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Existence and Unique Coupled Solution in $S_b$-Metric Spaces by Rational Contraction with Application

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Abstract: In this paper, we prove a unique common coupled fixed point theorem for two pairs of $w$-compatible mappings in $S_b$-metric spaces. We also furnish an example to support our main result.

Keywords: $S_b$-metric space; $w$-compatible pairs; $S_b$-completeness.

1. Introduction

In 2012, Sedghi et al. [1] introduced the notion of $S$-metric space and proved several results. Some other authors also worked on this (e.g., [2–6]). On the other hand, the concept of $b$-metric space was introduced by Bakhtin [7] and Czerwik [8] (see also [9–11]).

Recently, Sedghi et al. [1] defined $S_b$-metric spaces using the concepts of $S$ and $b$-metric spaces and proved common fixed point theorem for four maps in $S_b$-metric spaces (see also [12]). Bhaskar and Lakshmikantham [13] introduced the notion of coupled fixed point and proved some coupled fixed point results as well.

The aim of this paper is to prove a unique common coupled fixed point theorem for four mappings in $S_b$-metric spaces. Throughout this paper, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of all non-negative real numbers and positive integers, respectively.

First, we recall some definitions, lemmas and examples.

Definition 1. Ref. [4] Let $X$ be a nonempty set. A $S$-metric on $X$ is a function $S : X^3 \to \mathbb{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$:

- (S1) $S(x, y, z) = 0$ if and only if $x = y = z$,
- (S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then, the pair $(X, S)$ is called a $S$-metric space.

Definition 2. (8) Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is called a $b$-metric if the following axioms are satisfied for all $x, y, z \in X$:

- (b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) \( d(x, y) = d(y, x) \); and
(b3) \( d(x, y) \leq s[d(x, z) + d(z, y)] \).

The pair \((X, d)\) is called a b b-metric space.

**Definition 3.** ([1]) Let \( X \) be a nonempty set and \( b \geq 1 \) be a given real number. Suppose that a mapping \( S_b : X^3 \to \mathbb{R}^+ \) is a function satisfying the following properties:

\( S_b(x, y, z) = 0 \) if and only if \( x = y = z \); and
\( S_b(x, y, z) \leq b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)) \) for all \( x, y, z, a \in X \).

Then, the function \( S_b \) is called a \( S_b \)-metric on \( X \) and the pair \((X, S_b)\) is called a \( S_b \)-metric space.

**Remark 1.** ([1]) It should be noted that the class of \( S_b \)-metric spaces is effectively larger than that of \( S \)-metric spaces. Indeed, each \( S \)-metric space is a \( S_b \)-metric space with \( b = 1 \).

The following example shows that a \( S_b \)-metric on \( X \) need not be a \( S \)-metric on \( X \).

**Example 1.** ([1]) Let \((X, S_b)\) be a \( S \)-metric space and \( S_b(x, y, z) = S^p(x, y, z) \), where \( p > 1 \) is a real number.

Note that \( S_b \) is a \( S_b \)-metric with \( b = 2^{(p-1)} \). In addition, \((X, S_b)\) is not necessarily a \( S \)-metric space.

**Definition 4.** ([1]) Let \((X, S_b)\) be a \( S_b \)-metric space. Then, for \( x \in X, r > 0 \), we defined the open ball \( B_{S_b}(x, r) \) and closed ball \( B_{S_b}[x, r] \) with center \( x \) and radius \( r \) as follows, respectively:

\[
B_{S_b}(x, r) = \{ y \in X : S_b(y, y, x) < r \},
\]
\[
B_{S_b}[x, r] = \{ y \in X : S_b(y, y, x) \leq r \}.
\]

**Lemma 1.** ([1]) In a \( S_b \)-metric space, we have

\[
S_b(x, x, y) \leq bS_b(y, y, x)
\]

and

\[
S_b(y, y, x) \leq bS_b(x, x, y).
\]

**Lemma 2.** ([1]) In a \( S_b \)-metric space we have

\[
S_b(x, x, z) \leq 2bS_b(x, y, x) + b^2S_b(y, y, z).
\]

**Definition 5.** ([1]) Let \((X, S_b)\) be a \( S_b \)-metric space. A sequence \( \{x_n\} \) in \( X \) is said to be:

(1) \( S_b \)-Cauchy sequence if, for each \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( S_b(x_m, x_n, x_m) < \epsilon \) for each \( m, n \geq n_0 \).

(2) \( S_b \)-convergent to a point \( x \in X \) if, for each \( \epsilon > 0 \), there exists a positive integer \( n_0 \) such that \( S_b(x_n, x_n, x) < \epsilon \) or \( S_b(x_n, x, x_n) < \epsilon \) for all \( n \geq n_0 \) and we denote that by \( \lim_{n \to \infty} x_n = x \).

**Definition 6.** ([1]) A \( S_b \)-metric space \((X, S_b)\) is called complete if every \( S_b \)-Cauchy sequence is \( S_b \)-convergent in \( X \).

