Article

Generalized Hyers-Ulam Stability of the Pexider Functional Equation

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Abstract: In this paper, we investigate the generalized Hyers-Ulam stability of the Pexider functional equation

\[ f(x + y, z + w) = g(x, z) + h(y, w). \]

Keywords: Pexider functional equation; additive functional equation; additive mappings

MSC: 39B82; 39B52

1. Introduction

Throughout this paper, let \( V \) and \( W \) be real vector spaces and \( Y \) be a real Banach space. In 1940, Ulam [1] raised the question about the stability of group homomorphisms. In 1941, for a given mapping \( f : V \to Y \), Hyers [2] solved the stability problem for the Cauchy additive functional equation

\[ f(x + y) - f(x) - f(y) = 0, \]


For given mappings \( f, g, h : V \to Y \), the stability of the Pexider functional equation

\[ f(x + y) - g(x) - h(y) = 0 \quad \forall x, y \in V \]

was investigated by Lee and Jun [5] (see also [6–9]).

Now, for given mappings \( f, g, h : V \times V \to Y \), we consider the Pexider functional equation

\[ f(x + y, z + w) - g(x, z) - h(y, w) = 0, \quad (1) \]

for all \( x, y, z, w \) in the vector space \( V \). One of typical examples of solutions of the functional Equation (1) are the mappings \( f, g, h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) given by \( f(x, z) = ax + bz + c + d \), \( g(x, z) = ax + bz + c \) and \( h(x, z) = ax + bz + d \) with real constants \( a, b, c, d \).

In this paper, we will investigate the generalized Hyers-Ulam stability of the Pexider functional Equation (1).

2. Main Results

For given mappings \( f, g, h : V \times V \to Y \), we use the following abbreviations:

\[ D_{fgh}(x, y, z, w) := f(x + y, z + w) - g(x, z) - h(y, w), \]

for all \( x, y, z, w \in V \). We need the following lemma to prove the main theorems.
**Theorem 1.** If the mappings \( f, g, h : V \times V \to W \) satisfy (1) for all \( x, y, z, w \in V \), then there exists an additive mapping \( k : V \times V \to W \) such that 
\[
k(x, z) = f(x, z) - f(0, 0) = g(x, z) - g(0, 0) = h(x, z) - h(0, 0)
\]
for all \( x, z \in V \).

**Proof.** From the equalities 
\[
f(0, 0) = g(0, 0) + h(0, 0), \quad f(x, z) = g(x, z) + h(x, z), \quad f(x, z) = g(0, 0) + h(x, z)
\]
for all \( x, z \in V \), we obtain the equalities 
\[
f(x, z) - f(0, 0) = g(x, z) - g(0, 0) = h(x, z) - h(0, 0)
\]
for all \( x, z \in V \). Put 
\[
k(x, z) = f(x, z) - f(0, 0) = g(x, z) - g(0, 0) = h(x, z) - h(0, 0)
\]
for all \( x, y, z, w \in V \), then
\[
k(x + y, z + w) = f(x + y, z + w) - f(0, 0) = g(x, z) - g(0, 0) + h(y, w) - h(0, 0) = k(x, z) + k(y, w)
\]
for all \( x, y, z, w \in V \). \( \square \)

Using the previous lemma, we obtain the following generalized Hyers-Ulam stability results for Equation (1).

**Theorem 1.** Suppose that \( f, g, h : V \times V \to Y \) are mappings for which there exists a function \( \varphi : V^2 \to [0, \infty) \) such that
\[
\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z, 2^i w)}{2^i} < \infty
\]
and
\[
\|f(x + y, z + w) - g(x, z) - h(y, w)\| \leq \varphi(x, y, z, w)
\]
for all \( x, y, z, w \in V \). Then, there exists a unique additive mapping \( F : V \times V \to Y \) such that
\[
\|f(x, z) - f(0, 0) - F(x, z)\| \leq \sum_{n=0}^{\infty} \frac{\mu(2^n x, 2^n z)}{2^{n+1}},
\]
\[
\|g(x, z) - g(0, 0) - F(x, z)\| \leq \sum_{n=0}^{\infty} \frac{\nu(2^n x, 2^n z)}{2^{n+1}},
\]
\[
\|h(x, z) - h(0, 0) - F(x, z)\| \leq \sum_{n=0}^{\infty} \frac{\xi(2^n x, 2^n z)}{2^{n+1}},
\]
for all \( x, z \in V \), where the functions \( \mu, \nu, \xi : V^2 \to \mathbb{R} \) are defined by
\[
\mu(x, z) = \varphi(x, x, z, z) + \varphi(x, 0, 0, z) + \varphi(0, x, 0, 0) + \varphi(0, 0, 0, 0),
\]
\[
\nu(x, z) = \varphi(2x, x, -x, z) + \varphi(x, 0, 0, z) + \varphi(x, -x, z, z) + \varphi(0, 0, 0, 0),
\]
\[
\xi(x, z) = \varphi(-x, 2x, -x, z) + \varphi(0, 0, 0, z) + \varphi(-x, x, z, z) + \varphi(0, 0, 0, 0).
\]

