\right], \\ \qquad 1<\beta\leq 2, \quad 0<\delta<1, \end{cases}$$

supplemented with the coupled nonlocal and integral boundary conditions of the form

$$\begin{cases} x\left(0\right)=h\left(y\right), \quad \int_0^T y\left(s\right)ds=\mu_1 x\left(\eta\right) \\ y\left(0\right)=\phi\left(x\right), \quad \int_0^T x\left(s\right)ds=\mu_2 y\left(\xi\right), \ \eta,\xi\in\left(0,T\right) \end{cases}$$

where D^i denotes the Caputo fractional derivatives of order $i = \alpha, \beta, \gamma, \delta$ and $f, g : [0, T] \times R \times R \times R \to R,\ h, \phi : C([0, T], R) \to R$ are given continuous functions, and $\mu_{1,}\mu_2$are real constants.
In [27], the authors investigated the existence and uniqueness of solutions for the coupled system of nonlinear fractional differential equations with three-point boundary conditions, given below:

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{p}v(t)), & t \in (0,1), \\ D^{\beta}v(t) = g(t, u(t), D^{q}u(t)), & t \in (0,1), \\ u(0) = 0,\ u(1) = \gamma u(\eta),\ v(0) = 0,\ v(1) = \gamma v(\eta), \end{cases} .$$

where $1 < \alpha, \beta < 2,\ p, q, \gamma > 0,\ 0 < \eta < 1,\ \alpha - q \geq 1,\ \beta - p \geq 1,\ \gamma\eta^{\alpha-1} < 1,\ \gamma\eta^{\beta-1} < 1$,and D is the standard Riemann–Liouville fractional derivative and $f, g : [0, 1] \times R \times R \to \times R$ are given continuous functions. It is worth mentioning that the nonlinear terms in the coupled system contain the fractional derivatives of the unknown functions.

Moreover, in a study [28], the following coupled system of nonlinear fractional differential equations, with the given boundary conditions was studied:

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)), & 0 < t < 1, \\ D^{\beta}v(t) = g(t, u(t), D^{\nu}u(t)), & 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases}$$

where $1 < \alpha, \beta < 2, \mu, \nu > 0, \alpha - \nu \geq 1, \beta - \mu \geq 1$, and $f, g : [0, 1] \times R \times R \to R$ are given functions and D is the standard Riemann–Liouville differentiation.

The present paper is motivated by the above papers and is aimed to study a coupled system of nonlinear fractional differential equations:

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t), y(t), D^{\gamma}y(t)), & t \in [0, T] \\ \quad 1 < \alpha \leq 2, \quad 0 < \gamma < 1 \\ D^{\beta}y(t) = g(t, x(t), D^{\sigma}x(t), y(t)), & t \in [0, T] \\ \quad 1 < \beta \leq 2, \quad 0 < \sigma < 1 \end{cases} \tag{1}$$

supported by integral boundary conditions of the form

$$\begin{cases} \int_0^T x(s)\,ds = \rho_1 y(\zeta_1), \int_0^T x'(s)\,ds = \rho_2 y'(\zeta_2) \\ \int_0^T y(s)\,ds = \mu_1 x(\eta_1), \int_0^T y'(s)\,ds = \mu_2 x'(\eta_2) \\ \eta_1, \eta_2, \zeta_1, \zeta_2 \in [0, T], \end{cases} , \tag{2}$$

where D^k denote the Caputo fractional derivatives of order k,and $f, g : [0, T] \times R^3 \to R$, are given continuous functions, and $\rho_1, \rho_2, \mu_1, \mu_2$ are real constants.

The paper is organized as follows. In Section 2, we recall some definitions from fractional calculus, and state and prove an auxiliary lemma, which gives an explicit formula for a solution of nonhomogeneous equation correspond to our problem. The main results for the coupled system of Caputo fractional differential equations with integral boundary conditions, using the Banach fixed point theorem and Leray-Schauder alternative, are presented in Section 3. The paper concludes with concrete examples.

2. Preliminaries

Firstly, we recall definitions of fractional derivative and integral [29,30].

Definition 1. *The Riemann-Liouville fractional integral of order α for a continuous function h is given by*

$$(I_s^{\alpha}h)(s) = \frac{1}{\Gamma(\alpha)} \int_0^s \frac{h(t)}{(s-t)^{1-\alpha}} dt, \quad \alpha > 0$$

provided that the integral exists on R^+.

Definition 2. *The Caputo fractional derivatives of order* α *for* $(m-1)$ *−times absolutely continuous function* $h:[0,\infty)\to R$ *is defined as*

$$(D^\alpha h)(s) = \frac{1}{\Gamma(m-\alpha)}\int_0^s (s-t)^{m-\alpha-1} h^{(m)}(t)\,dt, \qquad m-1<\alpha<m, \quad m=[\alpha]+1,$$

where $[\alpha]$ *is the integer part of the real number* α.

We use the following notations.

$$\Delta_1 = T^2-\mu_1\rho_1 \neq 0, \;\; \Delta_2 = T^2-\mu_2\rho_2 \neq 0,$$

$$\begin{aligned}
\Theta_1(t) &:= \frac{2T\rho_1\zeta_1\mu_2\rho_2 - T^4\rho_2 + 2T\rho_1\mu_1\eta_1\rho_2 - T^2\mu_2\rho_1\rho_2}{\Delta_1\Delta_2} + \frac{\rho_2 Tt}{\Delta_2}.\\
\Theta_2(t) &:= \frac{-2T^2\rho_1\zeta_1 + T^3\rho_2 - 2\rho_1\mu_1\eta_1\rho_2 + \rho_1 T^3}{\Delta_1\Delta_2} - \frac{\rho_2 t}{\Delta_2},\\
\Theta_3 &:= \frac{T\rho_1}{\Delta_1}, \;\; \Theta_4 := -\frac{\rho_1}{\Delta_1}.
\end{aligned}$$