**Lemma 3.** ([14]) If \((X, S_b)\) is a \( S_b \)-metric space with \( b \geq 1 \) and \( \{x_n\} \) is a \( S_b \)-convergent to \( x \), then for all \( y \in X \), we have

(i) \[ \frac{1}{2b} S_b(y, y, x) \leq \liminf_{n \to \infty} S_b(y, y, x_n) \leq \limsup_{n \to \infty} S_b(y, y, x_n) \leq 2bS_b(y, y, x) ; \] and

(ii) \[ \frac{1}{b^2} S_b(x, x, y) \leq \liminf_{n \to \infty} S_b(x_n, x_n, y) \leq \limsup_{n \to \infty} S_b(x_n, x_n, y) \leq b^2S_b(x, x, y) . \]
In particular, if \( x = y \), then we have \( \lim_{n \to \infty} S_b(x_n, x_n, y) = 0 \).

**Definition 7.** ([13]) Let \( X \) be a nonempty set. An element \( (x, y) \in X \times X \) is called a coupled fixed point of a mapping \( F : X \times X \to X \) if \( x = F(x, y) \) and \( y = F(y, x) \).

**Definition 8.** ([15]) Let \( X \) be a nonempty set. An element \( (x, y) \in X \times X \) is called:

(i) a coupled coincident point of mappings \( F : X \times X \to X \) and \( f : X \to X \) if \( f(x) = F(x, y) \) and \( f(y) = F(y, x) \); and

(ii) a common coupled fixed point of mappings \( F : X \times X \to X \) and \( f : X \to X \) if \( x = f(x) = F(x, y) \) and \( y = f(y) = F(y, x) \).

**Definition 9.** ([16]) Let \( X \) be a nonempty set. An element \( x \in X \) is called a coupled fixed point of a mapping \( F : X \to X \) if \( x = F(x, x) \).

For more details of other generalized metric spaces as well as on some rational contraction, see [9,17–19].

Now, we give our main result.

### 2. Main Results

Let \( \Phi \) denote the class of all functions \( \phi : [0, \infty) \to [0, \infty) \) such that \( \phi \) is a non-decreasing, continuous, \( \phi(t) < \frac{t}{4t^4} \) for all \( t > 0 \) and \( \phi(0) = 0 \).

**Theorem 1.** Let \((X, S_b)\) be a \( S_b \)-metric space. Suppose that \( A, B : X \times X \to X \) and \( P, Q : X \to X \) satisfy:

1. \( A(X \times X) \subseteq Q(X) \), \( B(X \times X) \subseteq P(X) \);
2. \( \{A, P\} \) and \( \{B, Q\} \) are \( w \)-compatible pairs;
3. one of \( P(X) \) or \( Q(X) \) is \( S_b \)-complete subspace of \( X \); and
4. \( 2b^5 S_b(A(x, y), A(x, y), B(u, v)) \)

\[
\begin{align*}
\phi 
\leq \max \left\{ S_b(P(x), P(x), Q(u)), S_b(P(y), P(y), Q(v)), S_b(A(x, y), A(x, y), P(x)), S_b(A(y, x), A(y, x), P(y)), S_b(B(u, v), B(u, v), Q(u)), S_b(B(v, u), B(v, u), Q(v)), S_b(A(x, y), A(x, y), Q(u)), S_b(B(u, v), B(u, v), P(x)) \right\} \leq \frac{S_b(A(x, y), A(x, y), Q(u)) S_b(B(u, v), B(u, v), P(x))}{1 + S_b(P(x), P(x), Q(u))}, \frac{S_b(A(y, x), A(y, x), Q(v)) S_b(B(v, u), B(v, u), P(y))}{1 + S_b(P(y), P(y), Q(v))},
\end{align*}
\]

for all \( x, y, u, v \in X, \phi \in \Phi. \)

Then, \( A, B, P \) and \( Q \) have a unique common coupled fixed point in \( X \times X \).

**Proof of Theorem.** Let \( x_0, y_0 \in X \). From Equation (1), we can construct the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) and \( \{w_n\} \) such that

\[
\begin{align*}
A(x_{2n}, y_{2n}) &= Q(x_{2n+1}) = z_{2n}, \\
A(y_{2n}, x_{2n}) &= Q(y_{2n+1}) = w_{2n}, \\
B(x_{2n+1}, y_{2n+1}) &= P(x_{2n+2}) = z_{2n+1}, \\
B(y_{2n+1}, x_{2n+1}) &= P(y_{2n+2}) = w_{2n+1}, \quad n = 0, 1, 2, \ldots
\end{align*}
\]

**Case (i).** Suppose \( z_{2m} = z_{2m+1} \) and \( w_{2m} = w_{2m+1} \) for some \( m \). Assume that \( z_{2m+1} \neq z_{2m+2} \) or \( w_{2m+1} \neq w_{2m+2} \).
From Equation (4), we have
\[ S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}) \]
\[ \leq 2b^5 S_b(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), B(x_{2m+1}, y_{2m+1})) \]
\[
\begin{align*}
\leq \phi \left\{ \max \left\{ 
\begin{array}{l}
S_b(p(x_{2m+2}), p(x_{2m+2}), q(x_{2m+1})), S_b(p(y_{2m+2}), p(y_{2m+2}), q(y_{2m+1})), \\
S_b(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), p(x_{2m+2})), \\
S_b(A(y_{2m+2}, y_{2m+2}), A(y_{2m+2}, y_{2m+2}), p(y_{2m+2})), \\
S_b(B(x_{2m+1}, y_{2m+1}), B(x_{2m+1}, y_{2m+1}), q(x_{2m+1})), \\
S_b(B(x_{2m+1}, y_{2m+1}), B(x_{2m+1}, y_{2m+1}), q(y_{2m+1})), \\
S_b(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), q(x_{2m+1})), S_b(B(x_{2m+1}, y_{2m+1}), B(x_{2m+1}, y_{2m+1}), p(x_{2m+2})), \\
S_b(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), p(y_{2m+2})), S_b(B(x_{2m+1}, y_{2m+1}), B(x_{2m+1}, y_{2m+1}), q(y_{2m+1})), \\
1 + S_b(p(x_{2m+2}, p(x_{2m+2}), q(x_{2m+1})), S_b(p(y_{2m+2}, p(y_{2m+2}), q(y_{2m+1}))) \\
\end{array} \right\} \right. \\
= \phi \left\{ \max \left\{ 
\begin{array}{l}
S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), \\
S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}), S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), \\
S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}) S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \\
1 + S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \\
\end{array} \right\} \right. \\
= \phi \left\{ \max \left\{ 
\begin{array}{l}
S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \end{array} \right\} \right. \\
= \phi \left( \max \left\{ 0, 0, S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \right\} \right).
\end{align*}
\]