**Proof.** Let 
\[
f'(x, z) := f(x, z) - f(0, 0), \quad g'(x, z) := g(x, z) - g(0, 0), \quad \text{and} \quad h'(x, z) := h(x, z) - h(0, 0).
\]
Since the equalities
\[
f'(2x, 2z) - 2f'(x, z) = D_{fgh}(x, x, z, z) - D_{fgh}(x, 0, 0, 0),
\]
\[
g'(2x, 2z) - 2g'(x, z) = D_{fgh}(2x, -x, 2z, -z) - D_{fgh}(x, 0, 0, 0),
\]
\[
h'(2x, 2z) - 2h'(x, z) = D_{fgh}(-x, 2x, -x, z) - D_{fgh}(0, 0, 0, 0),
\]
where
\[
D_{fgh}(x, y, z, w) = f(x + y, z + w) - f(x, z) - f(y, w) - f(0, 0) - g(x, z) - g(0, 0) - h(y, w) - h(0, 0),
\]
for all \( x, y, z, w \in V \), then
\[
\|f'(2x, 2z) - 2f'(x, z)\| \leq \sum_{n=0}^{\infty} \frac{\mu(2^n x, 2^n z)}{2^{n+1}},
\]
\[
\|g'(2x, 2z) - 2g'(x, z)\| \leq \sum_{n=0}^{\infty} \frac{\nu(2^n x, 2^n z)}{2^{n+1}},
\]
\[
\|h'(2x, 2z) - 2h'(x, z)\| \leq \sum_{n=0}^{\infty} \frac{\xi(2^n x, 2^n z)}{2^{n+1}},
\]
for all \( x, z \in V \).
hold for all \( x, z \in V \), the inequalities
\[
\| f'(x, z) - \frac{f'(2^n x, 2^n z)}{2^n} \| \leq \frac{1}{2} h(x, z),
\]
\[
\| g'(x, z) - \frac{g'(2^n x, 2^n z)}{2^n} \| \leq \frac{1}{2} v(x, z),
\]
\[
\| h'(x, z) - \frac{h'(2^n x, 2^n z)}{2^n} \| \leq \frac{1}{2} \xi(x, z),
\]
for all \( x, z \in V \), follow from Inequality (3). Using the above inequalities, and the equalities
\[
\frac{f'(2^n x, 2^n z)}{2^n} - \frac{f'(2^{n+m} x, 2^{n+m} z)}{2^{n+m}} = \sum_{k=n}^{n+m-1} \left( \frac{f'(2^k x, 2^k z)}{2^k} - \frac{f'(2^{k+1} x, 2^{k+1} z)}{2^{k+1}} \right),
\]
\[
\frac{g'(2^n x, 2^n z)}{2^n} - \frac{g'(2^{n+m} x, 2^{n+m} z)}{2^{n+m}} = \sum_{k=n}^{n+m-1} \left( \frac{g'(2^k x, 2^k z)}{2^k} - \frac{g'(2^{k+1} x, 2^{k+1} z)}{2^{k+1}} \right),
\]
\[
\frac{h'(2^n x, 2^n z)}{2^n} - \frac{h'(2^{n+m} x, 2^{n+m} z)}{2^{n+m}} = \sum_{k=n}^{n+m-1} \left( \frac{h'(2^k x, 2^k z)}{2^k} - \frac{h'(2^{k+1} x, 2^{k+1} z)}{2^{k+1}} \right),
\]
we get the following inequalities
\[
\left\| \frac{f'(2^n x, 2^n z)}{2^n} - \frac{f'(2^{n+m} x, 2^{n+m} z)}{2^{n+m}} \right\| \leq \sum_{k=n}^{n+m-1} \left( \frac{f'(2^k x, 2^k z)}{2^k} - \frac{f'(2^{k+1} x, 2^{k+1} z)}{2^{k+1}} \right),
\]
(10)
\[
\left\| \frac{g'(2^n x, 2^n z)}{2^n} - \frac{g'(2^{n+m} x, 2^{n+m} z)}{2^{n+m}} \right\| \leq \sum_{k=n}^{n+m-1} \left( \frac{g'(2^k x, 2^k z)}{2^k} - \frac{g'(2^{k+1} x, 2^{k+1} z)}{2^{k+1}} \right),
\]
(11)
\[
\left\| \frac{h'(2^n x, 2^n z)}{2^n} - \frac{h'(2^{n+m} x, 2^{n+m} z)}{2^{n+m}} \right\| \leq \sum_{k=n}^{n+m-1} \left( \frac{h'(2^k x, 2^k z)}{2^k} - \frac{h'(2^{k+1} x, 2^{k+1} z)}{2^{k+1}} \right),
\]
(12)
for all \( x, z \in V \) and all \( n, m \in \mathbb{N} \cup \{0\} \). So, the sequences \( \left\{ \frac{f'(2^n x, 2^n z)}{2^n} \right\}_{n \in \mathbb{N}} \), \( \left\{ \frac{g'(2^n x, 2^n z)}{2^n} \right\}_{n \in \mathbb{N}} \), and \( \left\{ \frac{h'(2^n x, 2^n z)}{2^n} \right\}_{n \in \mathbb{N}} \) are Cauchy sequences for all \( x, z \in V \setminus \{0\} \). As \( Y \) is a real Banach space, we can define the mappings \( F, G, H : V \times V \rightarrow Y \) by
\[
F(x, z) = \lim_{n \to \infty} \frac{f'(2^n x, 2^n z)}{2^n},
\]
\[
G(x, z) = \lim_{n \to \infty} \frac{g'(2^n x, 2^n z)}{2^n},
\]
\[
H(x, z) = \lim_{n \to \infty} \frac{h'(2^n x, 2^n z)}{2^n},
\]
for all \( x, z \in V \). Since
\[
\lim_{n \to \infty} \left\| \frac{g'(2^n x, 2^n z)}{2^n} - \frac{f'(2^n x, 2^n z)}{2^n} \right\| = \lim_{n \to \infty} \left\| -D_{fgh}(2^n x, 0, 2^n z, 0) + D_{fgh}(0, 0, 0, 0) \right\| \leq \lim_{n \to \infty} \left( \frac{\varphi(2^n x, 0, 2^n z, 0) + \varphi(0, 0, 0, 0)}{2^n} \right) = 0,
\]
\[
\lim_{n \to \infty} \left\| \frac{h'(2^n x, 2^n z)}{2^n} - \frac{f'(2^n x, 2^n z)}{2^n} \right\| = \lim_{n \to \infty} \left\| -D_{fgh}(0, 2^n x, 0, 2^n z) + D_{fgh}(0, 0, 0, 0) \right\| \leq \lim_{n \to \infty} \left( \frac{\varphi(0, 2^n x, 0, 2^n z) + \varphi(0, 0, 0, 0)}{2^n} \right) = 0,
\]
for all $x, z \in V$, we have $F(x, z) = G(x, z) = H(x, z)$ for all $x, z \in V$. By putting $n = 0$ and letting $m \to \infty$ in Inequalities (10)–(12), we obtain Inequalities (4)–(6) for all $x, z \in V$.

