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Hyers–Ulam Stability and Existence of Solutions for Differential Equations with Caputo–Fabrizio Fractional Derivative

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Abstract: In this paper, the Hyers–Ulam stability of linear Caputo–Fabrizio fractional differential equation is established using the Laplace transform method. We also derive a generalized Hyers–Ulam stability result via the Gronwall inequality. In addition, we establish existence and uniqueness of solutions for nonlinear Caputo–Fabrizio fractional differential equations using the generalized Banach fixed point theorem and Schaefer's fixed point theorem. Finally, two examples are given to illustrate our main results.

Keywords: Caputo–Fabrizio fractional differential equations; Hyers–Ulam stability

MSC: 34A08; 34D20

1. Introduction

Fractional differential operators describe mechanical and physical processes with historical memory and spatial global correlation and for the basic theory—see [1–3]. Results on existence, stability and controllability for differential equations with Caputo, Riemann–Liouville and Hilfer type fractional derivatives can be found, for example, in [4–19]. Caputo and Fabrizio [20] introduced a new nonlocal derivative without a singular kernel and Atangana and Nieto [21] studied the numerical approximation of this new fractional derivative and established a modified resistance loop capacitance (RLC) circuit model. Losada and Nieto [22] presented a fractional integral corresponding to the Caputo–Fabrizio fractional derivative and introduced Caputo–Fabrizio fractional differential equations and established existence and uniqueness results. Baleanu et al. [23] extended the study to Caputo–Fabrizio fractional integro-differential equations and obtained the approximate solution. Franc and Goufo [24] established a new Korteweg–de Vries–Burgers equation involving the Caputo–Fabrizio fractional derivative with no singular kernel and presented existence and uniqueness results and also gave numerical approximations.

Hyers–Ulam stability is a concept that provides an approximate solution for the exact solution in a simple form for differential equations. A Laplace transform method is applied to show the Hyers–Ulam stability for integer order differential equations in [25,26] and Wang and Li [27] adopted the idea and applied a Laplace transform method to show the Hyers–Ulam stability for fractional order differential equations involving Caputo derivatives. There are many papers on differential equations involving fractional derivatives–see, for example, [28–36]. However, there are only a few papers on the Hyers–Ulam stability for differential equations with the Caputo–Fabrizio fractional derivative. In [37], Wang et al. offered the Ulam stability for the fractional differential equations with the Caputo derivative.

First, we recall the well-known Caputo fractional derivative [2] of order β, given by

$$(\mathbb{D}^{\beta}y)(x) = \frac{1}{\Gamma(1-\beta)}\int_a^x \frac{\dot{f}(s)}{(x-s)^{\beta}}ds,\ 0<\beta<1,$$

where $f \in C^1(a,b), b > a$. By changing the kernel $(x-s)^{-\beta}$ with the function $\exp(-\frac{\beta}{1-\beta}(x-s))$ and $\frac{1}{\Gamma(1-\beta)}$ by $\frac{1}{\sqrt{2\pi(1-\alpha^2)}}$, we obtain the new definition of fractional derivative without a singular kernel $({}^{CF}\mathbb{D}^{\alpha}y)(x)$—see Definition 1 for details.

In this paper, we study Hyers–Ulam stability and existence and uniqueness of solutions for the following Caputo–Fabrizio fractional derivative equations:

$$({}^{CF}\mathbb{D}^{\alpha}y)(x) - \lambda({}^{CF}\mathbb{D}^{\beta}y)(x) = u(x),\ x \in [0,T],\ 0<\alpha,\beta<1, \tag{1}$$

and

$$({}^{CF}\mathbb{D}^{\alpha}y)(x) = f(x,y(x)),\ x \in [0,T],\ 0<\alpha<1, \tag{2}$$

where $({}^{CF}\mathbb{D}^{\gamma}y)(\cdot)$ denotes the Caputo–Fabrizio derivative for y with the order $0 < \gamma < 1$ (see Definition 1), $\lambda \in \mathbb{R}$, $u : [0,T] \to \mathbb{R}$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ will be specified later.

The main contributions are as follows: we obtain a simple result to check whether the approximate solution is near the exact solution for linear Equation (1), which implies Hyers–Ulam stability and generalized Hyers–Ulam stability on the finite time interval. In addition, we present a condition to derive existence and uniqueness of solutions for nonlinear Equation (2) using the generalized Banach fixed point theorem (this improves the result in (Theorem 1, [22])). In addition, we establish sufficient conditions to guarantee the existence of solutions for nonlinear Equation (2) using Schaefer's fixed point theorem. Based on the existence and uniqueness result, we prove the Hyers–Ulam stability of (2) via the Gronwall inequality.

2. Preliminaries

Let $C(I,\mathbb{R})$ be the Banach space of all continuous functions from I into $\mathbb{R}$ with the norm $\|y\|_C := \sup\{|y(x)| : x \in I\}$.

Definition 1 (see [22]). *Let $0<\alpha<1$, $h \in C^1[0,b)$ and $b > 0$. The Caputo–Fabrizio fractional derivative for a function h of order α is defined by*

$${}^{CF}\mathbb{D}^{\alpha}h(\tau) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^{\tau}\exp(-\frac{\alpha}{1-\alpha}(\tau-x))h'(x)dx,\ \tau\geq 0,$$

where $M(\alpha)$ is a normalization constant depending on α. Note that $({}^{CF}\mathbb{D}^{\alpha})(h) = 0$ if and only if h is a constant function.

Definition 2 (see Definition 1, [22]). *Let $0<\alpha<1$. The Caputo–Fabrizio fractional integral for a function h of order α is defined by*

$${}^{CF}I^{\alpha}h(\tau) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}h(\tau) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^{\tau}h(x)dx,\ \tau \geq 0.$$

Theorem 1 (see [20,22]). *Let $\alpha \in (0,1)$. Then,*

$$\mathcal{L}[{}^{CF}\mathbb{D}^{\alpha}h(\tau)](s) = \frac{(2-\alpha)M(\alpha)}{2(s+\alpha(1-s)}(s\mathcal{L}[h(\tau)](s) - h(0)),\ s > 0.$$

Motivated by (Definition 2.3, [37]), we introduce the following definition.