Similarly, we can prove that
\[ S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \leq \phi \left( \max \left\{ S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \right\} \right). \]

It follows that
\[ \max \left\{ S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \right\} \leq \phi \left( \max \left\{ S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \right\} \right). \]

Hence, \( z_{2m+2} = z_{2m+1} \) and \( w_{2m+2} = w_{2m+1} \).

Continuing in this process, we can conclude that \( z_{2m+k} = z_{2m} \) and \( w_{2m+k} = w_{2m} \) for all \( k \geq 0 \).

It follows that \( \{z_m\} \) and \( \{w_m\} \) are Cauchy sequences.

**Case (ii).** Assume that \( z_{2n} \neq z_{2n+1} \) or \( w_{2n} \neq w_{2n+1} \) for all \( n \).

Put \( S_n = \max \left\{ S_b(z_{n+1}, z_{n+1}, z_n), S_b(w_{n+1}, w_{n+1}, w_n) \right\} \).

From Equation (4), we have
\[ S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) \]
\[ \leq 2b^5 S_b(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), B(x_{2n+1}, y_{2n+1})) \]
\[
\begin{align*}
\leq \phi \left\{ \max \left\{ 
\begin{array}{l}
S_b(p(x_{2n+2}), p(x_{2n+2}), q(x_{2n+1})), S_b(p(y_{2n+2}), p(y_{2n+2}), q(y_{2n+1})), \\
S_b(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), p(x_{2n+2})), \\
S_b(A(y_{2n+2}, y_{2n+2}), A(y_{2n+2}, y_{2n+2}), p(y_{2n+2})), \\
S_b(B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1}), q(x_{2n+1})), \\
S_b(B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1}), q(y_{2n+1})), \\
S_b(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), q(x_{2n+1})), S_b(B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1}), p(x_{2n+2})), \\
S_b(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), p(y_{2n+2})), S_b(B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1}), q(y_{2n+1})), \\
1 + S_b(p(x_{2n+2}, p(x_{2n+2}), q(x_{2n+1})), S_b(p(y_{2n+2}, p(y_{2n+2}), q(y_{2n+1}))) \\
\end{array} \right\} \right. \\
= \phi \left\{ \max \left\{ 
\begin{array}{l}
S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), \\
S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}), S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\
S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \\
1 + S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \\
\end{array} \right\} \right. \\
= \phi \left\{ \max \left\{ 
\begin{array}{l}
S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \end{array} \right\} \right. \\
= \phi \left( \max \left\{ 0, 0, S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \right\} \right).
\end{align*}
\]
From Equations (4) and (5), we have

\[
= \phi \left( \max \left\{ S_b(z_{n+1}, z_{n+1}, z_n), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(z_{n+2}, z_{n+2}, z_{n+1}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}), S_b(z_{n+1}, z_{n+1}, z_{n+1}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(z_{n+2}, z_{n+2}, z_{n+1}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \right\} \right)
\]

\[
= \phi \left( \max \left\{ S_b(z_{n+1}, z_{n+1}, z_n), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \right\} \right)
\]

\[
= \phi \left( \max \left\{ S_{2n+1}, S_{2n} \right\} \right).
\]

Similarly, we can prove

\[
S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \leq \phi \left( \max \{ S_{2n+1}, S_{2n} \} \right).
\]

Thus,

\[
S_{2n+1} \leq \phi(\max \{ S_{2n}, S_{2n+1} \}).
\]

If \( S_{2n+1} \) is maximum, then we get contradiction so that \( S_{2n} \) is maximum.

Thus,

\[
S_{2n+1} \leq \phi(S_{2n}) < S_{2n}.
\]

Similarly, we can conclude that \( S_{2n} < S_{2n-1} \).

It is clear that \( \{ S_n \} \) is a non-increasing sequence of non-negative real numbers and must converge to a real number, say \( r \geq 0 \).

Suppose \( r > 0 \). Letting \( n \to \infty \), in Equation (1), we have \( r \leq \phi(r) \leq r \).

It is a contradiction. Hence, \( r = 0 \).