From Inequality (3), we get

$$\left\| F\left(x, \frac{2^k x}{2^k}, \frac{2^k y}{2^k}, \frac{2^k z}{2^k}, \frac{2^k w}{2^k}\right) \right\| \leq \left\| F\left(2^k x, 2^k y, 2^k z, 2^k w\right) \right\|,$$

for all $x, y, z, w \in V$. Since the right-hand side of the above equality tends to zero as $n \to \infty$, we obtain that $F, G,$ and $H$ satisfy the functional Equation (1). Thus, by Lemma 1, $F$ is an additive mapping.

If $F' : V \times V \to Y$ is another additive mapping satisfying (4)–(6), we obtain

$$\left\| F'\left(x, z\right) - F\left(x, z\right) \right\| \leq \left\| \frac{F'\left(2^k x, 2^k y\right) - f\left(2^k x, 2^k y\right)}{2^k} \right\| + \left\| \frac{f\left(2^k x, 2^k y\right)}{2^k} - \frac{F\left(2^k x, 2^k y\right)}{2^k} \right\|$$

for all $x, z \in V$ and all $k \in \mathbb{N}$. As $\sum_{n=k}^{\infty} \frac{\mu(2^k x, 2^k y)}{2^k} \to 0$ as $k \to \infty$, we have $F'(x, z) = F(x, z)$ for all $x, z \in V$. Hence, the mapping is the unique additive mapping, as desired. □

**Corollary 1.** Suppose that $f : V \times V \to Y$ is a mapping for which there exists a function $\varphi : V^2 \to [0, \infty)$ satisfying inequality (2) and

$$\left\| 2f\left(\frac{x + y}{2}, \frac{z + w}{2}\right) - f(x, z) - f(y, w) \right\| \leq \varphi(x, y, z, w),$$

for all $x, y, z, w \in V$. Then, there exists a unique additive mapping $F : V \times V \to Y$ such that

$$\left\| f\left(x, z\right) - f(0, 0) - F\left(x, z\right) \right\| \leq \min \left\{ \sum_{n=0}^{\infty} \frac{\mu(2^n x, 2^n y)}{2^{n+1}}, \sum_{n=0}^{\infty} \frac{\nu(2^n x, 2^n y)}{2^{n+1}}, \sum_{n=0}^{\infty} \frac{\zeta(2^n x, 2^n y)}{2^{n+1}} \right\}$$

for all $x, z \in V$.

**Theorem 2.** Suppose that $f : V \times V \to Y$ is a mapping for which there exists a function $\varphi : V^2 \to [0, \infty)$, such that

$$\sum_{n=0}^{\infty} 2^n \varphi\left(\frac{z}{2^n}, \frac{z}{2^n}, \frac{z}{2^n}, \frac{z}{2^n}\right) < \infty$$

and (3) for all $x, y, z, w \in V$. Then, there exists a unique additive mapping $F : V \times V \to Y$ such that

$$\left\| f\left(x, z\right) - f(0, 0) - F\left(x, z\right) \right\| \leq \sum_{n=0}^{\infty} 2^n \mu\left(\frac{x}{2^{n+1}}, \frac{z}{2^{n+1}}\right),$$

$$\left\| g\left(x, z\right) - g(0, 0) - F\left(x, z\right) \right\| \leq \sum_{n=0}^{\infty} 2^n \nu\left(\frac{x}{2^{n+1}}, \frac{z}{2^{n+1}}\right),$$

$$\left\| h\left(x, z\right) - h(0, 0) - F\left(x, z\right) \right\| \leq \sum_{n=0}^{\infty} 2^n \zeta\left(\frac{x}{2^{n+1}}, \frac{z}{2^{n+1}}\right),$$

for all $x, z \in V$, where the functions $\mu, \nu, \zeta : V^2 \to \mathbb{R}$ are defined as in Theorem 1.
Proof. The inequalities

$$
\|f'(x, z) - 2f' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) \| \leq \mu \left( \frac{x}{2^n}, \frac{z}{2^n} \right),
$$

$$
\|g'(x, z) - 2g' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) \| \leq \nu \left( \frac{x}{2^n}, \frac{z}{2^n} \right),
$$

$$
\|h'(x, z) - 2h' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) \| \leq \xi \left( \frac{x}{2^n}, \frac{z}{2^n} \right),
$$

for all $x, z \in V$ follow from equalities (7)–(9) for all $x, z \in V$ and inequality (3). Using the above inequalities, we easily get the following inequalities

$$
\left\| 2^n f' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) - 2^{n+m} f' \left( \frac{x}{2^{n+m}}, \frac{z}{2^{n+m}} \right) \right\| \leq \sum_{k=0}^{n+m-1} 2^k \mu \left( \frac{x}{2^k}, \frac{z}{2^k} \right),
$$