Definition 3. *Let $0 < \alpha, \beta < 1$ and $u : [0,T] \to \mathbb{R}$ be a continuous function. Then, (1) is Hyers–Ulam stable if there exists $K > 0$ and $\epsilon > 0$ such that, for each solution $y \in C([0,T],\mathbb{R})$ of (1),*

$$|{}^{CF}\mathbb{D}^{\alpha}y(x) - \lambda {}^{CF}\mathbb{D}^{\beta}y(x) - u(x)| \leq \epsilon,\ \forall x \in [0,T], \tag{3}$$

and there exists a solution $z \in C([0,T),\mathbb{R})$ of (2) with

$$|y(x) - z(x)| \leq K\epsilon,\ \forall x \in [0,T].$$

Definition 4. *Let $0 < \alpha, \beta < 1$, $u : [0,T] \to \mathbb{R}$ be a continuous function and $G : [0,T] \to \mathbb{R}_+$ be continuous functions. Then, (1) is generalized Hyers–Ulam–Rassias stable with respect to G if there exists a constant $c_{f,G} > 0$ such that for each solution $y \in C([0,T],\mathbb{R})$ of (1),*

$$|{}^{CF}\mathbb{D}^{\alpha}y(x) - \lambda {}^{CF}\mathbb{D}^{\beta}y(x) - u(x))| \leq G(x),\ \forall x \in [0,T], \tag{4}$$

and there exists a solution $z \in C([0,T],\mathbb{R})$ of (2) with

$$|y(x) - z(x)| \leq c_{f,G}G(x),\ \forall x \in [0,T].$$

Definition 5. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, (2) is Hyers–Ulam stable if there exists $K > 0$ and $\epsilon > 0$ such that for each solution $y \in C([0,T],\mathbb{R})$ of (2),*

$$|{}^{CF}\mathbb{D}^{\alpha}y(x) - f(x,y(x))| \leq \epsilon,\ \forall x \in [0,T], \tag{5}$$

and there exists a solution $z \in C([0,T),\mathbb{R})$ of (2) with

$$|y(x) - z(x)| \leq K\epsilon,\ \forall x \in [0,T].$$

Definition 6. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $G : [0,T] \to \mathbb{R}_+$ be continuous functions. Then, (2) is generalized Hyers–Ulam–Rassias stable with respect to G if there exists a constant $c_{f,G} > 0$ such that, for each solution $y \in C([0,T],\mathbb{R})$ of (2),*

$$|{}^{CF}\mathbb{D}^{\alpha}y(x) - f(x,y(x))| \leq G(x),\ \forall x \in [0,T], \tag{6}$$

and there exists a solution $z \in C([0,T],\mathbb{R})$ of (2) with

$$|y(x) - z(x)| \leq c_{f,G}G(x),\ \forall x \in [0,T].$$

3. Stability Results for the Linear Equation

In this section, we study Hyers–Ulam and generalized Hyers–Ulam-Rassias stability of (1).

Theorem 2. *Let $0 < \beta, \alpha < 1$, $\lambda \in \mathbb{R}$, and $u(x)$ be a given real function on $[0,T]$. If a function $y : [0,T] \to \mathbb{R}$ satisfies the inequality*

$$|({}^{CF}\mathbb{D}^{\alpha}y)(x) - \lambda({}^{CF}\mathbb{D}^{\beta}y)(x) - u(x)| \leq \varepsilon \tag{7}$$

for each $x \in [0,T]$ and $\varepsilon > 0$, then there exists a solution $y_a : [0,T] \to \mathbb{R}$ of (1) such that

$$|y(x) - y_a(x)| \leq 2|\frac{C}{A}|\varepsilon + 2\left|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right| \max\{1, \exp(-\frac{B}{A}T)\}x\varepsilon + 2|\frac{\alpha\beta}{B}|x\varepsilon, \tag{8}$$

where

$$\begin{cases} A = (1-\beta)(2-\alpha)M(\alpha) - \lambda(2-\beta)M(\beta)(1-\alpha), \\ B = (2-\alpha)M(\alpha)\beta - \lambda(2-\beta)M(\beta)\alpha, \\ C = (1-\beta)(1-\alpha), \\ D = \alpha + \beta - 2\alpha\beta. \end{cases} \tag{9}$$

Proof. Let

$$F(x) = ({}^{CF}\mathbb{D}^\alpha y)(x) - \lambda({}^{CF}\mathbb{D}^\beta y)(x) - u(x),\ x \in [0,T]. \tag{10}$$

Taking the Laplace transform of (10) via Theorem 1, and we have

$$\begin{aligned} \mathcal{L}\{F(x)\}(s) &= \mathcal{L}\{({}^{CF}\mathbb{D}^\alpha y)(x) - \lambda({}^{CF}\mathbb{D}^\beta y)(x) - u(x)\}(s) \\ &= \mathcal{L}\{({}^{CF}\mathbb{D}^\alpha y)(x)\}(s) - \lambda\mathcal{L}\{({}^{CF}\mathbb{D}^\beta y)(x)\}(s) - \mathcal{L}\{u(x)\}(s) \\ &= [\frac{(2-\alpha)M(\alpha)}{2(s+\alpha(1-s))} - \lambda\frac{(2-\beta)M(\beta)}{2(s+\beta(1-s))}]s\mathcal{L}\{y(x)\}(s) \\ &\quad +[-\frac{(2-\alpha)M(\alpha)}{2(s+\alpha(1-s))} + \lambda\frac{(2-\beta)M(\beta)}{2(s+\beta(1-s))}]y(0) - \mathcal{L}\{u(x)\}(s), \end{aligned} \tag{11}$$

where $\mathcal{L}\{F\}$ denotes the Laplace transform of the function F. From (11), one has

$$\begin{aligned} &\mathcal{L}\{y(x)\}(s) \\ =\ &\frac{1}{s}y(0) + \frac{1}{s}\frac{2(s+\alpha(1-s))(s+\beta(1-s))}{(2-\alpha)M(\alpha)(s+\beta(1-s)) - \lambda(2-\beta)M(\beta)(s+\alpha(1-s))} \\ &\times\Big(\mathcal{L}\{u(x)\}(s) + \mathcal{L}\{F(x)\}(s)\Big) \\ =\ &\frac{1}{s}y(0) + 2\left(\frac{C}{A} + \frac{AD-BC}{A^2}\frac{1}{s+\frac{B}{A}} + \frac{\alpha\beta}{B}\frac{1}{s} - \frac{\alpha\beta}{B}\frac{1}{s+\frac{B}{A}}\right)(\mathcal{L}\{u(x)\}(s) + \mathcal{L}\{F(x)\}(s)), \end{aligned} \tag{12}$$

where A, B, C, D are defined as in (9). Set

$$y_a(x) = y(0) + 2\frac{C}{A}u(x) + 2\left(\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right)\int_0^x \exp(-\frac{B}{A}t)u(x-t)dt + 2\frac{\alpha\beta}{B}\int_0^x u(x-t)dt. \tag{13}$$