Thus,

\[
\lim_{n \to \infty} S_b(z_{n+1}, z_{n+1}, z_n) = 0
\]

and

\[
\lim_{n \to \infty} S_b(w_{n+1}, w_{n+1}, w_n) = 0.
\]

Now, we prove that \( \{ z_{2n} \} \) and \( \{ w_{2n} \} \) are Cauchy sequences in \((X, \mathcal{B})\). On the contrary, we suppose that \( \{ z_{2n} \} \) or \( \{ w_{2n} \} \) is not Cauchy. Then, there exist \( \epsilon > 0 \) and monotonically increasing sequence of natural numbers \( \{ 2m_k \} \) and \( \{ 2n_k \} \) such that for \( n_k > m_k \)

\[
\max \{ S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k}) \} \geq \epsilon
\]

and

\[
\max \{ S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k}) \} < \epsilon.
\]

From Equations (4) and (5), we have
\[ c \leq \max \{ S_b(z_{2m_l}, z_{2m_l}, z_{2m_l}), S_b(w_{2m_l}, w_{2m_l}, w_{2m_l}) \} \]
\[ \leq 2b \max \{ S_b(z_{2m_l}, z_{2m_l}, z_{2m_l} + 2), S_b(w_{2m_l}, w_{2m_l}, w_{2m_l} + 2) \} + b \max \{ S_b(z_{2m_l}, z_{2m_l}, z_{2m_l} + 2), S_b(w_{2m_l}, w_{2m_l}, w_{2m_l} + 2) \} \]
\[ \leq 2b^5 \max \{ S_b(z_{2m_l} + 1, z_{2m_l} + 1, z_{2m_l}), S_b(w_{2m_l} + 1, w_{2m_l} + 1, w_{2m_l}), S_b(z_{2m_l} + 2, z_{2m_l} + 2, z_{2m_l} + 1), \]
\[ S_b(w_{2m_l} + 2, w_{2m_l} + 2, w_{2m_l} + 1), S_b(z_{2m_l} + 1, z_{2m_l} + 1, z_{2m_l}), S_b(w_{2m_l} + 1, w_{2m_l} + 1, w_{2m_l}), \]
\[ S_b(z_{2m_l} + 2, z_{2m_l} + 2, z_{2m_l}), S_b(w_{2m_l} + 2, w_{2m_l} + 2, w_{2m_l}), S_b(z_{2m_l} + 1, z_{2m_l} + 1, z_{2m_l} + 1), \]
\[ \frac{1 + S_b(z_{2m_l} + 1, z_{2m_l} + 1, z_{2m_l})}{1 + S_b(w_{2m_l} + 1, w_{2m_l} + 1, w_{2m_l})} \} \]
\[ \leq \phi \]

Similarly,
\[ 2b^5 S_b(w_{2m_l} + 2, w_{2m_l} + 2, w_{2m_l} + 1) \]
\[ \leq \phi \]

Thus,
\[ 2b^5 \max \{ S_b(z_{2m_l} + 2, z_{2m_l} + 2, z_{2m_l} + 1), S_b(w_{2m_l} + 2, w_{2m_l} + 2, w_{2m_l} + 1) \} \]
\[ \leq \phi \]

(7)
However,
\[
\max \{ S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1}), S_b(w_{2m_1 + 1}, w_{2m_1}, w_{2m_1}) \} 
\leq 2b \max \{ S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1}), S_b(w_{2m_1 + 1}, w_{2m_1}, w_{2m_1}) \} 
+ b \max \{ S_b(z_{2m_1}, z_{2m_1}, z_{2m_1}), S_b(w_{2m_1}, w_{2m_1}, w_{2m_1}) \} 
\leq 2b \max \{ S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1}), S_b(w_{2m_1 + 1}, w_{2m_1}, w_{2m_1}) \} 
+ b^2 \max \{ S_b(z_{2m_1}, z_{2m_1}, z_{2m_1}), S_b(w_{2m_1}, w_{2m_1}, w_{2m_1}) \} 
\leq 2b \max \{ S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1}), S_b(w_{2m_1 + 1}, w_{2m_1}, w_{2m_1}) \} 
+ b^2 \left( 2b \max \{ S_b(z_{2m_1}, z_{2m_1}, z_{2m_1}), S_b(w_{2m_1}, w_{2m_1}, w_{2m_1}) \} \right) 
+ b^3 \left( b \max \{ S_b(z_{2m_1}, z_{2m_1}, z_{2m_1}), S_b(w_{2m_1}, w_{2m_1}, w_{2m_1}) \} \right) 
\leq 2b \max \{ S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1}), S_b(w_{2m_1 + 1}, w_{2m_1}, w_{2m_1}) \} 
+ 2b^3 + b^4 \max \{ S_b(z_{2m_1}, z_{2m_1}, z_{2m_1}), S_b(w_{2m_1}, w_{2m_1}, w_{2m_1}) \} 
+ b^3 \left( b \max \{ S_b(z_{2m_1}, z_{2m_1}, z_{2m_1}), S_b(w_{2m_1}, w_{2m_1}, w_{2m_1}) \} \right) 
\leq 2b \max \{ S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1}), S_b(w_{2m_1 + 1}, w_{2m_1}, w_{2m_1}) \} 
+ 2b^3 + b^4 \max \{ S_b(z_{2m_1}, z_{2m_1}, z_{2m_1}), S_b(w_{2m_1}, w_{2m_1}, w_{2m_1}) \} 
+ b^5 \max \{ S_b(z_{2m_1}, z_{2m_1}, z_{2m_1}), S_b(w_{2m_1}, w_{2m_1}, w_{2m_1}) \}. 
\]