(18)

$$
\left\| 2^n g' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) - 2^{n+m} g' \left( \frac{x}{2^{n+m}}, \frac{z}{2^{n+m}} \right) \right\| \leq \sum_{k=0}^{n+m-1} 2^k \nu \left( \frac{x}{2^k}, \frac{z}{2^k} \right),
$$

(19)

$$
\left\| 2^n h' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) - 2^{n+m} h' \left( \frac{x}{2^{n+m}}, \frac{z}{2^{n+m}} \right) \right\| \leq \sum_{k=0}^{n+m-1} 2^k \xi \left( \frac{x}{2^k}, \frac{z}{2^k} \right),
$$

(20)

for all $x, z \in V$ and all $n, m \in \mathbb{N} \cup \{0\}$. Thus, the sequences $\{2^n f' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) \}_{n \in \mathbb{N}}$, $\{2^n g' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) \}_{n \in \mathbb{N}}$, and $\{2^n h' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) \}_{n \in \mathbb{N}}$ are Cauchy sequences for all $x, z \in V$. Since $V$ is a real Banach space, we can define the mappings $F, G, H : V \times V \to Y$ by

$$
F(x) = \lim_{n \to \infty} 2^n f' \left( \frac{x}{2^n}, \frac{z}{2^n} \right),
$$

$$
G(x) = \lim_{n \to \infty} 2^n g' \left( \frac{x}{2^n}, \frac{z}{2^n} \right),
$$

$$
H(x) = \lim_{n \to \infty} 2^n h' \left( \frac{x}{2^n}, \frac{z}{2^n} \right)
$$

for all $x, z \in V$. Since

$$
\lim_{n \to \infty} \left\| 2^n g' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) - 2^n f' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) \right\| = \lim_{n \to \infty} \left\| 2^n \left( \frac{x}{2^n}, 0, 0, 0 \right) \right\| = 0,
$$

$$
\lim_{n \to \infty} \left\| 2^n h' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) - 2^n f' \left( \frac{x}{2^n}, \frac{z}{2^n} \right) \right\| = \lim_{n \to \infty} \left\| 2^n \left( \frac{x}{2^n}, 0, 0, 0 \right) \right\| = 0,
$$

for all $x, z \in V$, we have $F(x, z) = G(x, z) = H(x, z)$ for all $x, z \in V$. By putting $n = 0$ and letting $m \to \infty$ in Inequalities (18)–(20), we obtain Inequalities (15)–(17) for all $x, z \in V$.

From Inequality (3), we get

$$
\left\| 2^n D_{fg} \left( \frac{x}{2^n}, \frac{z}{2^n}, \frac{u}{2^n} \right) \right\| \leq 2^n \varphi \left( \frac{x}{2^n}, \frac{z}{2^n}, \frac{u}{2^n} \right)
$$

for all $x, y, z, w \in V$. Since the right-hand side of the above equality tends to zero as $n \to \infty$, we obtain that $F, G, \text{ and } H$ satisfy the functional Equation (1). Hence, by Lemma 1, $F$ is an additive mapping.
If $F : V \times V \to Y$ is another additive mapping satisfying (15)–(17), we obtain $G(0,0) = 0 = F(0,0)$ and
\[
\|F'(x,z) - F(x,z)\| \leq \left|\sum_{n=0}^{\infty} 2^{n+1} \left(\frac{x}{2^{n+1}} \cdot \frac{z}{2^{n+1}}\right)\right| + \left|\sum_{n=0}^{\infty} 2^n v \left(\frac{x}{2^{n+1}} \cdot \frac{z}{2^{n+1}}\right)\right| + \left|\sum_{n=0}^{\infty} 2^n \nu \left(\frac{x}{2^{n+1}} \cdot \frac{z}{2^{n+1}}\right)\right|
\]
for all $x, z \in V$ and all $k \in \mathbb{N}$. Since $\sum_{n=0}^{\infty} 2^{n+1} \left(\frac{x}{2^{n+1}} \cdot \frac{z}{2^{n+1}}\right) \to 0$ as $k \to \infty$, we have $F'(x,z) = F(x,z)$ for all $x, z \in V$. Hence, the mapping $F$ is the unique additive mapping, as desired. \hfill \square

**Corollary 2.** Suppose that $f : V \times V \to Y$ is a mapping for which there exists a function $\varphi : V^2 \to [0, \infty)$ satisfying inequality (14) and $f$ satisfies inequality (13) for all $x, y, z, w \in V$. Then, there exists a unique additive mapping $F : V \times V \to Y$, such that
\[
\|f(x,z) - f(0,0) - F(x,z)\|
\leq \min \left\{\sum_{n=0}^{\infty} 2^n \mu \left(\frac{x}{2^{n+1}} \cdot \frac{z}{2^{n+1}}\right), \sum_{n=0}^{\infty} 2^n v \left(\frac{x}{2^{n+1}} \cdot \frac{z}{2^{n+1}}\right), \sum_{n=0}^{\infty} 2^n \xi \left(\frac{x}{2^{n+1}} \cdot \frac{z}{2^{n+1}}\right)\right\},
\]
for all $x, z \in V$.