$$\begin{aligned}
\Xi_1(t) &:= \frac{2T^2\rho_1\zeta_1\mu_2 - T^3\rho_2\mu_2 + 2\rho_1\mu_1\eta_1\rho_2\mu_2 - \rho_1\mu_2 T^3}{\Delta_1\Delta_2} + \frac{t\mu_2\rho_2}{\Delta_2}\\
\Xi_2(t) &:= \frac{-2T\rho_1\zeta_1\mu_2 + T^4 - 2T\rho_1\mu_1\eta_1 + T^2\rho_1\mu_2}{\Delta_1\Delta_2} - \frac{Tt}{\Delta_2},\\
\Xi_3 &:= \frac{\rho_1\mu_1}{\Delta_1}, \quad \Xi_4 := -\frac{T}{\Delta_1}.
\end{aligned}$$

$$\begin{aligned}
\widehat{\Theta}_1(t) &:= \frac{1}{\Delta_1}\left(\rho_2 T\left(T\mu_1\eta_1\frac{1}{\Delta_2} - \mu_1\frac{T^2}{2}\frac{1}{\Delta_2}\right) + \mu_2\rho_2\left(\mu_1\rho_1\zeta_1\frac{1}{\Delta_2} - \frac{T^3}{2}\frac{1}{\Delta_2}\right)\right) + \frac{1}{\Delta_2}\mu_2\rho_2 t.\\
\widehat{\Theta}_2(t) &:= \frac{1}{\Delta_1}\left(-\rho_2\left(T\mu_1\eta_1\frac{1}{\Delta_2} - \mu_1\frac{T^2}{2}\frac{1}{\Delta_2}\right) - T\left(\mu_1\rho_1\zeta_1\frac{1}{\Delta_2} - \frac{T^3}{2}\frac{1}{\Delta_2}\right)\right) - \frac{1}{\Delta_2}Tt,\\
\widehat{\Theta}_3 &:= \frac{1}{\Delta_1}\rho_1\mu_1, \;\; \widehat{\Theta}_3 := \frac{-T}{\Delta_1}.
\end{aligned}$$

$$\begin{aligned}
\widehat{\Xi}_1(t) &:= \frac{1}{\Delta_1}\left(\mu_2\rho_2\left(T\mu_1\eta_1\frac{1}{\Delta_2} - \mu_1\frac{T^2}{2}\frac{1}{\Delta_2}\right) + \mu_2 T\left(\mu_1\rho_1\zeta_1\frac{1}{\Delta_2} - \frac{T^3}{2}\frac{1}{\Delta_2}\right)\right) + \frac{1}{\Delta_2}\mu_2 Tt\\
\widehat{\Xi}_2(t) &:= \frac{1}{\Delta_1}\left(-T\left(T\mu_1\eta_1\frac{1}{\Delta_2} - \mu_1\frac{T^2}{2}\frac{1}{\Delta_2}\right) - \mu_2\left(\mu_1\rho_1\zeta_1\frac{1}{\Delta_2} - \frac{T^3}{2}\frac{1}{\Delta_2}\right)\right) - \frac{1}{\Delta_2}\mu_2 t,\\
\widehat{\Xi}_3 &:= \frac{\mu_1 T}{\Delta_1}, \quad \widehat{\Xi}_4 := \frac{-\mu_1}{\Delta_1}.
\end{aligned}$$

To show that the problem (1) and (2) is equivalent to the problem of finding solutions to the Volterra integral equation, we need the following auxiliary lemma

Lemma 1. *Let $w, z \in C([0,T], R)$. Then the unique solution for the problem*

$$\begin{cases} D^{\alpha}x(t) = w(t), \quad t \in [0,T], \qquad 1 < \alpha \leq 2, \\ D^{\beta}y(t) = z(t), \quad t \in [0,T], \qquad 1 < \beta \leq 2, \\ \int_0^T x(s)\,ds = \rho_1 y(\zeta_1), \quad \int_0^T x'(s)\,ds = \rho_2 y'(\zeta_2) \\ \int_0^T y(s)\,ds = \mu_1 x(\eta_1), \quad \int_0^T y'(s)\,ds = \mu_2 x'(\eta_2) \end{cases} \tag{3}$$

is

$$\begin{aligned} x(t) = {} & \Theta_1(t)\left(I^{\beta-1}z\right)(\zeta_2) + \Theta_2(t)\int_0^T \left(I^{\beta-1}z\right)(s)\,ds + \Theta_3\left(I^{\beta}z\right)(\zeta_1) - \Theta_4\int_0^T \left(I^{\beta}z\right)(s)\,ds \\ & + \Xi_1(t)\left(I^{\alpha-1}w\right)(\eta_2) + \Xi_2(t)\int_0^T \left(I^{\alpha-1}w\right)(s)\,ds + \Xi_3(I^{\alpha}w)(\eta_1) - \Xi_4\int_0^T (I^{\alpha}w)(s)\,ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}w(s)\,ds \end{aligned} \tag{4}$$

and

$$\begin{aligned} y(t) = {} & \widehat{\Theta}_1(t)\left(I^{\beta-1}z\right)(\zeta_2) + \widehat{\Theta}_2(t)\int_0^T \left(I^{\beta-1}z\right)(s)\,ds + \widehat{\Theta}_3\left(I^{\beta}z\right)(\zeta_1) - \widehat{\Theta}_4\int_0^T \left(I^{\beta}z\right)(s)\,ds \\ & + \widehat{\Xi}_1(t)\left(I^{\alpha-1}w\right)(\eta_2) + \widehat{\Xi}_2(t)\int_0^T \left(I^{\alpha-1}w\right)(s)\,ds + \widehat{\Xi}_3(I^{\alpha}w)(\eta_1) - \widehat{\Xi}_4\int_0^T (I^{\alpha}w)(s)\,ds \\ & + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}z(s)\,ds \end{aligned} \tag{5}$$

Proof. We know that, see [30] Lemma 2.12, the general solutions for the FDE in (3) is defined as

$$\begin{aligned} x(t) &= c_1 t + c_2 + (I^{\alpha}w)(t) \\ y(t) &= d_1 t + d_2 + \left(I^{\beta}z\right)(t), \end{aligned} \tag{6}$$

where $c_1, c_2, d_1, d_2 \in R$ are arbitrary constants. It follows that

$$\begin{aligned} x'(t) &= c_1 + \left(I^{\alpha-1}w\right)(t), \\ y'(t) &= d_1 + \left(I^{\beta-1}z\right)(t). \end{aligned}$$