Letting \( k \to \infty \), we have
\[
\lim_{k \to \infty} \max \{ S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1}), S_b(w_{2m_1 + 1}, w_{2m_1}, w_{2m_1}) \} \leq 2b^3 e. 
\]

In addition,
\[
\lim_{k \to \infty} \frac{S_b(z_{2m_1 + 2}, z_{2m_1 + 2}, z_{2m_1 + 2}) S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1 + 1})}{1 + S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1 + 1})} 
\leq \lim_{k \to \infty} \frac{1}{1 + S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1 + 1})} \cdot \left[ 2b S_b(z_{2m_1 + 2}, z_{2m_1 + 2}, z_{2m_1 + 2}) + b S_b(z_{2m_1 + 2}, z_{2m_1 + 2}, z_{2m_1 + 2}) \right] 
= \lim_{k \to \infty} \frac{b^3 S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1 + 1}) S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1 + 1})}{1 + S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1 + 1})} 
\leq \lim_{k \to \infty} b^3 S_b(z_{2m_1 + 1}, z_{2m_1 + 1}, z_{2m_1 + 1}) 
\leq 2b^6 e. 
\]

Similarly,
\[
\lim_{k \to \infty} \frac{S_b(z_{2m_1 + 2}, z_{2m_1 + 2}, z_{2m_1 + 2}) S_b(w_{2m_1 + 1}, w_{2m_1 + 1}, w_{2m_1 + 1})}{1 + S_b(w_{2m_1 + 1}, w_{2m_1 + 1}, w_{2m_1 + 1})} \leq 2b^6 e. 
\]

Letting \( k \to \infty \) in Equation (7), we have
\[
\lim_{k \to \infty} \max \{ S_b(z_{2m_1 + 2}, z_{2m_1 + 2}, z_{2m_1 + 2}), S_b(w_{2m_1 + 2}, w_{2m_1 + 2}, w_{2m_1 + 2}) \} 
\leq \frac{1}{2b} \phi \left( \max \{ 2b^3 e, 0, 0, 0, 0, 2b^6 e, 2b^6 e \} \right) 
= \frac{1}{2b} \phi(2b^6 e). 
\]
Now, letting $n \to \infty$ in Equation (6), from Equations (2), (3) and (8), we have

$$\epsilon \leq 0 + 0 + b^2 \frac{1}{2b^5} \phi(2b^6 \epsilon) < \epsilon.$$ 

It is a contradiction. Hence, $\{z_{2n}\}$ and $\{w_{2n}\}$ are $S_b$-Cauchy sequences in $(X, S_b)$. In addition,

$$\max\{S_b(z_{2n+1}, z_{2n+2}, z_{2m+1}), S_b(w_{2n+1}, w_{2n+2}, w_{2m+1})\}$$

$$\leq 2b \max\{S_b(z_{2m+1}, z_{2n+1}, z_{2n}), S_b(w_{2m+1}, w_{2n+1}, w_{2n})\}$$

$$+ b \max\{S_b(z_{2m+1}, z_{2n+1}, z_{2n}), S_b(w_{2m+1}, w_{2m}, w_{2n})\}$$

$$\leq 2b^2 \max\{S_b(z_{2m+1}, z_{2n+1}, z_{2n}), S_b(w_{2m+1}, w_{2n+1}, w_{2n})\}$$

$$+ 2b^2 \max\{S_b(z_{2m+1}, z_{2n+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m})\}$$

Since $\{z_{2n}\}$ and $\{w_{2n}\}$ are $S_b$-Cauchy sequences, from Equations (2) and (3), it follows that $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are also $S_b$-Cauchy sequences in $(X, S_b)$. Hence, $\{z_n\}$ and $\{w_n\}$ are $S_b$-Cauchy sequences in $(X, S_b)$.

Suppose $P(X)$ is a $S_b$-complete subspace of $(X, S_b)$. Then, the sequences $\{z_n\}$ and $\{w_n\}$ converge to $a$ and $b$ in $P(X)$. Thus, there exist $a$ and $b$ in $P(X)$ such that

$$\lim_{n \to \infty} z_n = a = P(a) \quad \text{and} \quad \lim_{n \to \infty} w_n = b = P(b). \quad (9)$$

Now, we have to prove that $A(a, b) = a$ and $A(b, a) = b$. On the contrary, suppose that $A(a, b) \neq a$ or $A(b, a) \neq b$.

From Equation (4) and Lemma 3, we obtain that

$$\frac{1}{2b} S_b(A(a, b), A(a, b), a)$$

$$\leq \lim_{n \to \infty} \inf_\phi \frac{1}{2b} S_b(A(a, b), A(a, b), B(x_{2n+1}, y_{2n+1}))$$

$$= \phi \left( \max \left\{ \begin{array}{c} 0, 0, S_b(A(a, b), A(a, b), a), S_b(A(a, b), A(a, b), \beta), S_b(z_{2n+1}, z_{2n+2}, z_{2n}), S_b(w_{2n+1}, w_{2n+2}, w_{2n}), S_b(A(a, b), A(a, b), Q(x_{2n+1})), S_b(z_{2m+1}, z_{2n+1}, z_{2n}), S_b(w_{2m+1}, w_{2n+1}, w_{2n}), S_b(A(a, b), A(a, b), Q(y_{2n+1})), S_b(z_{2m+1}, z_{2n+1}, z_{2n}), S_b(w_{2m+1}, w_{2n+1}, w_{2n}) \end{array} \right\} \right).$$
Similarly,

\[
\frac{1}{2b} S_b(A(b,a), A(b,a), \beta) \leq \phi \left( \max \left\{ S_b(A(a,b), A(a,b), a), S_b(A(b,a), A(b,a), \beta) \right\} \right).
\]