Aoki ([10]) and Gajda ([11]) proved the generalized Hyers-Ulam stability for an additive mapping in the cases where $0 \leq p < 1$ and $1 < p$, respectively. It was also proved by Gajda ([11]) and Rassias and Semrl ([12]) that the generalized Hyers-Ulam stability for an additive mapping does not holds for the case $p \neq 1$.

The above results for the cases $p > 0$ and $p \neq 1$ can be applied to the next results, due to Theorems 1 and 2.

**Corollary 3.** Let $X$ be a normed space and $p, \theta$ be positive real numbers with $p \neq 1$. Suppose that $f, g, h : X^2 \to Y$ are mappings, such that
\[
\|f(x+y,z+w) - g(x,z) - h(y,w)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)
\]
for all $x, y, z, w \in X$. Then, there exists a unique additive mapping $F : X \times X \to Y$, such that
\[
\|f(x,z) - f(0,0) - F(x,z)\| \leq \frac{4}{|2 - 2^p|} \theta(\|x\|^p + \|z\|^p),
\]
\[
\|g(x,z) - g(0,0) - F(x,z)\| \leq \frac{4 + 2^p}{|2 - 2^p|} \theta(\|x\|^p + \|z\|^p),
\]
\[
\|h(x,z) - h(0,0) - F(x,z)\| \leq \frac{4 + 2^p}{|2 - 2^p|} \theta(\|x\|^p + \|z\|^p),
\]
for all $x, z \in X$.

**Corollary 4.** Let $X$ be a normed space and $p, \theta$ be positive real numbers with $p \neq 1$. If $f : X^2 \to Y$ is a mapping satisfying Inequality (13) for all $x, y, z, w \in X$. Then, there exists a unique additive mapping $F : X \times X \to Y$, such that
\[
\|f(x,z) - f(0,0) - F(x,z)\| \leq \frac{4}{|2 - 2^p|} \theta(\|x\|^p + \|z\|^p),
\]
for all $x, z \in X$. 
Lemma 2. If the odd mappings \( f, g, h : V \times V \to W \) satisfy the equality (1) for all \( x, y, z, w \in V \setminus \{0\} \). Then, the mapping \( f \) satisfies the equality \( D_{fff}(x, y, z, w) = 0 \), for all \( x, y, z, w \in V \).

Proof. Choose any \( u \in V \setminus \{0\} \). Notice that the equality

\[
f(2x, 2z) = 2f(x, z)
\]

for all \( x, z \in V \) follows from the equality

\[
f(2x, 2z) - 2f(x, z) = D_{fg}(\frac{3x}{2}, \frac{x}{2}, \frac{3z}{2}, \frac{z}{2}) + D_{fg}(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2})
\]

(21)

\[
f(0, 2z) - 2f(0, z) = D_{fg}(\frac{u - u}{2}, \frac{u}{2}, \frac{z}{2}, \frac{z}{2}) + D_{fg}(\frac{u}{2}, \frac{u}{2}, \frac{z}{2}, \frac{z}{2})
\]

(22)

\[
f(2x, 0) - 2f(x, 0) = D_{fg}(\frac{3x}{2}, \frac{x}{2}, \frac{u}{2}, \frac{u}{2}) + D_{fg}(\frac{x}{2}, \frac{x}{2}, \frac{u}{2}, \frac{u}{2})
\]

(23)

for all \( x, z, u \in V \setminus \{0\} \). Using the equalities \( f(2x, 2z) = 2f(x, z) \),

\[
2f(\frac{x + y + z + w}{2}) - f(x, z) - f(y, w) = D_{fg}(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}) + D_{fg}(\frac{y}{2}, \frac{x}{2}, \frac{z}{2}, \frac{w}{2})
\]

(24)

\[
2f(\frac{y + z + w}{2}) - f(0, z) - f(y, w) = D_{fg}(\frac{3y}{4}, \frac{y}{4}, \frac{z}{2}, \frac{w}{2}) + D_{fg}(\frac{y}{4}, \frac{y}{4}, \frac{z}{2}, \frac{w}{2})
\]

(25)

\[
2f(\frac{z + w}{2}) - f(0, z) - f(0, w) = D_{fg}(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{w}{2}) + D_{fg}(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{w}{2})
\]

(26)

\[
2f(\frac{y + z}{2}) - f(0, z) - f(y, 0) = D_{fg}(\frac{3y}{4}, \frac{y}{4}, \frac{z}{4}, \frac{w}{4}) + D_{fg}(\frac{y}{4}, \frac{y}{4}, \frac{z}{4}, \frac{w}{4})
\]