Taking the Laplace transform of (13), one has

$$\begin{aligned} &\mathcal{L}\{y_a(x)\}(s) \\ =\ &\frac{1}{s}y(0) + 2\frac{C}{A}\mathcal{L}\{u(x)\}(s) + 2\left(\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right)\frac{1}{s+\frac{B}{A}}\mathcal{L}\{u(x)\}(s) + 2\frac{\alpha\beta}{B}\frac{1}{s}\mathcal{L}\{u(x)\}(s) \\ =\ &\frac{1}{s}y(0) + 2\left(\frac{C}{A} + \frac{AD-BC}{A^2}\frac{1}{s+\frac{B}{A}} + \frac{\alpha\beta}{B}\frac{1}{s} - \frac{\alpha\beta}{B}\frac{1}{s+\frac{B}{A}}\right)\mathcal{L}\{u(x)\}(s). \end{aligned} \tag{14}$$

Note that

$$\begin{aligned} &\mathcal{L}\{({}^{CF}\mathbb{D}^\alpha y_a)(x) - \lambda({}^{CF}\mathbb{D}^\beta y_a)(x)\}(s) \\ =\ &\frac{(2-\alpha)M(\alpha)(s+\beta(1-s)) - \lambda(2-\beta)M(\beta)(s+\alpha(1-s))}{2(s+\alpha(1-s))(s+\beta(1-s))}(s\mathcal{L}\{y_a(x)\}(s) - y(0)). \end{aligned} \tag{15}$$

Substituting (14) into (15), we obtain

$$\mathcal{L}\{({}^{CF}\mathbb{D}^{\alpha}y_a)(x) - \lambda({}^{CF}\mathbb{D}^{\beta}y_a)(x)\}(s) = \mathcal{L}\{u(x)\},$$

which yields that $y_a(x)$ is a solution of Equation (1) since $\mathcal{L}$ is one-to-one. From (12) and (14), we have

$$\mathcal{L}\{y(x) - y_a(x)\}(s) = 2\left(\frac{C}{A} + \frac{AD-BC}{A^2}\frac{1}{s+\frac{B}{A}} + \frac{\alpha\beta}{B}\frac{1}{s} - \frac{\alpha\beta}{B}\frac{1}{s+\frac{B}{A}}\right)\mathcal{L}\{F(x)\}.$$

This implies that

$$y(x) - y_a(x) = 2\frac{C}{A}F(x) + 2(\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B})(\exp(-\frac{B}{A}x) * F(x) + 2\frac{\alpha\beta}{B}(1 * F(x)),$$

so

$$\begin{aligned}
& |y(x) - y_a(x)| \\
= \quad & \left|2\frac{C}{A}F(x) + 2(\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B})(\exp(-\frac{B}{A}x) * F(x) + 2\frac{\alpha\beta}{B}(1 * F(x))\right| \\
\leq \quad & 2|\frac{C}{A}F(x)| + 2\left|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right|| \exp(-\frac{B}{A}x) * F(x)| + 2|\frac{\alpha\beta}{B}||1 * F(x)| \\
\leq \quad & 2|\frac{C}{A}||F(x)| + 2\left|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right| \int_0^x |\exp(-\frac{B}{A}t)||F(x-t)|dt + 2|\frac{\alpha\beta}{B}| \int_0^x |F(x-t)|dt) \\
\leq \quad & 2|\frac{C}{A}||F(x)| + 2\left|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right|\varepsilon \int_0^x \max\{1, \exp(-\frac{B}{A}(T))\}dt + 2|\frac{\alpha\beta}{B}|\varepsilon \int_0^x 1dt) \\
\leq \quad & 2|\frac{C}{A}|\varepsilon + 2\left|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right|x\max\{1, \exp(-\frac{B}{A}T)\}\varepsilon + 2|\frac{\alpha\beta}{B}|x\varepsilon.
\end{aligned}$$

The proof is complete. □

Remark 1. *If $T < \infty$, then (1) is Hyers–Ulam stable with the constant*

$$K = 2|\frac{C}{A}| + 2|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}|\max\{1, \exp(-\frac{B}{A}T)\}T + 2|\frac{\alpha\beta}{B}|T.$$

Remark 2. *Let $0 < \beta, \alpha < 1$, $\lambda \in \mathbb{R}$, and $u(x)$ be a given real function on $[0, T]$. If a function $y : [0, T] \to \mathbb{R}$ satisfies the inequality*

$$|({}^{CF}\mathbb{D}^{\alpha}y)(x) - \lambda({}^{CF}\mathbb{D}^{\beta}y)(x) - u(x)| \leq G(x), \tag{16}$$

this implies that

$$|F(x)| \leq G(x)$$

for each $x \in [0, T]$ and some function $G(x) > 0$, where F is defined in (10).

From Theorem 2, then there exists a solution $y_a : [0, T] \to \mathbb{R}$ of (1) such that

$$y(x) - y_a(x) = 2\frac{C}{A}F(x) + 2(\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B})(\exp(-\frac{B}{A}x) * F(x) + 2\frac{\alpha\beta}{B}(1 * F(x)),$$

and

$$
\begin{aligned}
& |y(x) - y_a(x)| \\
\leq\ & 2|\frac{C}{A}F(x)| + 2\left|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right|| \exp(-\frac{B}{A}x) * F(x)| + 2|\frac{\alpha\beta}{B}||1 * F(x)| \\
\leq\ & 2|\frac{C}{A}||F(x)| + 2\left|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right| \max\{1, \exp(-\frac{B}{A}T)\}| \int_0^x F(x-t)dt| + 2|\frac{\alpha\beta}{B}|| \int_0^x F(x-t)dt| \\
\leq\ & 2|\frac{C}{A}||F(x)| + 2\left|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right| \max\{1, \exp(-\frac{B}{A}T)\}|F(x)| + 2|\frac{\alpha\beta}{B}||F(x)| \\
\leq\ & 2\left[|\frac{C}{A}| + \left|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right| \max\{1, \exp(-\frac{B}{A}(T))\} + |\frac{\alpha\beta}{B}|\right] G(x)
\end{aligned}
$$

provided that

$$\int_0^x F(t)dt \leq F(x)$$

for any $x \in [0, T]$, where F is defined in (10) and A, B, C, D are defined as in (9). Thus, (2) is generalized Hyers–Ulam stable with respect to G on $[0, T]$.

