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Some Remarks and New Results in Ordered Partial *b*-Metric Spaces

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Abstract: We improve and generalize some new results in fixed point theory in the context of partial *b*-metric spaces.

1. Introduction

Fixed point theory and different forms of generalizations of the usual metric space are significant topics for many researchers. This can be witnessed from the vast literature available in this topic. In order to study some forms of generalizations of metric spaces, one can see the results in [1–41]. Let X be a non-empty set and f be a mapping on X.

If there exists a point $z \in X$ such that $fz = z$, then such z is said a fixed point of f and the set of all fixed points of f is denoted by $F(f)$. Otherwise, the fixed point theory is one of the most significant, as well as important as famous theory in mathematics, since it has applications to very different types of problems in enough areas of science. One of the well known fixed point result is the Banach fixed point theorem proved by Banach in 1922 (see also [2,21]). It is worth to mention that this principle has been generalized in two directions, by acting on the contraction (expansive) condition, or changing the topology of the space. Among these generalizations, Matthews [22] introduced a new distance on a non-empty set X, which is called a partial metric. Here, the distance of a point to itself need not be equal to zero. Further, Bakhtin (1989) and Czerwik (1993) replaced the standard triangular inequality by $d(\varrho, \delta) \leq s[d(\varrho, \varsigma) + d(\varsigma, \delta)]$ with $s \geq 1$.

2. Definitions, Notations and Preliminaries

Definition 1. *(References [8,9]) Given $s \geq 1$ and X a non-empty set. If the function $d : X \times X \to [0, \infty)$ is such that for all $\varrho, \delta, \varsigma \in X$, we have:*

(b1) $d(\varrho, \delta) = 0$ iff $\varrho = \delta$;

(b2) $d(\varrho, \delta) = d(\delta, \varrho)$;

(b3) $d(\varrho, \delta) \leq s[d(\varrho, \varsigma) + d(\varsigma, \delta)]$.

then d is a b-metric on X (with coefficient s).

Definition 2. *(References [22,27]). Let X be a non-empty set. If $p : X \times X \to [0, \infty)$ is such that*

(p1) $\varrho = \delta$ iff $p(\varrho, \varrho) = p(\varrho, \delta) = p(\delta, \delta)$;

(p2) $p(\varrho, \varrho) \leq p(\varrho, \delta)$;

(p3) $p(\varrho, \delta) = p(\delta, \varrho)$;

(p4) $p(\varrho, \delta) \leq p(\varrho, \varsigma) + p(\varsigma, \delta) - p(\varsigma, \varsigma)$,

for all $\varrho, \delta, \varsigma \in X$, then p is a partial metric.

The above definition was generalized by Shukla [35].

Definition 3. *(Reference [35]) Given a non-empty set X and $s \geq 1$. If $p_b : X \times X \to [0, \infty)$ is such that*

(pb1) $\varrho = \delta$ iff $p(\varrho, \varrho) = p_b(\varrho, \delta) = p_b(\delta, \delta)$;

(pb2) $p_b(\varrho, \varrho) \leq p_b(\varrho, \delta)$;

(pb3) $p_b(\varrho, \delta) = p_b(\delta, \varrho)$;

(pb4) $p_b(\varrho, \delta) \leq s[p_b(\varrho, \varsigma) + p_b(\varsigma, \delta)] - p_b(\varsigma, \varsigma)$,

for all $\varrho, \delta, \varsigma \in X$, then p_b is a partial metric with a coefficient $s \geq 1$.

In the following, a partial *b*-metric on *X* is neither a *b*-metric, nor a partial metric (you may see ([15,29,35]).

Example 1. *[35] Define on $X = [0, \infty)$,*

$$p_b(\varrho, \delta) = [\max(\varrho, \delta)]^2 + |\varrho - \delta|^2.$$

Here, p_b is a partial b-metric on X and the coefficient $s = 2$. Note that p_b is neither a b-metric, nor a partial metric on X.

The following two propositions are very useful in the context of partial *b*-metric spaces.

Proposition 1. *Reference [35] Let X be a non-empty set. Let p (resp. d) be a partial metric (resp. a b-metric), then $p_b(\varrho, \delta) = p(\varrho, \delta) + d(\varrho, \delta)$ is a partial b-metric on X.*

Proposition 2. *Reference [35] If p is a partial metric, then $p_b(\varrho, \delta) = [p(\varrho, \delta)]^r$ is a partial b-metric on X with $s = 2^{r-1}$ for $r > 1$.*

On the other hand, Mustafa et al. [29] modify (*pb*4) in Definition 3.