(27)

for all \( x, y, z, w \in V \setminus \{0\} \), we have the equalities

\[
D_{fff}(x, y, z, w) = f(x + y, z + w) - f(x, z) - f(y, w) = 0,
\]

\[
D_{fff}(0, y, z, w) = f(y + z, w + 0) - f(0, z) - f(y, w) = 0,
\]

\[
D_{fff}(0, 0, z, w) = f(0, z + w) - f(0, z) - f(0, w) = 0,
\]

\[
D_{fff}(0, 0, 0, 0) = f(y, z) - f(0, z) - f(y, 0) = 0,
\]
Theorem 3. Suppose that $f, g, h : V \times V \to Y$ are odd mappings, for which there exists a function $\varphi : (V\{0\})^4 \to [0, \infty)$ such that
\begin{equation}
\sum_{i=0}^{\infty} \varphi(2^i x, 2^i y, 2^i z, 2^i w) \leq \frac{\mu(2^n x, 2^n y, 2^n z, 2^n w)}{2^{n+1}} < \infty \tag{24}
\end{equation}
and
\begin{equation}
\|f(x + y, z + w) - g(x, z) - h(y, w)\| \leq \varphi(x, y, z, w), \tag{25}
\end{equation}
for all $x, y, z, w \in V\{0\}$. Then, there exists a unique mapping $F : V \times V \to Y$, such that $D_{FFF}(x, y, z, w) = 0$ for all $x, y, z, w \in V$ and
\begin{align*}
\|f(x, z) - F(x, z)\| &\leq \sum_{n=0}^{\infty} \frac{\mu(2^n x, 2^n y, 2^n z, 2^n w)}{2^{n+1}}, \tag{26} \\
\|f(x, 0) - F(x, 0)\| &\leq \sum_{n=0}^{\infty} \frac{\nu(2^n x, 2^n y, 2^n z, 2^n w)}{2^{n+1}}, \tag{27} \\
\|f(0, z) - F(0, z)\| &\leq \sum_{n=0}^{\infty} \frac{\zeta(2^n x, 2^n y, 2^n z, 2^n w)}{2^{n+1}}, \tag{28}
\end{align*}
for all $x, z, u \in V\{0\}$, where the functions $\mu, \nu, \xi : V^2 \to \mathbb{R}$ are defined by
\begin{align*}
\mu(x, z) &= \varphi\left(\frac{3x}{2}, \frac{x}{2}, \frac{3z}{2}, \frac{z}{2}\right) + \varphi\left(\frac{x}{2}, \frac{z}{2}, \frac{3z}{2}, \frac{z}{2}\right) + \varphi\left(\frac{3x}{2}, \frac{z}{2}, \frac{x}{2}, \frac{3z}{2}\right) + \varphi(\frac{3x}{2}, \frac{z}{2}, \frac{x}{2}, \frac{z}{2}), \\
\nu(x, u) &= \varphi\left(\frac{3x}{2}, \frac{x}{2}, \frac{3u}{2}, \frac{z}{2}, \frac{u}{2}\right) + \varphi\left(\frac{x}{2}, \frac{z}{2}, \frac{3u}{2}, \frac{u}{2}\right) + \varphi(\frac{3x}{2}, \frac{z}{2}, \frac{x}{2}, \frac{3u}{2}, \frac{u}{2}) + \varphi(\frac{3x}{2}, \frac{z}{2}, \frac{x}{2}, \frac{3u}{2}, \frac{u}{2}), \\
\zeta(u, z) &= \varphi\left(\frac{u}{2}, \frac{3z}{2}, \frac{z}{2}, \frac{z}{2}\right) + \varphi\left(\frac{u}{2}, \frac{3z}{2}, \frac{z}{2}, \frac{z}{2}\right) + \varphi\left(\frac{u}{2}, \frac{3z}{2}, \frac{z}{2}, \frac{z}{2}\right) + \varphi\left(\frac{u}{2}, \frac{3z}{2}, \frac{z}{2}, \frac{z}{2}\right),
\end{align*}
for all $x, z, u \in V\{0\}$. 

for all $x, y, z, w \in V\{0\}$. By the same method, we obtain the equalities
\begin{align*}
D_{fff}(x, 0, z, w) &= f(x, z + w) - f(x, z) - f(0, w) = 0, \\
D_{fff}(x, y, 0, w) &= f(x + y, w) - f(x, 0) - f(y, w) = 0, \\
D_{fff}(x, y, z, 0) &= f(x + y, z) - f(x, z) - f(y, 0) = 0, \\
D_{fff}(x, 0, 0, w) &= f(x, w) - f(x, 0) - f(0, w) = 0, \\
D_{fff}(x, y, 0, 0) &= f(x + y, 0) - f(x, 0) - f(y, 0) = 0,
\end{align*}
for all $x, y, z, w \in V\{0\}$. As $f(0, 0) = 0$, the equalities $D_{fff}(x, 0, z, 0) = 0, D_{fff}(0, y, 0, w) = 0, D_{fff}(0, 0, 0, 0) = 0, D_{fff}(0, x, 0, 0) = 0, D_{fff}(0, 0, z, 0) = 0, D_{fff}(0, 0, 0, w) = 0,$ $D_{fff}(0, 0, 0, 0) = 0$ hold for all $x, y, z, w \in V\{0\}$. Hence, we conclude that $D_{fff}(x, y, z, w) = 0$ for all $x, y, z, w \in V$. \qed