Applying the conditions

$$\int_0^T x'(s)\,ds = \rho_2 y'(\zeta_2), \int_0^T y'(s)\,ds = \mu_2 x'(\eta_2)$$

we get

$$\begin{aligned} c_1 T + \int_0^T \left(I^{\alpha-1}w\right)(s)\,ds &= \rho_2 d_1 + \rho_2\left(I^{\beta-1}z\right)(\zeta_2), \\ d_1 T + \int_0^T \left(I^{\beta-1}z\right)(s)\,ds &= \mu_2 c_1 + \mu_2\left(I^{\alpha-1}w\right)(\eta_2). \end{aligned}$$

Solving the above equations together for c_1 and d_1 we get

$$c_1 = \frac{1}{\Delta_2}\left(\rho_2 T\left(I^{\beta-1}z\right)(\zeta_2) - \rho_2\int_0^T \left(I^{\beta-1}z\right)(s)\,ds + \mu_2\rho_2\left(I^{\alpha-1}w\right)(\eta_2) - T\int_0^T \left(I^{\alpha-1}w\right)(s)\,ds\right)$$

$$d_1 = \frac{1}{\Delta_2}\left(\mu_2 T\left(I^{\alpha-1}w\right)(\eta_2) - \mu_2\int_0^T \left(I^{\alpha-1}w\right)(s)\,ds + \mu_2\rho_2\left(I^{\beta-1}z\right)(\zeta_2) - T\int_0^T \left(I^{\beta-1}z\right)(s)\,ds\right)$$

Considering the following boundary conditions not involving derivatives

$$\int_0^T x(s)\,ds = \rho_1 y(\zeta_1), \quad \int_0^T y(s)\,ds = \mu_1 x(\eta_1),$$

we get

$$c_2 T - \rho_1 d_2 = \rho_1 d_1 \zeta_1 + \rho_1 \left(I^{\beta} z\right)(\zeta_1) - c_1 \frac{T^2}{2} - \int_0^T (I^{\alpha} w)(s)\,ds,$$
$$d_2 T - \mu_1 c_2 = \mu_1 c_1 \eta_1 + \mu_1 (I^{\alpha} w)(\eta_1) - d_1 \frac{T^2}{2} - \int_0^T \left(I^{\beta} z\right)(s)\,ds.$$

This implies

$$c_2 = \frac{1}{\Delta_1}\left(T\rho_1 d_1 \zeta_1 + \rho_1 T \left(I^{\beta} z\right)(\zeta_1) - c_1 \frac{T^3}{2} - T\int_0^T (I^{\alpha} w)(s)\,ds \right.$$
$$\left. + \rho_1 \mu_1 c_1 \eta_1 + \rho_1 \mu_1 (I^{\alpha} w)(\eta_1) - \rho_1 d_1 \frac{T^2}{2} - \rho_1 \int_0^T \left(I^{\beta} z\right)(s)\,ds \right),$$

$$d_2 = \frac{1}{\Delta_1}\left(T\mu_1 c_1 \eta_1 + \mu_1 T (I^{\alpha} w)(\eta_1) - d_1 \frac{T^3}{2} - T\int_0^T \left(I^{\beta} z\right)(s)\,ds + \mu_1 \rho_1 d_1 \zeta_1 \right.$$
$$\left. + \rho_1 \mu_1 \left(I^{\beta} z\right)(\zeta_1) - c_1 \mu_1 \frac{T^2}{2} - \mu_1 \int_0^T (I^{\alpha} w)(s)\,ds \right).$$

Inserting the values of c_1 and d_1 we get

$$\begin{aligned} c_2 = {} & \frac{2T\rho_1\zeta_1\mu_2\rho_2 - T^4\rho_2 + 2T\rho_1\mu_1\eta_1\rho_2 - T^2\mu_2\rho_1\rho_2}{2\Delta_1\Delta_2}\left(I^{\beta-1} z\right)(\zeta_2) \\ & + \frac{-2T^2\rho_1\zeta_1 + T^3\rho_2 - 2\rho_1\mu_1\eta_1\rho_2 + \rho_1 T^3}{2\Delta_1\Delta_2}\int_0^T \left(I^{\beta-1} z\right)(s)\,ds \\ & + \frac{T\rho_1}{\Delta_1}\left(I^{\beta} z\right)(\zeta_1) - \frac{\rho_1}{\Delta_1}\int_0^T \left(I^{\beta} z\right)(s)\,ds \\ & + \frac{2T^2\rho_1\zeta_1\mu_2 - T^3\rho_2\mu_2 + 2\rho_1\mu_1\eta_1\rho_2\mu_2 - \rho_1\mu_2 T^3}{2\Delta_1\Delta_2}\left(I^{\alpha-1} w\right)(\eta_2) \\ & + \frac{-2T\rho_1\zeta_1\mu_2 + T^4 - 2T\rho_1\mu_1\eta_1 + T^2\rho_1\mu_2}{2\Delta_1\Delta_2}\int_0^T \left(I^{\alpha-1} w\right)(s)\,ds \\ & + \frac{\rho_1\mu_1}{\Delta_1}(I^{\alpha} w)(\eta_1) - \frac{T}{\Delta_1}\int_0^T (I^{\alpha} w)(s)\,ds, \end{aligned}$$

$$
\begin{aligned}
d_2 = &\frac{2\rho_2 T^2\mu_1\eta_1 - \rho_2\mu_1 T^3 + 2\mu_2\rho_2\mu_1\rho_1\zeta_1 - \mu_2\rho_2 T^3}{2\Delta_1\Delta_2}\left(I^{\beta-1}z\right)(\zeta_2) \\
&+ \frac{-2T\rho_2\mu_1\eta_1 + \rho_2\mu_1 T^2 - 2T\mu_1\rho_1\zeta_1 - T^4}{2\Delta_1\Delta_2}\int_0^T \left(I^{\beta-1}z\right)(s)\,ds \\
&+ \frac{1}{\Delta_1}\rho_1\mu_1\left(I^{\beta}z\right)(\zeta_1) - \frac{1}{\Delta_1}T\int_0^T \left(I^{\beta}z\right)(s)\,ds \\
&+ \frac{2T\mu_2\rho_2\mu_1\eta_1 - \mu_2\rho_2\mu_1 T^2 + 2\mu_2 T\mu_1\rho_1\zeta_1 - \mu_2 T^4}{2\Delta_1\Delta_2}\left(I^{\alpha-1}w\right)(\eta_2) \\
&+ \frac{-2T^2\mu_1\eta_1 + \mu_1 T^3 - 2\mu_2\rho_2\mu_1\rho_1\zeta_1 + \mu_2 T^3}{2\Delta_1\Delta_2}\int_0^T \left(I^{\alpha-1}w\right)(s)\,ds \\
&+ \mu_1 T\frac{1}{\Delta_1}\left(I^{\alpha}w\right)(\eta_1) - \mu_1\frac{1}{\Delta_1}\int_0^T \left(I^{\alpha}w\right)(s)\,ds.
\end{aligned}
$$