Definition 4. *Reference [29] Let X be a non-empty set and $s \geq 1$. A function $p_b : X \times X \to [0, \infty)$ is a partial b-metric if for all $\varrho, \delta, \varsigma \in X$: (pb1), (pb2), (pb3) are the same as in the Definition 3, while (pb4) is modified with*

(pb4') $p_b(\varrho, \delta) \leq s[p_b(\varrho, \varsigma) + p_b(\varsigma, \delta) - p_b(\varsigma, \varsigma)] + \frac{1-s}{2}(p_b(\varrho, \varrho) + p_b(\delta, \delta))$.

The pair (X, p_b) is called a partial b-metric space if it satisfies conditions (pb1), (pb2), (pb3) and (pb4'). The real $s \geq 1$ is the coefficient of (X, p_b). Clearly, (pb4') implies (pb4).

Example 2. *Reference [29] Define $p_b(\varrho, \delta) = (\varrho, \delta)^2 + 5$ for all $\varrho, \delta \in X = \mathbb{R}$. Then p_b is a partial b-metric on X with $s = 2$ (in the sense of Definition 4). Here p_b is not a partial metric on X. Indeed, for $x = 1, y = 4$ and $z = 2$, we have*

$$p_b(1, 4) = 14 \not\leq p_b(1, 2) + p_b(2, 4) - p_b(2, 2) = 10.$$

Proposition 3. *Reference [29] Each partial b-metric p_b defines a b-metric d_b, where*

$$d_b(\varrho, \delta) = 2p_b(\varrho, \delta) - p_b(\varrho, \varrho) - p_b(\delta, \delta), \text{ for all } \varrho, \delta \in X. \tag{1}$$

Definition 5. *Reference [29] Given $\{\varrho_n\}$ a sequence in a partial b-metric space (X, p_b).*

(i) $\{\varrho_n\}$ p_b-converges to $\varrho \in X$ if $\lim_{n \to \infty} p_b(\varrho, \varrho_n) = p_b(\varrho, \varrho)$;

(ii) $\{\varrho_n\}$ is p_b-Cauchy if $\lim_{n,m \to \infty} p_b(\varrho_n, \varrho_m)$ exists (and is finite).

(iii)   *Also, (X, p_b) is said to be p_b-complete if each p_b-Cauchy sequence $\{\varrho_n\}$ in X, p_b-converges to $\varrho \in X$ so that*

$$\lim_{n,m \to \infty} p_b(\varrho_n, \varrho_m) = \lim_{n \to \infty} p_b(\varrho_n, \varrho) = p_b(\varrho, \varrho). \tag{2}$$

Lemma 1. *Reference [29] A sequence $\{\varrho_n\}$ is p_b-Cauchy in a partial b-metric space (X, p_b) iff it is d_b-Cauchy in the b-metric space (X, d_b).*

Lemma 2. *Reference [29] A partial b-metric space (X, p_b) is p_b-complete iff the b-metric space (X, d_b) is d_b-complete. Further, $\lim_{n,m \to \infty} d_b(\varrho_n, \varrho_m) = 0$ iff*

$$\lim_{n \to \infty} p_b(\varrho, \varrho_n) = \lim_{n,m \to \infty} d_{p_b}(\varrho_n, \varrho_m) = p_b(\varrho, \varrho). \tag{3}$$

Further, in ([12], Definition 2.1.) authors introduced the following notions on a partial b-metric space (for some other details, see also [17]).

Definition 6. *Let (X, p_b) be a partial b -metric space.*

1.   *A sequence $\{\varrho_n\}$ is called $0 - p_b$-Cauchy if $\lim_{n,m \to \infty} p_b(\varrho_n, \varrho_m) = 0$.*

2.   *(X, p_b) is called $0 - p_b$-complete if for each $0 - p_b$-Cauchy sequence $\{\varrho_n\}$ in X, there is $\varrho \in X$ such that*

$$\lim_{n,m \to \infty} p_b(\varrho_n, \varrho_m) = \lim_{n \to \infty} p_b(\varrho_n, \varrho) = p_b(\varrho, \varrho) = 0.$$

The relation between p_b-completeness and $0 - p_b$-completeness of a partial b-metric space is given in the following.