**Proof.** Since \( f, g, h : V \times V \rightarrow Y \) are odd mappings, the equalities 
\( f(x, y) = -f(-x, -y), g(x, y) = -g(-x, -y), h(x, y) = -h(-x, -y) \), and 
\( f(0, 0) = g(0, 0) = h(0, 0) = 0 \) hold for all \( x, z \in V \). Choose any \( u \in \{0\} \). Since the equalities (21)–(23) hold for all \( x, z \in V \setminus \{0\} \), the inequalities

\[
\|f(x) - f(2x, 2z)\| \leq \frac{1}{2} \mu(x),
\]

\[
\|f(x, 0) - f(2x, 0)\| \leq \frac{1}{2} \mu(x),
\]

\[
\|f(0, z) - f(0, 2z)\| \leq \frac{1}{2} \zeta(u),
\]

for all \( x, z, u \in V \setminus \{0\} \) follow from the inequality (25). Using the above inequalities, we get the following inequalities

\[
\left\| \frac{f(2^n x, 2^n z)}{2^n} - \frac{f(2^{n+m} x, 2^{n+m} z)}{2^{n+m}} \right\| \leq \sum_{k=n}^{n+m-1} \frac{\mu(2^k x, 2^k z)}{2^{k+1}},
\]

(29)

\[
\left\| \frac{f(2^n x, 0)}{2^n} - \frac{f(2^{n+m} x, 0)}{2^{n+m}} \right\| \leq \sum_{k=n}^{n+m-1} \frac{\nu(2^k x, 2^k u)}{2^{k+1}},
\]

(30)

\[
\left\| \frac{f(0, 2^n z)}{2^n} - \frac{f(0, 2^{n+m} z)}{2^{n+m}} \right\| \leq \sum_{k=n}^{n+m-1} \frac{\zeta(2^k u, 2^k z)}{2^{k+1}},
\]

(31)

for all \( x, z, u \in V \setminus \{0\} \) and all nonnegative integers \( n, m \). So, the sequence \( \left\{ \frac{f(2^n x, 2^n z)}{2^n} \right\}_{n \in \mathbb{N}} \) is a Cauchy sequence for all \( x, z \in V \). As \( Y \) is a real Banach space, we can define a mapping \( F : V \times V \rightarrow Y \) by

\[
F(x, z) = \lim_{n \to \infty} \frac{f(2^n x, 2^n z)}{2^n}.
\]

By putting \( n = 0 \) and letting \( m \to \infty \) in the inequalities (29)–(31), we obtain the inequalities (26)–(28) for all \( x, z \in V \).

Since the equalities

\[
\lim_{n \to \infty} \frac{2g(2^n x, 2^n z) - f(2^{n+1} x, 2^{n+1} z)}{2^{n+1}} = \lim_{n \to \infty} \left( - \frac{D_{fg}(2^n x, 2^n x, 2^n z, 2^n z)}{2^{n+1}} - \frac{D_{gh}(2^n x, 2^n x, 2^n z, -2^n z)}{2^{n+1}} \right) = 0,
\]

\[
\lim_{n \to \infty} \frac{2h(2^n x, 2^n z) - f(2^{n+1} x, 2^{n+1} z)}{2^{n+1}} = \lim_{n \to \infty} \left( - \frac{D_{fg}(2^n x, 2^n x, 2^n z, 2^n z)}{2^{n+1}} - \frac{D_{gh}(2^n x, 2^n x, -2^n z, 2^n z)}{2^{n+1}} \right) = 0,
\]

hold for all \( x, z \in V \setminus \{0\} \), we know that the limits of \( \frac{g(2^n x, 2^n z)}{2^n} \) and \( \frac{h(2^n x, 2^n z)}{2^n} \) are given by

\[
\lim_{n \to \infty} \frac{g(2^n x, 2^n z)}{2^n} = F(x, z),
\]

\[
\lim_{n \to \infty} \frac{h(2^n x, 2^n z)}{2^n} = F(x, z),
\]

for all \( x, z \in V \setminus \{0\} \). From inequality (25), we get

\[
D_{FFF}(x, y, z, w) = \lim_{n \to \infty} \left\| \frac{D_{fg}(2^n x, 2^n y, 2^n z, 2^n w)}{2^n} \right\| \leq \lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y, 2^n z, 2^n w)}{2^n},
\]
for all \(x, y, z, w \in V \setminus \{0\}\). Since the right-hand side in the above equality equals zero, we obtain that the equality \(D_{F,F,F}(x, y, z, w) = 0\) holds for all \(x, y, z, w \in V \setminus \{0\}\). By Lemma 2, \(F\) satisfies the equality \(D_{F,F,F}(x, y, z, w) = 0\) for all \(x, y, z, w \in V\). If \(F' : V \times V \to Y\) is another mapping satisfying inequalities (26)–(28) and the equality \(D_{F,F,F}(x, y, z, w) = 0\) for all \(x, y, z, w \in V\), then we obtain the inequalities

\[
\|F'(x, z) - F(x, z)\| \leq \sum_{n=k}^{\infty} \mu(2^n x, 2^n z) / 2^n,
\]

\[
\|F'(x, 0) - F(x, 0)\| \leq \sum_{n=k}^{\infty} \nu(2^n x, 2^n u) / 2^n,
\]

\[
\|F'(0, z) - F(0, z)\| \leq \sum_{n=k}^{\infty} \zeta(2^n u, 2^n z) / 2^n,
\]

for all \(x, z, u \in V \setminus \{0\}\) and all \(k \in \mathbb{N}\). As \(\sum_{n=k}^{\infty} \mu(2^n x, 2^n z) / 2^n, \sum_{n=k}^{\infty} \nu(2^n x, 2^n u) / 2^n, \sum_{n=k}^{\infty} \zeta(2^n u, 2^n z) / 2^n \to 0\) as \(k \to \infty\), we have \(F'(x, z) = F(x, z)\) for all \(x, z \in V\). Hence, the mapping \(F\) is the unique additive mapping, as desired. \(\square\)

Lee (Theorem 5 in [13] and Corollary 1 in [14]) proved the generalized Hyers-Ulam stability for an additive mapping in the case \(n = 1\) and \(p < 0\).