Substituting c_1, c_2, d_1, d_2 in (6) we get (4) and (5). □

Remark 1. *In (4) and (5) $x(t)$ and $y(t)$ depend on η_i, ζ_i, μ_i, ρ_i, $i = 1,2$.*

3. Existence Results

Consider the space

$$C_\gamma([0,T],R) = \{x(t) : x(t) \in C([0,T],R) \text{ and } D^\gamma x(t) \in C([0,T],R)\},$$

with the norm

$$\|x\|_\gamma = \|x\| + \|D^\gamma x\| = \max_{0\le t\le T}|x(t)| + \max_{0\le t\le T}|D^\gamma x(t)|.$$

It is clear that $\left(C_\gamma([0,T],R), \|\cdot\|_\gamma\right)$ is a Banach space. Consequently, the product space $\left(C_\sigma([0,T],R) \times C_\gamma([0,T],R), \|\cdot\|_{\sigma\times\gamma}\right)$ is a Banach Space with the norm $\|(x,y)\|_{\sigma\times\gamma} = \|x\|_\sigma + \|y\|_\gamma$ for $(x,y) \in C_\sigma([0,T],R) \times C_\gamma([0,T],R)$.

Next, using Lemma 1, we define the operator $G : C_\sigma([0,T],R) \times C_\gamma([0,T],R) \to C_\sigma([0,T],R) \times C_\gamma([0,T],R)$ as follows

$$G(x,y)(t) = (G_1(x,y)(t), G_2(x,y)(t)),$$

where

$$
\begin{aligned}
G_1(x,y)(t) = &\,\Theta_1(t)\left(I^{\beta-1}g(\cdot,x(\cdot),y(\cdot),D^\sigma x(\cdot))\right)(\zeta_2) + \Theta_2(t)\int_0^T\left(I^{\beta-1}g(\cdot,x(\cdot),y(\cdot),D^\sigma x(\cdot))\right)(s)\,ds \\
&+ \Theta_3\left(I^{\beta}g(\cdot,x(\cdot),y(\cdot),D^\sigma x(\cdot))\right)(\zeta_1) - \Theta_4\int_0^T\left(I^{\beta}g(\cdot,x(\cdot),y(\cdot),D^\sigma x(\cdot))\right)(s)\,ds \\
&+ \Xi_1(t)\left(I^{\alpha-1}f(\cdot,x(\cdot),y(\cdot),D^\gamma y(\cdot))\right)(\eta_2) + \Xi_2(t)\int_0^T\left(I^{\alpha-1}f(\cdot,x(\cdot),y(\cdot),D^\gamma y(\cdot))\right)(s)\,ds \\
&+ \Xi_3\left(I^{\alpha}f(\cdot,x(\cdot),y(\cdot),D^\gamma y(\cdot))\right)(\eta_1) - \Xi_4\int_0^T\left(I^{\alpha}f(\cdot,x(\cdot),y(\cdot),D^\gamma y(\cdot))\right)(s)\,ds \\
&+ \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s),y(s),D^\gamma y(s))\,ds,
\end{aligned}
$$

and

$$
\begin{aligned}
G_2(x,y)(t) = {} & \widehat{\Theta}_1(t)\left(I^{\beta-1}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(\zeta_2)+\widehat{\Theta}_2(t)\int_0^T\left(I^{\beta-1}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(s)\,ds \\
& +\widehat{\Theta}_3\left(I^{\beta}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(\zeta_1)-\widehat{\Theta}_4\int_0^T\left(I^{\beta}g(\cdot,x(\cdot),y(\cdot),D^{\sigma}x(\cdot))\right)(s)\,ds \\
& +\widehat{\Xi}_1(t)\left(I^{\alpha-1}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(\eta_2)+\widehat{\Xi}_2(t)\int_0^T\left(I^{\alpha-1}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(s)\,ds \\
& +\widehat{\Xi}_3\left(I^{\alpha}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(\eta_1)-\widehat{\Xi}_4\int_0^T\left(I^{\alpha}f(\cdot,x(\cdot),y(\cdot),D^{\gamma}y(\cdot))\right)(s)\,ds \\
& +\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s,x(s),y(s),D^{\sigma}x(s))\,ds.
\end{aligned}
$$

In what follows we use the following notations.