Lemma 3. *(Reference [12], Lemma 2.2.) Let (X, p_b) be a partial b-metric space. If (X, p_b) is p_b-complete, then it is $0 - p_b$-complete.*

The converse of Lemma 3 does not hold as shown in Example 2.3 in [12]. Also, in [12] (similarly as in [17] for partial metric spaces), authors state the relation between a partial b-metric p_b and a certain b-metric on (X, d_{p_b}) as follows.

Theorem 1. *(References [1], Lemma 2.1, [12], Theorem 2.4.) Let (X, p_b) be a partial b-metric with coefficient $s \geq 1$. For all $x, y \in X$, put*

$$d_{p_b}(\varrho, \delta) = \begin{cases} 0 \text{ if } \varrho = \delta \\ p_b(\varrho, \delta) \text{ if } \varrho \neq \delta. \end{cases}$$

Then

1.   *d_{p_b} is a b-metric with coefficient s on X.*

2.   *If $\lim_{n \to \infty} \varrho_n = x$ in (X, d_{p_b}), then $\lim_{n \to \infty} \varrho_n = x$ in (X, p_b).*

3.   *(X, p_b) is $0 - p_b$-complete iff (X, d_{p_b}) is d_{p_b}- complete.*

Remark 1. *For more significant and important results in the context of partial b-metric spaces, readers also can see ([1], Lemma 2.1, Lemma 2.2 and final Conclusion). Example 2.5 from [12] shows that the converse of statement 2 from Theorem 1 does not hold.*

Lemma 4. *Reference [29] Let* (X, p_b) *be a partial b-metric space with* $s > 1$. *If* $\{\varrho_n\}$ *and* $\{\delta_n\}$ *are* p_b*-convergent to* ϱ *and* δ, *respectively, then*

$$\frac{1}{s^2} p_b (\varrho, \delta) - \frac{1}{s} p_b (\varrho, \varrho) - p_b (\delta, \delta) \leq \underline{\lim}_{n \to \infty} p_b (\varrho_n, \delta_n)$$

$$\leq \overline{\lim}_{n \to \infty} p_b (\varrho_n, \delta_n) \leq s p_b (\varrho, \varrho) + s^2 p_b (\delta, \delta) + s^2 p_b (\varrho, \delta).$$

Definition 7. *Reference [42] The function* $\psi : [0, \infty) \to [0, \infty)$ *is said an altering distance if it is continuous, nondecreasing and* $\psi(t) = 0$ *iff* $t = 0$.

Let Θ be the set of altering distance functions.

In this manuscript, we discuss and improve many known results in literature.

3. Improvement Results and Remarks on Recent Ones

In 2014, Mukheimer [Definition 2.1] [27] introduced the $\alpha - \psi - \varphi$-contractive self maps on partial b-metric spaces as follows.

Definition 8. *Let* (X, p_b) *be a partial b-metric space with* $s \geq 1$. $f : X \to X$ *is said an* $\alpha - \psi - \varphi$*-contractive map if there are* $\psi, \varphi \in \Theta$ *and* $\alpha : X \times X \to [0, \infty)$ *so that*

$$\alpha(\varrho, \delta) \psi (s p_b (f\varrho, f\delta)) \leq \psi \left(M_s^f (\varrho, \delta) \right) - \varphi \left(M_s^f (\varrho, \delta) \right) \tag{4}$$

for all $\varrho, \delta \in X$, *where*

$$M_s^f (\varrho, \delta) = \max \left\{ p_b (\varrho, \delta), p_b (\varrho, f\varrho), p_b (\delta, f\delta), \frac{p_b (\varrho, f\delta) + p_b (\delta, f\varrho)}{2s} \right\}. \tag{5}$$

Definition 9. *Reference [29] Given* $f : X \to X$ *on a partial b-metric space* (X, p_b). *Such* f *is* α*-admissible if* $\varrho, \delta \in X, \alpha(\varrho, \delta) \geq 1$ *implies that* $\alpha(f\varrho, f\delta) \geq 1$. f *is* L_α*-admissible (resp.* R_α*-admissible) if* $\varrho, \delta \in X, \alpha(\varrho, \delta) \geq 1$ *implies that* $\alpha(f\varrho, \delta) \geq 1$ *(resp.* $\alpha(\varrho, f\delta) \geq 1$).