4. Existence and Stability Results for the Nonlinear Equation

We introduce the following conditions:

$[A1]$: $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

$[A2]$: There exists a $k_f > 0$ such that

$$|f(x, y) - f(x, g)| \leq k_f |y - g|, \ \ \forall y, g \in \mathbb{R}, x \in [0, T].$$

$[A3]$: There exists a constant $L > 0$ such that

$$|f(x, y)| \leq L(1 + |y|)$$

for each $x \in [0, T]$ and all $y \in \mathbb{R}$.

Let $a_\alpha = \frac{2(1-\alpha)}{2-\alpha} M(\alpha)$, $b_\alpha = \frac{2\alpha}{2-\alpha} M(\alpha)$, $y(0) = y_0$ and $C_0 = -a_\alpha f(0, y_0) + y_0$.

Theorem 3. *Let $0 < \alpha < 1$. Assume that $[A1]$ and $[A2]$ hold. If $a_\alpha k_f < 1$, then (2) with $y(0) = y_0$ has a unique solution.*

Proof. Consider $P : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$ as follows:

$$(Py)(x) = C_0 + a_\alpha f(x, y(x)) + b_\alpha \int_0^x f(s, y(s))ds. \tag{17}$$

Note P is well defined because of $[A1]$. For all $y_1, y_2 \in C([0, T], \mathbb{R})$ and all $x \in [0, T]$, using $[A2]$, we have

$$
\begin{aligned}
& |(Py_1)(x) - (Py_2)(x)| \\
\leq\ & a_\alpha |f(x, y_1(x)) - f(x, y_2(x))| + b_\alpha \int_0^x |f(s, y_1(s)) - f(x, y_2(x))|ds \\
\leq\ & a_\alpha k_f |y_1(x) - y_2(x)| + b_\alpha \int_0^x k_f |y_1(s) - y_2(s)|ds \\
=\ & a_\alpha k_f \|y_1 - y_2\|_C + b_\alpha k_f x \|y_1 - y_2\|_C.
\end{aligned}
$$

Denote $C_n^i = \frac{n!}{(n-i)!i!}$. Next,

$$\begin{aligned}
& |(P^2y_1)(x) - (P^2y_2)(x)| \\
\leq\ & a_\alpha |f(x,(Py_1)(x)) - f(x,(Py_2)(x))| + b_\alpha \int_0^x |f(s,(Py_1)(s)) - f(x,(Py_2)(x))| ds \\
\leq\ & a_\alpha k_f |Py_1(x) - Py_2(x)| + b_\alpha \int_0^x k_f |Py_1(s) - Py_2(s)| ds \\
\leq\ & a_\alpha k_f (a_\alpha k_f \|y_1 - y_2\|_C + b_\alpha k_f x \|y_1 - y_2\|_C) \\
& + b_\alpha k_f \int_0^x (a_\alpha k_f \|y_1 - y_2\|_C + b_\alpha k_f x \|y_1 - y_2\|_C) ds \\
\leq\ & \left((k_f a_\alpha)^2 + 2k_f a_\alpha (k_f b_\alpha x) + \frac{(k_f b_\alpha x)^2}{2!} \right) \|y_1 - y_2\|_C \\
=\ & \sum_{i=0}^{2} \frac{C_2^i (k_f a_\alpha)^{2-i} (k_f b_\alpha x)^i}{i!} \|y_1 - y_2\|_C.
\end{aligned}$$

For any $m \in \mathbb{N}^+$, suppose the following inequality hold

$$|(P^m y_1)(x) - (P^m y_2)(x)| \leq \sum_{i=0}^{m} \frac{C_m^i (k_f a_\alpha)^{m-i} (k_f b_\alpha x)^i}{i!} \|y_1 - y_2\|_C.$$

Then,

$$\begin{aligned}
& |(P^{m+1}y_1)(x) - (P^{m+1}y_2)(x)| \\
\leq\ & a_\alpha |f(x,(P^m y_1)(x)) - f(x,(P^m y_2)(x))| + b_\alpha \int_0^x |f(x,(P^m y_1)(s)) - f(x,(P^m y_2)(s))| ds \\
\leq\ & \left(k_f a_\alpha \sum_{i=0}^{m} \frac{C_m^i (k_f a_\alpha)^{m-i} (k_f b_\alpha x)^i}{i!} + k_f b_\alpha \int_0^x \sum_{i=0}^{m} \frac{C_m^i (k_f a_\alpha)^{m-i} (k_f b_\alpha s)^i}{i!} ds \right) \|y_1 - y_2\|_C \\
=\ & \sum_{i=0}^{m+1} \frac{C_{m+1}^i (k_f a_\alpha)^{m+1-i} (k_f b_\alpha x)^i}{i!} \|y_1 - y_2\|_C \\
\leq\ & S(m) \|y_1 - y_2\|_C,
\end{aligned}$$

where $S(m) := \sum_{i=0}^{m+1} \frac{C_{m+1}^i (k_f a_\alpha)^{m+1-i} (k_f b_\alpha T)^i}{i!}$. Thus, for any $m \in \mathbb{N}^+$,

$$\|P^{m+1}y_1 - P^{m+1}y_2\|_C \leq S(m) \|y_1 - y_2\|_C.$$

From the condition $k_f a_\alpha < 1$ via (Theorem 2.9, [38]), one has $S(m) \to 0$ as $m \to \infty$. This implies that for any large enough $m \in \mathbb{N}^+$, $S(m) < 1$. Thus, P^m is a contraction mapping. As a result, P has a fixed point. Thus, (2) with $y(0) = y_0$ has a unique solution. This proof is complete. □

Remark 3. *In (Theorem 1, [22]), an existence and uniqueness result for (2) with $y(0) = y_0$ is established by imposing a uniformly Lipschitz condition and applying Banach's fixed point theorem with the condition $a_\alpha k_f + b_\alpha T k_f < 1$, where k_f denotes the Lipschitz constant. Here, we use the generalized Banach fixed point theorem and we weaken the condition $a_\alpha k_f + b_\alpha T k_f < 1$ in (Theorem 1, [22]) to $a_\alpha k_f < 1$.*

Next, we show that the existence of solutions for (2) via Schaefer's fixed point theorem.

Theorem 4. *Assume that $[A1]$ and $[A3]$ hold. If $a_\alpha L < 1$, then (2) with $y(0) = y_0$ has at least one solution.*

Proof. Consider P as in (17). We divide our proof into several steps.

Step 1. P is continuous.