Thus,

\[
\frac{1}{2b} \max \left\{ S_b(A(a,b), A(a,b), a), S_b(A(b,a), A(b,a), \beta) \right\} \leq \phi \left( \max \left\{ S_b(A(a,b), A(a,b), a), S_b(A(b,a), A(b,a), \beta) \right\} \right).
\]

By the definition of \( \phi \), it follows that \( A(a,b) = a = P(a) \) and \( A(b,a) = \beta = P(b) \). Since \( (A, P) \) is \( \alpha \)-compatible pair, we have that \( A(\alpha, \beta) = P(\alpha) \) and \( A(\beta, \alpha) = P(\beta) \).

From Equation (4) and Lemma 3, we have

\[
\frac{1}{2b} S_b(A(\alpha, \beta), A(\alpha, \beta), a)
\leq \limsup_{n \to \infty} 2b S_b(A(a,b), A(a,b), B(x_{2n+1}, y_{2n+1}))
\leq \limsup_{n \to \infty} \phi \left( \max \left\{ S_b(P(a), P(a), Q(x_{2n+1})), S_b(P(\beta), P(\beta), Q(y_{2n+1})), S_b(A(\alpha, \beta), A(\alpha, \beta), a), S_b(A(\beta, \alpha), A(\beta, \alpha), \beta) \right\} \right)
\leq \limsup_{n \to \infty} \phi \left( \max \left\{ S_b(A(\alpha, \beta), A(\alpha, \beta), a), S_b(A(b,a), A(b,a), \beta) \right\} \right)
\leq \phi \left( \max \left\{ 2b S_b(A(\alpha, \beta), A(\alpha, \beta), a), 2b S_b(A(\beta, \alpha), A(\beta, \alpha), \beta), 0, 0, 2b S_b(a, a, A(\alpha, \beta), A(\beta, \alpha), a), 2b S_b(\beta, \beta, A(\alpha, \beta), A(\beta, \alpha)) \right\} \right).
\]
By the definition of \( \phi \), it follows that \( A(a, \beta) = a = P(a) \) and \( A(\beta, a) = \beta = P(\beta) \).

Therefore, \((a, \beta)\) is a common coupled fixed point of \(A\) and \(P\).

Since \( A(X \times X) \subseteq Q(X) \), there exist \( x \) and \( y \) in \( X \) such that \( A(a, \beta) = a = Q(x) \) and \( A(\beta, a) = \beta = Q(y) \).

From Equation (4), we have

\[
S_b(a, a, B(x, y)) = S_b(A(a, \beta), A(\alpha, \beta), B(x, y)) \leq 2b^2 S_b(A(a, \beta), A(\alpha, \beta), B(x, y)) \leq \phi \max \left\{ \frac{S_b(P(a), P(x), Q(y)), S_b(P(\beta), P(\beta), Q(y)), S_b(A(a, \beta), A(\alpha, \beta), P(x)), S_b(A(\beta, a), A(\alpha, \beta), P(\beta)), S_b(B(x, y), B(x, y), Q(y)), S_b(B(y, x), B(y, x), Q(y)), S_b(B(a, \beta), A(\alpha, \beta), Q(y)), S_b(B(\beta, a), B(\beta, a), Q(y)), S_b(A(\alpha, \beta), A(\beta, a), Q(y)), S_b(A(\alpha, \beta), B(\beta, a), B(\beta, a), P(\beta))}{1 + S_b(P(a), P(x), Q(y)), S_b(P(\beta), P(\beta), Q(y))} \right\}
\]

Similarly,

\[
S_b(\beta, \beta, B(x, y)) \leq \phi \left( b \max \left\{ S_b(a, a, B(x, y)), S_b(\beta, \beta, B(y, x)) \right\} \right).
\]

Thus,

\[
\max \left\{ S_b(a, a, B(x, y)), S_b(\beta, \beta, B(y, x)) \right\} \leq \phi \left( b \max \left\{ S_b(a, a, B(x, y)), S_b(\beta, \beta, B(y, x)) \right\} \right).
\]

It follows that \( B(x, y) = a = Q(x) \) and \( B(y, x) = \beta = Q(y) \).

Since \((B, Q)\) is \( \alpha \)-compatible pair, we have \( B(a, \beta) = Q(a) \) and \( B(\beta, a) = Q(\beta) \).