$$
\begin{aligned}
\Theta = {} & \|\Theta_1\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}+\|\Theta_2\|\frac{T^{\beta-1}}{\Gamma(\beta)}+|\Theta_3|\frac{\zeta_1^{\beta}}{\Gamma(\beta+1)}+|\Theta_4|\frac{T^{\beta}}{\Gamma(\beta+1)}+\frac{T^{1-\sigma}}{\Gamma(2-\sigma)}\left(\|\Theta_1'\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}+\|\Theta_2'\|\frac{T^{\beta-1}}{\Gamma(\beta)}\right), \\
\Xi = {} & \|\Xi_1\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)}+\|\Xi_2\|\frac{T^{\alpha-1}}{\Gamma(\alpha)}+|\Xi_3|\frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)}+|\Xi_4|\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\alpha}}{\Gamma(\alpha+1)} \\
& +\frac{T^{1-\sigma}}{\Gamma(2-\sigma)}\left(\|\Xi_1'\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)}+\|\Xi_2'\|\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\right).
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\Theta} = {} & \left\|\widehat{\Theta}_1\right\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}+\left\|\widehat{\Theta}_2\right\|\frac{T^{\beta-1}}{\Gamma(\beta)}+\left|\widehat{\Theta}_3\right|\frac{\zeta_1^{\beta}}{\Gamma(\beta+1)}+\left|\widehat{\Theta}_4\right|\frac{T^{\beta}}{\Gamma(\beta+1)}+\frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\left(\left\|\widehat{\Theta}_1'\right\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)}+\left\|\widehat{\Theta}_2'\right\|\frac{T^{\beta-1}}{\Gamma(\beta)}\right), \\
\widehat{\Xi} = {} & \left\|\widehat{\Xi}_1\right\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)}+\left\|\widehat{\Xi}_2\right\|\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\left|\widehat{\Xi}_3\right|\frac{\eta_1^{\alpha}}{\Gamma(\alpha+1)}+\left|\widehat{\Xi}_4\right|\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\beta}}{\Gamma(\beta+1)} \\
& +\frac{T^{1-\gamma}}{\Gamma(2-\gamma)}\left(\left\|\widehat{\Xi}_1'\right\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)}+\left\|\widehat{\Xi}_2'\right\|\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{T^{\beta-1}}{\Gamma(\beta)}\right),
\end{aligned}
$$

where $\Theta_i(t), \widehat{\Theta}_i(t), \Xi_i(t), \widehat{\Xi}_i(t), i=1,...,4$, are defined before Lemma 1.

Now we state and prove our first main result.

Theorem 1. *Let $f,g:[0,T]\times R^3\to R$ be jointly continuous functions. Assume that*

(i) there exist constants $l_f>0, l_g>0, \forall t\in[0,T]$ and $x_i,y_i\in R, i=1,2,3$

$$|f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)|\le l_f(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|),$$

$$|g(t,x_1,x_2,x_3)-g(t,y_1,y_2,y_3)|\le l_g(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|).$$

(ii)

$$1-2\left(\Theta l_g+\Xi l_f\right)>0,\ \ 1-2\left(\widehat{\Theta}l_g+\widehat{\Xi}l_f\right)>0.$$

Then the boundary value problem (1), (2) has a unique solution on $[0,T]$.

Proof. Assume that $\varepsilon>0$ is a real number satisfying

$$\varepsilon\ge\max\left(\frac{2(\Theta g_0+\Xi f_0)}{1-2\left(\Theta l_g+\Xi l_f\right)},\frac{2\left(\widehat{\Theta}g_0+\widehat{\Xi}f_0\right)}{1-2\left(\widehat{\Theta}l_g+\widehat{\Xi}l_f\right)}\right),$$

where

$$\max_{0\le t\le T}|f(t,0,0,0)|=f_0<\infty,\ \max_{0\le t\le T}|g(t,0,0,0)|=g_0<\infty.$$

Define

$$\Omega_{\varepsilon}=\left\{(x,y)\in C_{\sigma}\left([0,T],R\right)\times C_{\gamma}\left([0,T],R\right):\|(x,y)\|_{\sigma\times\gamma}\leq\varepsilon\right\}.$$

Step 1: Show that $G\Omega_{\varepsilon}\subset\Omega_{\varepsilon}$.

By our assumption, for $(x,y)\in\Omega_{\varepsilon}, t\in[0,T]$, we have

$$\begin{aligned}|f\left(t,x\left(t\right),y\left(t\right),D^{\gamma}y(t)\right)| &\leq |f\left(t,x\left(t\right),y\left(t\right),D^{\gamma}y(t)\right)-f\left(t,0,0,0\right)|+|f\left(t,0,0,0\right)| \\ &\leq l_f\left(|x(t)|+|y(t)|+|D^{\gamma}y(t)|\right)+f_0 \\ &\leq l_f\left(\|x\|_{\sigma}+\|y\|_{\gamma}\right)+f_0\leq l_f\varepsilon+f_0,\end{aligned} \tag{7}$$

similarly, we have

$$|g\left(t,x\left(t\right),D^{\sigma}x(t),y\left(t\right)\right)|\leq l_g\varepsilon+g_0. \tag{8}$$

Using these estimates, we get

$$\begin{aligned}|G_1\left(x,y\right)(t)|\leq &|\Theta_1\left(t\right)|\left(I^{\beta-1}|g|\right)(\zeta_2)+|\Theta_2\left(t\right)|\int_0^T\left(I^{\beta-1}|g|\right)(s)\,ds \\ &+|\Theta_3|\left(I^{\beta}|g|\right)(\zeta_1)+|\Theta_4|\int_0^T\left(I^{\beta}|g|\right)(s)\,ds \\ &+|\Xi_1\left(t\right)|\left(I^{\alpha-1}|f|\right)(\eta_2)+|\Xi_2\left(t\right)|\int_0^T\left(I^{\alpha-1}|f|\right)(s)\,ds \\ &+|\Xi_3|\left(I^{\alpha}|f|\right)(\eta_1)+|\Xi_4|\int_0^T\left(I^{\alpha}|f|\right)(s)\,ds \\ &+\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-s\right)^{\alpha-1}|f\left(s,x(s),y(s),D^{\gamma}y(s)\right)|\,ds.\end{aligned}$$

We use the following type inequalities

$$\begin{aligned}\left(I^{\beta-1}|g|\right)(\zeta_2)&=\frac{1}{\Gamma\left(\beta-1\right)}\int_0^{\zeta_2}\left(t-s\right)^{\beta-2}|g\left(s\right)|\,ds \\ &\leq\frac{1}{\Gamma\left(\beta-1\right)}\int_0^{\zeta_2}\left(t-s\right)^{\beta-2}ds\,\|g\|=\frac{\zeta_2^{\beta-1}}{\Gamma\left(\beta\right)}\|g\|,\end{aligned}$$