The next corollary, for the case \(p < 0\), follows from Theorem 3.

**Corollary 5.** Let \(X\) be a normed space and \(p\) be a negative real number. Suppose that \(f, g, h : X^2 \to Y\) are odd mappings, such that

\[
\|f(x + y, z + w) - g(x, z) - h(y, w)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)
\]

for all \(x, y, z, w \in X \setminus \{0\}\). Then \(f, g, h\) satisfy the equality \(D_{f,f,f}(x, y, z, w) = 0\) for all \(x, y, z, w \in X\) and the inequalities

\[
\|g(x, z) - f(x, z)\| \leq 2\theta(\|x\|^p + \|z\|^p),
\]

\[
\|h(x, z) - f(x, z)\| \leq 2\theta(\|x\|^p + \|z\|^p),
\]

for all \(x, z \in X \setminus \{0\}\).

**Proof.** By Theorem 3, there exists a mapping \(F : X \times X \to Y\), such that \(D_{F,F,F}(x, y, z, w) = 0\) for all \(x, y, z, w \in X\) and

\[
\|f(x, z) - F(x, z)\| \leq \frac{(2 \cdot 3^p + 6)\theta(\|x\|^p + \|z\|^p)}{2^p(2 - 2^p)},
\]

for all \(x, z, u \in X \setminus \{0\}\). Therefore, we obtain the inequalities

\[
\|g(x, z) - F(x, z)\| \leq \left(2 + \frac{3^p + 3}{2 - 2^p}\right)\theta(\|x\|^p + \|z\|^p)
\]

\[
\|h(x, z) - F(x, z)\| \leq \left(2 + \frac{3^p + 3}{2 - 2^p}\right)\theta(\|x\|^p + \|z\|^p),
\]
for all $x, z \in X \setminus \{0\}$, from the inequalities

\[
\|g(x, z) - F(x, z)\| \leq \|g(x, z) - \frac{f(2x, 2z)}{2}\| + \|\frac{f(2x, 2z)}{2}\| - \frac{F(2x, 2z)}{2} \| + (3^p + 3\theta(\|x\|^p + \|z\|^p)) \\frac{2}{2 - 2^p},
\]

\[
\|h(x, z) - F(x, z)\| \leq \|h(x, z) - \frac{f(2x, 2z)}{2}\| + \|\frac{f(2x, 2z)}{2}\| - \frac{F(2x, 2z)}{2} \| + (3^p + 3\theta(\|x\|^p + \|z\|^p)) \\frac{2}{2 - 2^p},
\]

for all $x, z \in X \setminus \{0\}$. Hence, we have the inequalities

\[
\|f(x, z) - F(x, z)\| = \|D_{fgh}(k + 1)x, x, x, x\| - \frac{D_{fgh}(k + 1)x, x, x, x}{2} - \frac{F(2x, 2z)}{2} \| + (3^p + 3\theta(\|x\|^p + \|z\|^p)) \\frac{2}{2 - 2^p},
\]

\[
\|f(x, 0) - F(x, 0)\| = \|D_{fgh}(k + 1)x, x, x, x\| - \frac{D_{fgh}(k + 1)x, x, x, x}{2} - \frac{F(2x, 2z)}{2} \| + (3^p + 3\theta(\|x\|^p + \|z\|^p)) \\frac{2}{2 - 2^p},
\]

\[
\|f(0, z) - F(0, z)\| = \|D_{fgh}(k + 1)x, x, x, x\| - \frac{D_{fgh}(k + 1)x, x, x, x}{2} - \frac{F(2x, 2z)}{2} \| + (3^p + 3\theta(\|x\|^p + \|z\|^p)) \\frac{2}{2 - 2^p},
\]

for all $x, z \in X \setminus \{0\}$ and $k \in \mathbb{N}$. Since the right-hand side of the above equalities tends to zero as $k \to \infty$ when $p < 0$, we know that $f(x, z) = F(x, z)$ for all $x, z \in X$. So, $f$ satisfies the equality $D_{fgh}(x, y, z, w) = 0$ for all $x, y, z, w \in X$, which implies the equality $f(x, z) = \frac{f(2x, 2z)}{2}$ for all $x, z \in X$, and the inequalities

\[
\|g(x, z) - f(x, z)\| = \|g(x, z) - \frac{f(2x, 2z)}{2}\|
\]

\[
\leq \|D_{fgh}(x, x, x, x)\| + \|\frac{f(2x, 2z)}{2}\| - \frac{F(2x, 2z)}{2} \| + (3^p + 3\theta(\|x\|^p + \|z\|^p)) \frac{2}{2 - 2^p},
\]

\[
\|h(x, z) - f(x, z)\| = \|h(x, z) - \frac{f(2x, 2z)}{2}\|
\]

\[
\leq \|D_{fgh}(x, x, x, x)\| + \|\frac{f(2x, 2z)}{2}\| - \frac{F(2x, 2z)}{2} \| + (3^p + 3\theta(\|x\|^p + \|z\|^p)) \frac{2}{2 - 2^p},
\]

for all $x, z \in X \setminus \{0\}$. □
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References


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