From Equation (4), we have

\[
S_b(a, a, B(\alpha, \beta)) = S_b(A(a, \beta), A(\alpha, \beta), B(\alpha, \beta)) \leq 2b^2 S_b(A(a, \beta), A(\alpha, \beta), B(\alpha, \beta)) \leq \phi \max \left\{ \frac{S_b(P(a), P(a), Q(a)), S_b(P(\beta), P(\beta), Q(\beta)), S_b(A(a, \beta), A(\alpha, \beta), P(a)), S_b(A(\beta, a), A(\alpha, \beta), P(\beta)), S_b(B(x, y), B(x, y), Q(y)), S_b(B(y, x), B(y, x), Q(y)), S_b(B(a, \beta), A(\alpha, \beta), Q(a)), S_b(B(\beta, a), B(\beta, a), Q(\beta)), S_b(A(\alpha, \beta), A(\beta, a), Q(y))S_b(B(a, \beta), B(\beta, a), P(\beta))}{1 + S_b(P(a), P(a), Q(a)), S_b(P(\beta), P(\beta), Q(\beta))} \right\}
\]

Similarly,

\[
S_b(\beta, \beta, B(\beta, a)) \leq \phi \left( b \max \left\{ S_b(a, a, B(\alpha, \beta)), S_b(\beta, \beta, B(\beta, a)) \right\} \right).
\]

Thus,

\[
\max \left\{ S_b(a, a, B(\alpha, \beta)), S_b(\beta, \beta, B(\beta, a)) \right\} \leq \phi \left( b \max \left\{ S_b(a, a, B(\alpha, \beta)), S_b(\beta, \beta, B(\beta, a)) \right\} \right).
\]

It follows that \( B(\alpha, \beta) = a = Q(a) \) and \( B(\beta, a) = \beta = Q(\beta) \).
Therefore, \((a, \beta)\) is a common coupled fixed point of \(A, B, P\) and \(Q\).

To prove uniqueness, let us take that \((a^1, \beta^1)\) is another common coupled fixed point of \(A, B, P\) and \(Q\).

From Equation (4), we have
\[
S_b(a, a, a^1) \leq 2b^2S_b(a, a, a^1) = 2b^2S_b(A(a, \beta), A(a, \beta), B(a, \beta)) \leq \phi \left( \max \left\{ \begin{array}{c}
S_b(a, a, a^1), S_b(\beta, \beta, \beta^1), S_b(a, a, a), \\
S_b(\beta, \beta, \beta), S_b(a^1, a^1, a^1), S_b(\beta^1, \beta^1, \beta^1), \\
S_b(a, a^1)S_b(a^1, a^1)S_b(\beta^1, \beta^1)S_b(1, 1, \beta^1), \\
1 + S_b(a, a^1), 1 + S_b(\beta, \beta^1)
\end{array} \right\} \right).
\]

Similarly,
\[
S_b(\beta, \beta, \beta^1) \leq \phi(\max\{bS_b(a, a, a^1), bS_b(\beta, \beta, \beta^1)\}).
\]

Thus,
\[
\max \left\{ S_b(a, a, a^1), S_b(\beta, \beta, \beta^1) \right\} \leq \phi \left( b \max \left\{ S_b(a, a, a^1), S_b(\beta, \beta, \beta^1) \right\} \right).
\]

It follows that \(a = a^1\) and \(\beta = \beta^1\). Hence, \((a, \beta)\) is a unique common coupled fixed point of \(A, B, P\) and \(Q\).

\(\square\)

Now, we give one example which support our main theoretical result.

**Example 2.** Let \(X = [0, 1]\) and let \(S_b : X \times X \times X \to \mathbb{R}^+\) be defined by \(S_b(x, y, z) = (|y + z| - 2x) + |y - z|)^2\). Then, \(S_b\) is a \(S_b\)-metric space with \(b = 4\). Define \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) by \(\phi(t) = \frac{t}{4^5}\), \(A, B : X \times X \to X\) and \(P, Q : X \to X\) by \(A(x, y) = \frac{x^2 + y^2}{4^5}\), \(B(x, y) = \frac{x^2 + y^2}{4^5}\), \(P(x) = \frac{x^2}{4}\) and \(Q(x) = \frac{x^2}{16}\), respectively. Then, we have
where

\[ S_b(A(x, y), A(x, y), B(u, v)) \]

\[ = 2 \left( 4^2 \right) (|A(x, y) + B(u, v) - 2A(x, y)| + |A(x, y) - B(u, v)|)^2 \]

\[ = 2 \left( 4^2 \right) (2|A(x, y) - B(u, v)|)^2 \]

\[ = 2 \left( 4^2 \right) (|A(x, y) - B(u, v)|)^2 \]

\[ = 2 \left( 4^4 \right) \frac{x^2 + y^2 - u^2 + v^2}{4^4} \]

\[ = 2 \left( 4^4 \right) \frac{|4x^2 - u^2 + 4y^2 - v^2|}{4^4} \]

\[ = \frac{2 \left( 4^4 \right)}{4^4} \left( \frac{1}{4} \left\{ \frac{|4x^2 - u^2|}{16} + \frac{|4y^2 - v^2|}{16} \right\} \right)^2 \]

\[ \leq \frac{2 \left( 4^4 \right)}{4^4} \left( \frac{1}{4} \left\{ \frac{|4x^2 - u^2|}{16} + \frac{|4y^2 - v^2|}{16} \right\} \right)^2 \]

\[ \leq \frac{2 \left( 4^4 \right)}{4^4} \left( \max \left\{ \frac{4x^2 - u^2}{16}, \frac{4y^2 - v^2}{16} \right\} \right)^2 \]

\[ \leq \frac{2 \left( 4^4 \right)}{4^4} \left( \max \left\{ \frac{4x^2 - u^2}{16}, \frac{4y^2 - v^2}{16} \right\} \right)^2 \]

\[ \leq \frac{2 \left( 4^4 \right)}{4^4} \max \left\{ \frac{4x^2 - u^2}{16}, \frac{4y^2 - v^2}{16} \right\} \]

\[ \leq \frac{2 \left( 4^4 \right)}{4^4} \max \left\{ \frac{4x^2 - u^2}{16}, \frac{4y^2 - v^2}{16} \right\} \]

It is clear that all conditions of Theorem 1 are satisfied and \((0, 0)\) is a unique common coupled fixed point of \(A, B, P\) and \(Q\).