to get

$$\begin{aligned}|G_1\left(x,y\right)(t)|\leq &\left(|\Theta_1\left(t\right)|\left(I^{\beta-1}1\right)(\zeta_2)+|\Theta_2\left(t\right)|\int_0^T\left(I^{\beta-1}1\right)(s)\,ds+|\Theta_3|\left(I^{\beta}1\right)(\zeta_1)+|\Theta_4|\int_0^T\left(I^{\beta}1\right)(s)\,ds\right)\|g\| \\ &+\left(|\Xi_1\left(t\right)|\left(I^{\alpha-1}1\right)(\eta_2)+|\Xi_2\left(t\right)|\int_0^T\left(I^{\alpha-1}1\right)(s)\,ds+|\Xi_3|\left(I^{\alpha}1\right)(\eta_1)+|\Xi_4|\int_0^T\left(I^{\alpha}1\right)(s)\,ds\right)\|f\| \\ &+\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-s\right)^{\alpha-1}ds\,\|f\| \\ &\leq\left(\|\Theta_1\|\frac{\zeta_2^{\beta-1}}{\Gamma\left(\beta\right)}+\|\Theta_2\|\frac{T^{\beta-1}}{\Gamma\left(\beta\right)}+|\Theta_3|\frac{\zeta_1^{\beta}}{\Gamma\left(\beta+1\right)}+|\Theta_4|\frac{T^{\beta}}{\Gamma\left(\beta+1\right)}\right)\|g\| \\ &+\left(\|\Xi_1\|\frac{\eta_2^{\alpha-1}}{\Gamma\left(\alpha\right)}+\|\Xi_2\|\frac{T^{\alpha-1}}{\Gamma\left(\alpha\right)}+|\Xi_3|\frac{\eta_1^{\alpha}}{\Gamma\left(\alpha+1\right)}+|\Xi_4|\frac{T^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)\|f\| \\ &+\frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)}\|f\|.\end{aligned} \tag{9}$$

Hence, by (7) and (8) we have

$$\|G_1\left(x,y\right)\|\leq\left(\Theta l_g+\Xi l_f\right)\varepsilon+\left(\Theta g_0+\Xi f_0\right).$$

On the other hand,

$$\frac{d}{dt}G_1(x,y)(t) = \Theta_1'(t)\left(I^{\beta-1}g\right)(\zeta_2) + \Theta_2'(t)\int_0^T \left(I^{\beta-1}g\right)(s)\,ds$$
$$+\Xi_1'(t)\left(I^{\alpha-1}f\right)(\eta_2) + \Xi_2'(t)\int_0^T \left(I^{\alpha-1}f\right)(s)\,ds$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f(s,x(s),y(s),D^\gamma y(s))\,ds.$$

and

$$\left|\frac{d}{dt}G_1(x,y)(t)\right| \le \left(\|\Theta_1'\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)} + \|\Theta_2'\|\frac{T^{\beta-1}}{\Gamma(\beta)}\right)\|g\|$$
$$+\left(\|\Xi_1'\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)} + \|\Xi_2'\|\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\right)\|f\|.$$

It follows that

$$\begin{aligned}|D^\sigma G_1(x,y)(t)| &\le \int_0^t \frac{(t-s)^{-\sigma}}{\Gamma(1-\sigma)}\left|\frac{d}{ds}G_1(x,y)(s)\right| ds \\ &\le \frac{T^{1-\sigma}}{\Gamma(2-\sigma)}\left(\|\Theta_1'\|\frac{\zeta_2^{\beta-1}}{\Gamma(\beta)} + \|\Theta_2'\|\frac{T^{\beta-1}}{\Gamma(\beta)}\right)\|g\| \\ &+ \frac{T^{1-\sigma}}{\Gamma(2-\sigma)}\left(\|\Xi_1'\|\frac{\eta_2^{\alpha-1}}{\Gamma(\alpha)} + \|\Xi_2'\|\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\right)\|f\|.\end{aligned} \tag{10}$$

Thus by (7)–(10) we obtain

$$\begin{aligned}\|G_1(x,y)\|_\sigma &= \|G_1(x,y)\| + \|D^\sigma G_1(x,y)\| \\ &\le \Theta\|g\| + \Xi\|f\| \\ &\le \left(\Theta l_g + \Xi l_f\right)\varepsilon + (\Theta g_0 + \Xi f_0) \le \frac{\varepsilon}{2}.\end{aligned} \tag{11}$$

In similar way we get

$$\begin{aligned}\|G_2(x,y)\|_\gamma &= \|G_2(x,y)\| + \|D^\gamma G_2(x,y)\| \\ &\le \left(\widehat{\Theta} l_g + \widehat{\Xi} l_f\right)\varepsilon + \left(\widehat{\Theta} g_0 + \widehat{\Xi} f_0\right) \le \frac{\varepsilon}{2}.\end{aligned} \tag{12}$$

From (11) and (12) we get

$$\|G_1(x,y)\|_\sigma + \|G_2(x,y)\|_\gamma \le \varepsilon.$$

Step 2: Show that G ia a contraction.

Now for $x_1, x_2, y_1, y_2 \in \Omega_\varepsilon, \forall t \in [0,T]$ we have

$$\|G_1(x_1,y_1) - G_1(x_2,y_2)\|_\sigma$$
$$\le \left(\Theta l_g + \Xi l_f\right)(\|x_1 - x_2\| + \|y_1 - y_2\| + \|D^\gamma y_1 - D^\gamma y_2\|),$$
$$\|G_2(x_1,y_1) - G_2(x_2,y_2)\|_\gamma$$
$$\le \left(\widehat{\Theta} l_g + \widehat{\Xi} l_f\right)(\|x_1 - x_2\| + \|y_1 - y_2\| + \|D^\sigma x_1 - D^\sigma x_2\|).$$

So we obtain

$$\begin{aligned}&\|(G_1, G_2)(x_1, y_1) - (G_1, G_2)(x_2, y_2)\|_{\sigma\times\gamma} \\ &\leq \left(\left(\Theta l_g + \Xi l_f\right) + \left(\widehat{\Theta} l_g + \widehat{\Xi} l_f\right)\right) \|(x_1, y_1) - (x_2, y_2)\|_{\sigma\times\gamma},\end{aligned}$$

which shows that G is a contraction. So, by the Banach fixed point theorem, the operator (G_1, G_2) has a unique fixed point in Ω_ε. $\square$

The second result is based on the Leray-Schauder alternative. Now we formulate and prove the second existence result.