Let y_n be a sequence such that $y_n \to y$ in $C([0,T],\mathbb{R})$. For all $x \in [0,T]$, we get

$$\begin{aligned} |Py_n(x) - Py(x)| &= |a_\alpha f(x, y_n(x)) + b_\alpha \int_0^x f(s, y_n(s))ds - a_\alpha f(x, y(x)) - b_\alpha \int_0^x f(s, y(s))ds| \\ &\leq a_\alpha |f(x, y_n(x)) - f(x, y(x))| + b_\alpha | \int_0^x f(s, y_n(s))ds - \int_0^x f(s, y(s))ds| \\ &\leq a_\alpha |f(x, y_n(x)) - f(x, y(x))| + b_\alpha \int_0^x |f(s, y_n(s)) - f(s, y(s))|ds. \\ &\leq (a_\alpha + b_\alpha T)\|f(\cdot, y_n) - f(\cdot, y)\|_C. \end{aligned}$$

This shows that P is continuous since $\|fy_n - fy\|_C \to 0$ when $n \to \infty$.

Step 2. P maps bounded sets into bounded sets of $C([0,T],\mathbb{R})$.

Indeed, we prove that for all $r > 0$, there exists a $k > 0$ such that for every $y \in B_r = \{y \in C([0,T],\mathbb{R}) : \|y\|_C \leq r\}$, we have $\|Py\|_C \leq k$. In fact, for any $x \in [0,T]$, from $[A3]$, we have

$$\begin{aligned} |Py(x)| &\leq |C_0| + a_\alpha |f(x, y(x))| + b_\alpha \int_0^x |f(s, y(s))|ds \\ &\leq |C_0| + a_\alpha L(1 + |y|) + b_\alpha L \int_0^x (1 + |y(s)|)ds \\ &\leq |C_0| + a_\alpha L(1 + \|y\|_C) + b_\alpha TL|(1 + \|y\|_C) \\ &\leq |C_0| + a_\alpha L(1 + r) + b_\alpha TL(1 + r) \\ &= |C_0| + (a_\alpha + b_\alpha T)L(1 + r), \end{aligned}$$

which implies that

$$\|Py\| \leq |C_0| + (a_\alpha + b_\alpha T)L(1 + r) := k.$$

Step 3. P maps bounded sets into equicontinuous sets in $C([0,T],\mathbb{R})$.

Let $x_1, x_2 \in [0,T]$, with $0 \leq x_1 < x_2 \leq T, y \in B_r$. From $[A3]$, we have

$$\begin{aligned} & |Py(x_1) - Py(x_2)| \\ = \;& |a_\alpha f(x_1, y(x_1)) + b_\alpha \int_0^{x_1} f(s, y(s))ds - a_\alpha f(x_2, y(x_2)) - b_\alpha \int_0^{x_2} f(s, y(s))ds| \\ \leq \;& a_\alpha |f(x_1, y(x_1)) - f(x_2, y(x_2))| + b_\alpha | \int_0^{x_1} f(s, y(s))ds - \int_0^{x_2} f(s, y(s))ds| \\ \leq \;& a_\alpha |f(x_1, y(x_1)) - f(x_1, y(x_2))| + a_\alpha |f(x_1, y(x_2)) - f(x_2, y(x_2))| + b_\alpha | \int_{x_1}^{x_2} f(s, y(s))ds| \\ \leq \;& a_\alpha |f(x_1, y(x_1)) - f(x_1, y(x_2))| + a_\alpha |f(x_1, y(x_2)) - f(x_2, y(x_2))| + b_\alpha L(1 + r)(x_2 - x_1). \end{aligned}$$

Then, as x_1 approaches x_2, the right-hand side of the above inequality tends to zero (because of $[A1]$) as $x_1 \to x_2$. Thus, P is equicontinuous.

We can conclude that P is completely continuous from Step 1–Step 3 with the Arzela–Ascoli theorem.

Step 4. A priori bounds.

Now, we show that the set $E(P) = \{y \in C([0,T],\mathbb{R}) : y = \lambda Py \text{ for some } \lambda \in (0,1)\}$ is bounded.

Let $y \in E(P)$. Then, $y = \lambda Py$ for some $\lambda \in (0,1)$. For each $x \in [0,T]$, we have

$$\begin{aligned}|y(x)| &\leq |C_0| + a_\alpha |f(x, y(x))| + b_\alpha \int_0^x |f(s, y(s))| ds \\ &\leq |C_0| + a_\alpha L(1 + |y(x)|) + b_\alpha L \int_0^x (1 + |y(s)|) ds \\ &\leq K + a_\alpha L|y(x)| + b_\alpha L \int_0^x |y(s)| ds \quad (K = |C_0| + a_\alpha L + b_\alpha LT).\end{aligned}$$

Using the condition $1 - a_\alpha L > 0$, one has

$$|y(x)| \leq \frac{K}{1 - a_\alpha L} + \frac{b_\alpha L}{1 - a_\alpha L} \int_0^x |y(s)| ds,$$

and Gronwall's inequality yields

$$|y(x)| \leq \frac{K}{1 - a_\alpha L} \exp\left(\frac{b_\alpha LT}{1 - a_\alpha L}\right) < \infty.$$

Then, the set $E(P)$ is bounded.

Schaefer's fixed point theorem guarantees that P has a fixed point, which is a solution of (2). The proof is finished. □

In the following, we consider (2) and (6) to discuss the generalized Ulam–Hyers–Rassias stability. We need the following condition.

$[A4]$: Let $G \in C([0, T], \mathbb{R}_+)$ be an increasing function and there exists $\lambda_G > 0$ such that

$$\int_0^x G(s) ds \leq \lambda_G G(x), \ \ \forall x \in [0, T].$$

Theorem 5. *Assumptions $[A1]$, $[A2]$ and $[A4]$ hold. If $a_\alpha k_f < 1$, then (2) is generalized Ulam–Hyers–Rassias stable with respect to G on $[0, T] (T < \infty)$.*

Proof. Let $g \in C([0, T], \mathbb{R})$ be a solution of (6). From Theorem 3,

$$\begin{cases} {}^{CF}\mathbb{D}^\alpha y(x) = f(x, y(x)), \ 0 < \alpha < 1, \ t \in [0, T), \\ y(0) = C_0, \end{cases} \tag{18}$$

has the unique solution

$$y(x) = C_0 + a_\alpha f(x, y(x)) + b_\alpha \int_0^x f(s, y(s)) ds, \ x \in [0, T].$$

From (6), we have

$$\begin{aligned}\left|g(x) - C_0 - a_\alpha f(x, g(x)) - b_\alpha \int_0^x f(s, g(s)) ds\right| &\leq a_\alpha G(x) + b_\alpha \int_0^x G(s) ds \\ &\leq (a_\alpha + b_\alpha \lambda_G) G(x), \ \ x \in [0, T].\end{aligned}$$