Putting \(A = B = P = Q\) in Theorem 1, we obtain the next important result on unique fixed point.

**Theorem 2.** Let \((X, S_b)\) be a complete \(S_b\)-metric space. Suppose that \(A : X \times X \to X\) satisfies condition

\[ 2b^5 S_b(A(x, y), A(x, y), A(u, v)) \]

\[ \leq \phi \left( \max \left\{ \frac{S_b(x, x, u), S_b(y, y, v), S_b(A(x, y), A(x, y), x), S_b(A(y, x), A(y, x), y), S_b(A(u, v), A(u, v), u), S_b(A(v, u), A(v, u), v), S_b(A(x, y), A(u, v), u), S_b(A(y, x), A(v, u), x), S_b(A(u, v), A(y, x), v), S_b(A(v, u), B(v, u), y)}{1 + S_b(x, x, u)} \right\} \right) \]

for all \(x, y, u, v \in X, \phi \in \Phi\). Then, \(A\) has a unique coupled fixed point in \(X \times X\).

**3. Application**

In this section, we study the existence of a unique solution to an initial value problem, as an application to Theorem 2. Consider the initial value problem:

\[ x^1(t) = f(t, x(t), x(t)), \quad t \in I = [0, 1], \quad x(0) = x_0 \]

(10)

where \(f : I \times \left[ \frac{x_0}{4}, \infty \right) \times \left[ \frac{x_0}{4}, \infty \right) \to \left[ \frac{x_0}{4}, \infty \right)\) and \(x_0 \in \mathbb{R}\).
Theorem 3. Consider the initial value problem in Equation (10) with 

\[ f \in C \left( I \times \left[ \frac{x_0}{4}, \infty \right) \times \left[ \frac{x_0}{4}, \infty \right) \right) \] 

and 

\[ \int_0^t f(s, x(s), y(s)) ds = \frac{1}{\sqrt{5}} \min \left\{ \int_0^t f(s, x(s), x(s)) ds, \int_0^t f(s, y(s), y(s)) ds \right\} . \]

Then, there exists a unique solution in 

\[ C \left( I \times \left[ \frac{x_0}{4}, \infty \right) \times \left[ \frac{x_0}{4}, \infty \right) \right) \] 

for the initial value problem in Equation (10).

Proof of Theorem. The integral equation corresponding to the initial value problem in Equation (10) is 

\[ x(t) = x_0 + \int_0^t f(s, x(s), x(s)) ds. \]

Let \( X = C \left( I \times \left[ \frac{x_0}{4}, \infty \right) \times \left[ \frac{x_0}{4}, \infty \right) \right) \) and let \( S(x, y, z) = (|y + z - 2x| + |y - z|)^2 \) for \( x, y \in X \). Define \( \phi : [0, \infty) \to [0, \infty) \) by \( \phi(t) = \frac{4t}{5} \) and \( A : X \times X \to X \) by

\[ A(x, y)(t) = x_0 + \int_0^t f(s, x(s), y(s)) ds. \] (11)

Now, we have 

\[ \begin{align*}
&= \{ |A(x, y)(t) + A(u, v)(t) - 2A(x, y)(t)| + |A(x, y)(t) - A(u, v)(t)| \}^2 \\
&= 4 \left| A(x, y)(t) - A(u, v)(t) \right|^2 \\
&= 4 \left| \int_0^t f(s, x(s), y(s)) ds - \int_0^t f(s, u(s), v(s)) ds \right|^2 \\
&= 4 \left| \frac{1}{\sqrt{5}} \min \left\{ \int_0^t f(s, x(s), x(s)) ds, \int_0^t f(s, y(s), y(s)) ds \right\} - \frac{1}{\sqrt{5}} \min \left\{ \int_0^t f(s, u(s), u(s)) ds, \int_0^t f(s, v(s), v(s)) ds \right\} \right|^2 \\
&\leq \frac{4}{5} \max \left\{ \left| \int_0^t f(s, x(s), x(s)) ds - \int_0^t f(s, u(s), u(s)) ds \right|^2, \left| \int_0^t f(s, y(s), y(s)) ds - \int_0^t f(s, v(s), v(s)) ds \right|^2 \right\} \\
&= \frac{4}{5} \max \left\{ \left| \int_0^t f(s, x(s), x(s)) ds - \int_0^t f(s, u(s), u(s)) ds \right|^2, \left| \int_0^t f(s, y(s), y(s)) ds - \int_0^t f(s, v(s), v(s)) ds \right|^2 \right\} \\
&= \frac{1}{5} \max \left\{ 4 \left| x(t) - u(t) \right|^2, 4 \left| y(t) - v(t) \right|^2 \right\} \\
&= \frac{1}{5} \max \{ S(x, u), S(y, v) \} \\
&\leq \phi(M(x, u, y, v)).
\end{align*} \]

Hence, from Theorem 2, we conclude that \( A \) has a unique coupled fixed point in \( X \). \( \Box \)
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