Theorem 2. *Let $f, g : [0, T] \times R^3 \to R$ be continuous functions. Assume that*

(i) there exist a positive real constants θ_i, λ_i $(i = 0, 1, 2, 3)$ such that $\forall x_i \in R$, $(i = 1, 2, 3)$

$$|f(t, x_1, x_2, x_3)| \leq \theta_0 + \theta_1 |x_1| + \theta_2 |x_2| + \theta_3 |x_3|,$$

$$|g(t, x_1, x_2, x_3)| \leq \lambda_0 + \lambda_1 |x_1| + \lambda_2 |x_2| + \lambda_3 |x_3|.$$

(ii) $\max\{A, B\} < 1$ *where*

$$\begin{aligned}A &= \left(\Theta + \widehat{\Theta}\right)\lambda_1 + \left(\Xi + \widehat{\Xi}\right)\max(\theta_1, \theta_3),\\ B &= \left(\Theta + \widehat{\Theta}\right)\max(\lambda_2, \lambda_3) + \left(\Xi + \widehat{\Xi}\right)\theta_2.\end{aligned}$$

Then there exists at least one solution for the problem (1), (2) on $[0, T]$.

Proof. The proof will be divided into several steps.

Step1: We show that $G : C_\sigma([0, T], R) \times C_\gamma([0, T], R) \to C_\sigma([0, T], R) \times C_\gamma([0, T], R)$ is completely continuous. The continuity of the operator holds true because of continuity of the function f, g.

Let $\Omega \subset C_\sigma([0, T], R) \times C_\gamma([0, T], R)$ be bounded. Then there exist $k_f, k_g > 0$ such that

$$|f(t, x(t), y(t), D^\gamma y(t))| \leq k_f, \quad |g(t, x(t), D^\sigma x(t), y(t))| \leq k_g, \quad \forall (x, y) \in \Omega,$$

also, from (11) it follows that

$$\begin{aligned}\|G_1(x, y)\|_\sigma &= \|G_1(x, y)\| + \|D^\sigma G_1(x, y)\| \\ &\leq \Theta\|g\| + \Xi\|f\| \\ &\leq \Theta k_g + \Xi k_f.\end{aligned} \tag{13}$$

Similarly, we obtain that

$$\begin{aligned}\|G_2(x, y)\|_\gamma &= \|G_2(x, y)\| + \|D^\gamma G_2(x, y)\| \\ &\leq \widehat{\Theta}\|g\| + \widehat{\Xi}\|f\| \\ &\leq \widehat{\Theta} k_g + \widehat{\Xi} k_f.\end{aligned} \tag{14}$$

So, from (13) and (14) we conclude that our operator G is uniformly bounded.

Now, let us show that G is equicontinuous. Consider $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then we have:

$$\begin{aligned}
|G_1(x,y)(t_2) - G_1(x,y)(t_1)| \leq & \ |\Theta_1(t_1) - \Theta_1(t_2)| \left(I^{\beta-1}|g|\right)(\zeta_2) \\
& + |\Theta_2(t_1) - \Theta_2(t_2)| \int_0^T \left(I^{\beta-1}|g|\right)(s)\, ds \\
& + |\Xi_1(t_1) - \Xi_1(t_2)| \left(I^{\alpha-1}|f|\right)(\eta_2) \\
& + |\Xi_2(t_1) - \Xi_2(t_1)| \int_0^T \left(I^{\alpha-1}|f|\right)(s)\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right| |f|\, ds \\
& + + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} |t_2 - s|^{\alpha-1} |f|\, ds
\end{aligned}$$

and

$$\left|G_1(x,y)'(t_2) - G_1(x,y)'(t_1)\right| \leq \frac{k_f}{\Gamma(\alpha)} \left[(t_2-t_1)^{\alpha-1} + \left|{t_2}^{\alpha-1} - {t_1}^{\alpha-1}\right|\right].$$

Thus

$$\begin{aligned}
|D^{\gamma}G_1(x,y)(t_2) - D^{\gamma}G_1(x,y)(t_1)| = & \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} \left|G_1(x,y)'t_2) - G_1(x,y)'(t_1)\right| ds \\
\leq & \frac{T^{1-\gamma}}{\Gamma(2-\gamma)} \frac{k_f}{\Gamma(\alpha)} \left[(t_2-t_1)^{\alpha-1} + \left|{t_2}^{\alpha-1} - {t_1}^{\alpha-1}\right|\right],
\end{aligned}$$

which implies that $\|G_1(x,y)(t_2) - G_1(x,y)(t_1)\| \to 0,$, independent of (x,y) as $t_2 \to t_1$. Similarly $\|G_2(x,y)(t_2) - G_2(x,y)(t_1)\| \to 0$,independent of (x,y) as $t_2 \to t_1$.Thus, $G(x,y)$ is equicontinuous, so by Arzela-Ascoli theorem $G(x,y)$ is completely continuous.

Step 2: Boundedness of $R = \{(x,y) \in C_\sigma([0,T],R) \times C_\gamma([0,T],R) : (x,y) = rG(x,y), r \in [0,1]\}$.
Let

$$x(t) = rG_1(x,y)(t), \ \ y(t) = rG_2(x,y)(t),$$

then

$$|x(t)| = r|G_1(x,y)(t)|.$$

By using our assumption we can easily get

$$\begin{aligned}
\|x\|_\sigma = r\|G_1(x,y)\|_\sigma = & \|G_1(x,y)\| + \|D^\sigma G_1(x,y)\| \\
\leq & \Theta\|g\| + \Xi\|f\| \\
\leq & \Theta\left(\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\| + \lambda_3\|y\|_\gamma\right) \\
& + \Xi(\theta_0 + \theta_1\|x\| + \theta_2\|y\| + \theta_3\|x\|_\sigma),
\end{aligned}$$

and in similar way, we can have

$$\begin{aligned}
\|y\|_\gamma = r\|G_2(x,y)\|_\gamma = & \|G_2(x,y)\| + \|D^\gamma G_2(x,y)\| \\
\leq & \widehat{\Theta}\|g\| + \widehat{\Xi}\|f\| \\
\leq & \widehat{\Theta}\left(\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\| + \lambda_3\|y\|_\gamma\right) \\
& + \widehat{\Xi}(\theta_0 + \theta_1\|x\| + \theta_2\|y\| + \theta_3\|x\|_\sigma).
\end{aligned}$$