Thus,

$$
\begin{aligned}
& |g(x)-y(x)| \\
\le\ & \left| g(x) - C_0 - a_\alpha f(x,y(x)) - b_\alpha \int_0^x f(s,y(s))ds \right| \\
\le\ & \Big| g(x) - C_0 - a_\alpha f(x,g(x)) - b_\alpha \int_0^x f(s,g(s))ds \\
& + a_\alpha f(x,y(x)) + b_\alpha \int_a^x f(s,y(s))ds - a_\alpha f(x,y(x)) - b_\alpha \int_0^x f(s,y(s))ds \Big| \\
\le\ & \left| g(x) - C_0 - a_\alpha f(x,g(x)) - b_\alpha \int_0^x f(s,g(s))ds \right| \\
& + a_\alpha |f(x,y(x)) - f(x,g(x))| + b_\alpha \int_0^x |f(s,y(s)) - f(s,g(s))|ds \\
\le\ & (a_\alpha + b_\alpha \lambda_G) G(x) + a_\alpha k_f |y(x)-g(x)| + b_\alpha k_f \int_0^x |y(s)-g(s)|ds.
\end{aligned}
$$

Note that $a_\alpha k_f < 1$, and so,

$$
|y(x)-g(x)| \le \frac{(a_\alpha + b_\alpha \lambda_G)G(x)}{1-a_\alpha k_f} + \frac{b_\alpha k_f}{1-a_\alpha k_f} \int_0^x |y(s)-g(s)|ds.
$$

From Gronwall's inequality, we have

$$
|y(x)-g(x)| \le \left[\frac{(a_\alpha + b_\alpha \lambda_G)}{1-a_\alpha k_f} \exp(x) \right] G(x),\ x \in [0,T]. \tag{19}
$$

Set $K^* = \frac{a_\alpha + b_\alpha \lambda_G}{1-a_\alpha k_f} \exp(T)$. Note that one has

$$
|y(x)-g(x)| \le K^* G(x),\ x \in [0,T].
$$

From Definition 6, (2) is generalized Ulam–Hyers–Rassias stable with respect to G on $[0,T]$. The proof is complete. □

5. Examples

In this section, two examples are given to illustrate our main results.

For convenience in calculating, we suppose that $M(\cdot)$ in Definition 2 is the roots of the following equation:

$$
\frac{2(1-\cdot)}{(2-\cdot)M(\cdot)} + \frac{2\cdot}{(2-\cdot)M(\cdot)} = 1.
$$

Then, one can derive an explicit formula $M(\alpha) = \frac{2}{2-\alpha}$ and $M(\beta) = \frac{2}{2-\beta}$ (see (p. 89, [22])).

Example 1. *Consider*

$$
({}^{CF}\mathbb{D}^{\frac{1}{2}} y)(x) - \frac{1}{3}({}^{CF}\mathbb{D}^{\frac{2}{3}} y)(x) = \frac{2}{3}e^x + \frac{1}{3}e^{-2x} - \frac{2}{3},\ x \in [0,T]. \tag{20}
$$

Set $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$, $u(x) = \frac{2}{3}e^x + \frac{1}{3}e^{-2x} - \frac{2}{3}$ *and* $\lambda = \frac{1}{3}$. *From (Definition 1, [22]),* $M(\frac{1}{2}) = \frac{4}{3}$ *and* $M(\frac{2}{3}) = \frac{3}{2}$.

Let $y_1(x) = e^x$, and we have

$$({}^{CF}\mathbb{D}^{\frac{1}{2}}y_1)(x) = 2\int_0^x e^{t-x}e^t dt = e^x - e^{-x},$$

$$({}^{CF}\mathbb{D}^{\frac{2}{3}}y_1)(x) = 3\int_0^x e^{-2(x-t)}e^t dt = e^x - e^{-2x}.$$

Choose $\varepsilon = \frac{2}{3}$. Note $y_1(x) = e^x$ satisfies

$$\begin{aligned} & \left|({}^{CF}\mathbb{D}^{\frac{1}{2}}y_1)(x) - \frac{1}{3}({}^{CF}\mathbb{D}^{\frac{2}{3}}y_1)(x) - \frac{2}{3}e^x - \frac{1}{3}e^{-2x} + \frac{2}{3}\right| \\ = & \left|e^x - e^{-x} - \frac{1}{3}e^x + \frac{1}{3}e^{-2x} - \frac{2}{3}e^x - \frac{1}{3}e^{-2x} + \frac{2}{3}\right| \\ = & |\frac{2}{3} - e^{-x}| \leq \frac{2}{3}. \end{aligned}$$

Note $y_1(0) = 1$ and with the formulas of A, B, C, D in (9) and (13), we obtain an exact solution of Equation (1) as

$$\begin{aligned} y_a(x) &= y(0) + 2\frac{C}{A}u(x) + 2\left(\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}\right)\int_0^x \exp(-\frac{B}{A}t)u(x-t)dt \\ &\quad +2\frac{\alpha\beta}{B}\int_0^x u(x-t)dt \\ &= 1 + \frac{2}{3}e^x - \frac{1}{3}e^{-2x} - \frac{2}{3} - \frac{4}{9}\int_0^x e^{-3t}(e^{x-t} + \frac{e^{-2(x-t)}}{2} - 1)dt \\ &\quad +\frac{4}{9}\int_0^x (e^{x-t} + \frac{e^{-2(x-t)}}{2} - 1)dt \\ &= e^x + \frac{4}{27} + \frac{5}{27}e^{-3x} - \frac{2}{3}e^{-2x} - \frac{4}{9}x. \end{aligned}$$

Clearly,

$$\begin{aligned} |y_1(x) - y_a(x)| &= |e^x + \frac{4}{27} + \frac{5}{27}e^{-3x} - \frac{2}{3}e^{-2x} - \frac{4}{9}x - e^x| \\ &= |\frac{4}{27} + \frac{5}{27}e^{-3x} - \frac{2}{3}e^{-2x} - \frac{4}{9}x| \\ &\leq |\frac{4}{27} - \frac{4}{9}x| \\ &\leq \frac{2}{3} + \frac{8}{9}x = (1 + \frac{4}{3}x)\frac{2}{3}. \end{aligned}$$

Note in Theorem 2 (see Remark 1) that we have $K = 2|\frac{C}{A}| + 2|\frac{AD-BC}{A^2} - \frac{\alpha\beta}{B}|\max\{1, \exp(-\frac{B}{A}T)\}T + 2|\frac{\alpha\beta}{B}|T = 1 + \frac{4}{3}T$ and $\varepsilon = \frac{2}{3}$. Thus, Equation (20) is Hyers–Ulam stable when $T < \infty$.

Example 2. *We consider the following fractional problem:*

$$({}^{CF}\mathbb{D}^{\frac{1}{3}}y)(x) = \frac{e^{-2x}}{1+e^x}\frac{|y|}{1+|y|},\ x \in [0,2], \tag{21}$$

and the inequality

$$\left|({}^{CF}\mathbb{D}^{\frac{1}{3}}y)(x) - \frac{e^{-2x}}{1+e^x}\frac{|y|}{1+|y|}\right| \leq G(x),\ x \in [0,2]. \tag{22}$$

Set $\alpha = \frac{1}{3}$, $T = 2$ *and* $f(x,y) = \frac{e^{-2x}}{1+e^x}\frac{|y|}{1+|y|}$, $(x,y) \in [0,2] \times \mathbb{R}$. *Clearly,* $[A1]$ *holds. Then,* $M(\frac{1}{3}) = \frac{6}{5}$, $a_{\frac{1}{3}} = \frac{24}{25}$, $b_{\frac{1}{3}} = \frac{12}{25}$. *Let* $G(x) = e^x \in C([0,2],\mathbb{R})$ *and* $\int_0^x G(s)ds = \int_0^x e^s ds = e^x - 1 \le e^x$. *Here,* $\lambda_G = 1 > 0$.

For any $x \in [0,2]$ *and* $y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned} |f(x,y_1) - f(x,y_2)| &= \frac{e^{-2x}}{1+e^x}\left|\frac{|y_1|}{1+|y_1|} - \frac{|y_2|}{1+|y_2|}\right| \le \frac{e^{-2x}|y_1-y_2|}{(1+e^x)(1+|y_1|)(1+|y_2|)} \\ &\le \frac{e^{-2x}}{(1+e^x)}|y_1-y_2| \le \frac{e^{-2x}}{2}|y_1-y_2| \le \frac{1}{2}|y_1-y_2|. \end{aligned}$$

For all $x \in [0,2]$ *and* $y \in \mathbb{R}$,

$$|f(x,y)| = \frac{e^{-2x}}{1+e^x}\frac{|y|}{1+|y|} \le \frac{e^{-2x}}{1+e^x}|y| \le \frac{e^{-2x}}{2}|y| \le \frac{1}{2}|y| \le \frac{1}{2}(1+|y|).$$

Thus, $[A2]$ *and* $[A3]$ *hold.*

Set $L = \frac{1}{2} = k_f$. *Then* $a_\alpha k_f = \frac{24}{25} \times \frac{1}{2} = \frac{12}{25} < 1$. *From Theorem 3, (21) has an unique solution.*

Thus, all the assumptions in Theorem 4 are satisfied, so our results can be applied to (21).

Let $g \in C([0,2],\mathbb{R})$ *be a solution of (22). We have*

$$\left|({}^{CF}\mathbb{D}^{\frac{1}{3}}g)(x) - f(x,g(x))\right| = \left|({}^{CF}\mathbb{D}^{\frac{1}{3}}g)(x) - \frac{e^{-2x}}{1+e^x}\frac{|g|}{1+|g|}\right| \le G(x),\ x \in [0,2]. \tag{23}$$

From Theorem 3, we see (21) with $y(0) = C_0$ *has the unique solution*

$$\begin{aligned} y(x) &= C_0 + a_{\frac{1}{3}} f(x,y(x)) + b_{\frac{1}{3}}\int_0^x f(s,y(s))ds \\ &= C_0 + \frac{24}{25}\frac{e^{-2x}}{1+e^x}\frac{|y|}{1+|y|} + \frac{12}{25}\int_0^x \frac{e^{-2s}}{1+e^s}\frac{|y|}{1+|y|}ds. \end{aligned}$$

Applying the fractional integrating operator ${}^{CF}I^\alpha(\cdot)$ *on both sides of (23), we have*

$$\begin{aligned} &\left|g(x) - C_0 - a_{\frac{1}{3}} f(x,g(x)) - b_{\frac{1}{3}}\int_0^x f(s,g(s))ds\right| \\ \le\ & a_{\frac{1}{3}}G(x) + b_{\frac{1}{3}}\int_0^x G(s)ds \\ \le\ & (a_{\frac{1}{3}} + b_{\frac{1}{3}}\lambda_G)G(x),\ x \in [0,2]. \end{aligned}$$

In addition,

$$|y(x) - g(x)| \le \left[\frac{(a_{\frac{1}{3}} + b_{\frac{1}{3}}\lambda_G)}{1 - a_{\frac{1}{3}}k_f}\exp(x)\right]G(x),\ x \in [0,2].$$

Set $K^* = \frac{a_{\frac{1}{3}} + b_{\frac{1}{3}}\lambda_G}{1 - a_{\frac{1}{3}}k_f}\exp(2) = \frac{\frac{24}{25} + \frac{12}{25}\times 1}{1 - \frac{24}{25}\times\frac{1}{2}}e^2 = \frac{36e^2}{13}$. *Note that one has*

$$|y(x) - g(x)| \le K^* G(x),\ x \in [0,2].$$

6. Conclusions

By applying the well-known Gronwall inequality and fixed point theorems, we obtain the Hyers–Ulam stability of linear and semilinear Caputo–Fabrizio fractional differential equations. Existence and uniqueness theorems of solution are established. In a forthcoming work, we shall consider the impulsive Cauchy problem with Caputo–Fabrizio fractional derivative.