So

$$\|x\|_\sigma + \|y\|_\gamma \leq \left(\Theta + \widehat{\Theta}\right)\lambda_0 + \left(\Xi + \widehat{\Xi}\right)\theta_0 + \max\{A,B\}\|(x,y)\|_{\sigma\times\gamma},$$

where

$$\|(x,y)\|_{\sigma\times\gamma} \leq \frac{\left(\Theta+\widehat{\Theta}\right)\lambda_0+\left(\Xi+\widehat{\Xi}\right)\theta_0}{1-\max\{A,B\}}.$$

As a result the set R is bounded. So, by Leray-Schauder alternative the operator G has at least one fixed point, which is the solution for the problem (1) with the boundary conditions (2) on $[0,T]$. □

4. Examples

Example 1. *Consider the following coupled system of fractional differential equation:*

$$\begin{cases} {}^cD^{6/5}x(t) = \dfrac{e^{-3t}}{12\sqrt{6400+t^4}}\left(\sin(x(t))+\cos(y(t))+\sin\left(D^{1/5}y(t)\right)\right) \\ {}^cD^{6/5}y(t) = \dfrac{1}{12\sqrt{3600+t^2}}\left(\cos(x(t))+\dfrac{|y(t)|}{2+|y(t)|}+\dfrac{\left|D^{1/3}x(t)\right|}{4+\left|D^{1/3}x(t)\right|}\right) \quad ,t\in[0,1] \end{cases}.$$

With the integral boundary conditions:

$$\begin{aligned} &\textstyle\int_0^1 x(s)\,ds = 3y(1/3), \int_0^1 x'(s)\,ds = -2y'(1/4), \\ &\textstyle\int_0^1 y(s)\,ds = x(1), \int_0^1 y'(s)\,ds = 2x'(1/2). \end{aligned}$$

It is clear that

$$\begin{aligned} f(t,x,y,z) &= \frac{e^{-3t}}{12\sqrt{6400+t^4}}(\sin x+\cos y+\sin z), \\ g(t,x,y,z) &= \frac{1}{12\sqrt{3600+t^2}}\left(\cos x+\frac{|y|}{2+|y|}+\frac{|z|}{4+|z|}\right) \end{aligned}$$

are jointly continuous and satisfy the Lipschitz condition with $l_f = 1/320, l_g = 1/240$. On the other hand

$$T=1, \rho_1=3, \zeta_1=1/3, \rho_2=-2, \zeta_2=1/4, \mu_1=1, \eta_1=1, \mu_2=2, \eta_2=1/2, \gamma=1/5, \sigma=1/3,$$

and Θ, Ξ, $\widehat{\Theta}$, $\widehat{\Xi}$ can be chosen as follows

$$\Theta = 3.4959,\ \Xi = 6.4324,\ \widehat{\Theta} = 5.1602,\ \widehat{\Xi} = 4.6058.$$

Then we obtain:

$$\begin{aligned} 1-2\left(\Theta l_g+\Xi l_f\right) &= 1-0.0693 = 0.9307 > 0, \\ 1-2\left(\widehat{\Theta} l_g+\widehat{\Xi} l_f\right) &= 1-0.0718 = 0.9282 > 0. \end{aligned}$$

Obviously, all the condition of Theorem 1 are satisfied so there exists unique solution for this problem.

Example 2. *Consider the following system:*

$$\begin{cases} {}^cD^{6/5}x(t) = \frac{1}{40+t^3}+\frac{y(t)}{115(1+x^2(t))}+\frac{1}{3(100+t^2)}\sin\left(D^{1/5}y(t)\right)+\frac{1}{3\sqrt{3600+t}}e^{-3t}\sin(x(t)) \\ {}^cD^{6/5}y(t) = \frac{1}{\sqrt{9+t^2}}\sin t+\frac{1}{180}e^{-2t}\sin(y(t))+\frac{1}{150}x(t)+\frac{1}{3(180+t)}D^{1/3}x(t), t\in[0,1], \end{cases}$$

with the following boundary conditions:

$$\begin{aligned} &\textstyle\int_0^1 x(s)\,ds = 3y(1/3), \int_0^1 x'(s)\,ds = -2y'(1/4), \\ &\textstyle\int_0^1 y(s)\,ds = x(1), \int_0^1 y'(s)\,ds = 2x'(1/2), \end{aligned}$$

$$T = 1, \rho_1 = 3, \zeta_1 = 1/3, \rho_2 = -2, \zeta_2 = 1/4, \mu_1 = 1, \eta_1 = 1, \mu_2 = 2, \eta_2 = 1/2, \gamma = 1/5, \sigma = 1/3,$$
$$\Theta = 3.4959, \Xi = 6.4324, \widehat{\Theta} = 5.1602, \widehat{\Xi} = 4.6058.$$

It is clear that:

$$\begin{aligned} |f(t, x_1, x_2, x_3)| &\leq \tfrac{1}{40} + \tfrac{1}{180}|x_1| + \tfrac{1}{115}|x_2| + \tfrac{1}{300}|x_3|, \\ |g(t, x_1, x_2, x_3)| &\leq \tfrac{1}{3} + \tfrac{1}{150}|x_1| + \tfrac{1}{180}|x_2| + \tfrac{1}{540}|x_3|. \end{aligned}$$

Thus

$$\begin{aligned} &\theta_0 = 1/40, \theta_1 = 1/180, \theta_2 = 1/115, \theta_3 = 1/300, \\ &\lambda_0 = 1/3, \lambda_1 = 1/150, \lambda_2 = 1/180, \lambda_3 = 1/540. \end{aligned}$$

We found A *and* B *such that:* $A = 0.1190, B = 0.1444$ *and that* $\max\{A, B\} = 0.1444 < 1$. *Since the conditions of Theorem 2 is achieved. So, there exists a solution for this problem.*

5. Conclusions

We studied the existence of solutions for a coupled system of fractional differential equations with integral boundary conditions. The first result was based on the Banach fixed point theorem. Secondly, by using Leray–Schauder's alternative, we proved the existence of solutions for Caputo fractional equations with integral boundary conditions. Finally, our results are supported by examples.

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