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References

1. Podlubny, I. *Fractional Differential Equations, Mathematics in Science and Engineering*; Academic Press: San Diego, CA, USA, 1999; Volume 198.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
3. Tarasov, V.E. *Fractional Dynamics: Application of Fractional Calculuts to Dynamics of Particles, Fields and Media*; Springer: Berlin/Heidelberg, Germany, 2011.
4. Abbas, S.; Benchohra, M.; Darwish, M.A. New stability results for partial fractional differential inclusions with not instantaneous impulses. *Fract. Calc. Appl. Anal.* **2015**, *18*, 172–191. [CrossRef]
5. Li, M.; Wang, J. Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. *Appl. Math. Comput.* **2018**, *324*, 254–265. [CrossRef]
6. Liu, S.; Wang, J.; Zhou, Y.; Fečkan, M. Iterative learning control with pulse compensation for fractional differential equations. *Math. Slov.* **2018**, *68*, 563–574. [CrossRef]
7. Wang, J.; Ibrahim, A.G.; O'Regan, D. Topological structure of the solution set for fractional non-instantaneous impulsive evolution inclusions. *J. Fixed Point Theory Appl.* **2018**, *20*, 59. [CrossRef]
8. Luo, D.; Wang, J.; Shen, D. Learning formation control for fractional-order multi-agent systems. *Math. Meth. Appl. Sci.* **2018**, *41*, 5003–5014. [CrossRef]
9. Peng, S.; Wang, J.; Yu, X. Stable manifolds for some fractional differential equations. *Nonlinear Anal. Model. Control* **2018**, *23*, 642–663. [CrossRef]
10. Chen, Y.; Wang, J. Continuous dependence of solutions of integer and fractional order non-instantaneous impulsive equations with random impulsive and junction points. *Mathematics* **2019**, *7*, 331. [CrossRef]
11. Zhang, J.; Wang, J. Numerical analysis for a class of Navier–Stokes equations with time fractional derivatives. *Appl. Math. Comput.* **2018**, *336*, 481–489
12. Zhu, B.; Liu, L.; Wu, Y. Local and global existence of mild solutions for a class of nonlinear fractional reaction-diffusion equation with delay. *Appl. Math. Lett.* **2016**, *61*, 73–79. [CrossRef]
13. Wang, Y.; Liu, L.; Wu, Y. Positive solutions for a nonlocal fractional differential equation. *Nonlinear Anal.* **2011**, *74*, 3599–3605. [CrossRef]
14. Zhang, X.; Liu, L.; Wu, Y. Existence results for multiple positive solutions of nonlinear higher order perturbed fractional differential equations with derivatives. *Appl. Math. Comput.* **2012**, *219*, 1420–1433. [CrossRef]
15. Wang, Y.; Liu, L.; Zhang, X.; Wu, Y. Positive solutions of a fractional semipositone differential system arising from the study of HIV infection models. *Appl. Math. Comput.* **2015**, *258*, 312–324.
16. Zhang, X.; Liu, L.; Wu, Y. Variational structure and multiple solutions for a fractional advection-dispersion equation. *Comput. Math. Appl.* **2014**, *68*, 1794–1805. [CrossRef]
17. Zhang, X.; Mao, C.; Liu, L.; Wu, Y. Exact iterative solution for an abstract fractional dynamic system model for bioprocess. *Qual. Theory Dyn. Syst.* **2017**, *16*, 205–222. [CrossRef]
18. Zhang, X.; Liu, L.; Wu, Y.; Wiwatanapataphee, B. Nontrivial solutions for a fractional advection dispersion equation in anomalous diffusion. *Appl. Math. Lett.* **2017**, *66*, 1–8. [CrossRef]
19. Jiang, J.; Liu, L.; Wu, Y. Multiple positive solutions of singular fractional differential system involving Stieltjes integral conditions. *Electron. J. Qual. Theory Differ. Equ.* **2012**, *43*, 1–18. [CrossRef]
20. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.

21. Atangana, A.; Nieto, J.J. Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel. *Adv. Mech. Eng.* **2015**, *7*, 1–7. [CrossRef]
22. Losada, J.; Nieto, J.J. Properties of a new fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 87–92.
23. Baleanu, D.; Mousalou, A.; Rezapour, S. On the existence of solutions for some infinite coefficient-symmetric Caputo–Fabrizio fractional integro-differential equations. *Bound. Value Prob.* **2017**, *2017*, 1–9. [CrossRef]
24. Franc, E.; Goufo, D. Application of the Caputo–Fabrizio fractional derivative without singular kernel to Korteweg–de Vries–Burgers equations. *Math. Model. Anal.* **2016**, *21*, 188–198.
25. Rezaei, H.; Jung, S.M.; Rassias, T.M. Laplace transform and Hyers–Ulam stability of linear differential equations. *J. Math. Anal. Appl.* **2013**, *403*, 244–251. [CrossRef]
26. Alqifiary, Q.H.; Jung, S.M. Laplace transform and generalized Hyers–Ulam stability of linear differential equations. *Electron. J. Diff. Equ.* **2014**, *2014*, 1–11.
27. Wang, J.; Li, X.Z. A uniform method to Ulam-Hyers stability for some Linear fractional equations. *Mediterr. J. Math.* **2016**, *13*, 625–635. [CrossRef]
28. Wang, J.; Zhang, Y. Ulam-Hyers-Mittag-Leffler stability of fractional-order delay differential equations. *Optimization* **2014**, *63*, 1181–1190. [CrossRef]
29. Capelas de Oliveira, E.; da C. Sousa, J.V. Ulam-Hyers-Rassias stability for a class of fractional integro-differential equations. *Result Math.* **2018**, *73*, 111. [CrossRef]
30. da C. Sousa, J.V.; Capelas de Oliveira, E. Ulam-Hyers stability of a nonlinear fractional Volterra integro-differential equation. *Appl. Math. Lett.* **2018**, *81*, 50–56.
31. da C. Sousa, J.V.; Kucche, K.D.; Capelas de Oliveira, E. Stability of ψ-Hilfer impulsive fractional differential equations. *Appl. Math. Lett.* **2018**, *88*, 73–80.
32. Wang, J.; Zhou, Y.; Fečkan, M. Nonlinear impulsive problems for fractional differential equations and Ulam stability. *Comput. Math. Appl.* **2012**, *64*, 3389–3405. [CrossRef]
33. da C. Sousa, J.V.; Capelas de Oliveira, E. On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator. *J. Fixed Point Theory Appl.* **2018**, *20*, 5–21.
34. Shah, K.; Ali, A.; Bushnaq, S. Hyers–Ulam stability analysis to implicit Cauchy problem of fractional differential equations with impulsive conditions. *Math. Meth. Appl. Sci.* **2018**, *41*, 8329–8343. [CrossRef]
35. Ali, Z.; Zada, A.; Shah, K. Ulam stability to a toppled systems of nonlinear implicit fractional order boundary value problem. *Bound. Value Prob.* **2018**, *2018*, 175. [CrossRef]
36. Liu, K.; Wang, J.; O'Regan, D. Ulam-Hyers-Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 50. [CrossRef]
37. Wang, J.; Lv, L.; Zhou, Y. Ulam stability and data depenaence for fractional differential equations with Caputo derivative. *Electron. J. Qual. Theory Differ. Equ.* **2011**, *63*, 1–10.
38. Wang, J.; Zhou, Y.; Fečkan, M. Abstract Cauchy problem for fractional differential equations. *Nonlinear Dyn.* **2013**, *71*, 685–700. [CrossRef]