In [27], we have

Theorem 2. *(Reference [27], Theorem 2.1.) Let* $(X, \preceq, p_b)$ *be a* p_b*-complete ordered partial b-metric space with* $s \geq 1$. *Let* $f : X \to X$ *be an* $\alpha - \psi - \varphi$*-contractive self mapping. Suppose that:*

(1)　*f is* α*-admissible and* L_α*-admissible (or* R_α*-admissible);*
(2)　*there is* $\varrho_1 \in X$ *so that* $\varrho_1 \preceq f\varrho_1$ *and* $\alpha(\varrho_1, f\varrho_1) \geq 1$;
(3)　*f is continuous, nondecreasing, with respect to* $\preceq$ *and if* $f^n \varrho_1 \to z$ *then* $\alpha(z, z) \geq 1$.

Then, f has a fixed point.

Mukheimer [27] omits the condition of continuity in the previous theorem.

Theorem 3. *(Reference [27], Theorem 2.2) Let* $(X, \preceq, p_b)$ *be a* p_b*-complete ordered partial b-metric space with* $s \geq 1$. *Let* $f : X \to X$ *be an* $\alpha - \psi - \varphi$- *contractive self mapping. Assume that:*

(1)　*f is* α*-admissible and* L_α*-admissible (or* R_α*-admissible);*
(2)　*there is* $\varrho_1 \in X$ *so that* $\varrho_1 \preceq f\varrho_1$ *and* $\alpha(\varrho_1, f\varrho_1) \geq 1$;
(3)　*f is nondecreasing, with respect to* $\preceq$;
(4)　*If* $\{\varrho_n\}$ *is such that* $\varrho_n \preceq \varrho$ *for each* $n \in \mathbb{N}$, $\alpha(\varrho_n, \varrho_{n+1}) \geq 1$ *and* $\varrho_n \to \varrho \in X$, *as* $n \to \infty$, *then* $\alpha(\varrho_n, \varrho) \geq 1$ *for each* $n \in \mathbb{N}$;

Then, f has a fixed point.

Now, we shall improve the proofs of Theorem 2 and Theorem 3. First of all, we prove the following:

Lemma 5. *Let (X, p_b) be a partial b-metric space with $s \geq 1$ and $T : X \to X$ be a mapping. If $\{\varrho_n\}$ is a sequence in X such that $\varrho_{n+1} = T\varrho_n$ and*

$$p_b(\varrho_n, \varrho_{n+1}) \leq \lambda p_b(\varrho_{n-1}, \varrho_n) \tag{6}$$

for each $n \in \mathbb{N}$, where $\lambda \in [0, 1)$, then $\{\varrho_n\}$ is $0 - p_b$- Cauchy.

Proof of Lemma. Let $\varrho_0 \in X$ and $\varrho_{n+1} = T\varrho_n$ for all $n \in \mathbb{N} \cup \{0\}$. We divide the proof into three cases.
    Case 1. Let $\lambda \in [0, \frac{1}{s})$ $(s > 1)$. By the hypotheses, we have

$$p_b(\varrho_n, \varrho_{n+1}) \leq \lambda p_b(\varrho_{n-1}, \varrho_n) \leq \lambda^2 p_b(\varrho_{n-2}, \varrho_{n-1}) \leq \cdots \leq \lambda^n p_b(\varrho_0, \varrho_1). \tag{7}$$

Thus, for $n > m$, we have

$$
\begin{aligned}
p_b(\varrho_m, \varrho_n) &\leq s\left[p_b(\varrho_m, \varrho_{m+1}) + p_b(\varrho_{m+1}, \varrho_n)\right] \\
&\leq sp_b(\varrho_m, \varrho_{m+1}) + s^2\left[p_b(\varrho_{m+1}, \varrho_{m+2}) + p_b(\varrho_{m+2}, \varrho_n)\right] \\
&\leq sp_b(\varrho_m, \varrho_{m+1}) + s^2 p_b(\varrho_{m+1}, \varrho_{m+2}) + s^3\left[p_b(\varrho_{m+2}, \varrho_{m+3}) + p_b(\varrho_{m+3}, \varrho_n)\right] \\
&\leq sp_b(\varrho_m, \varrho_{m+1}) + s^2 p_b(\varrho_{m+1}, \varrho_{m+2}) + s^3 p_b(\varrho_{m+2}, \varrho_{m+3}) \\
&\quad + \cdots + s^{n-m-1} p_b(\varrho_{n-2}, \varrho_{n-1}) + s^{n-m-1} p_b(\varrho_{n-1}, \varrho_n) \\
&\leq s\lambda^m p_b(\varrho_0, \varrho_1) + s^2 \lambda^{m+1} p_b(\varrho_0, \varrho_1) + s^3 \lambda^{m+2} p_b(\varrho_0, \varrho_1) \\
&\quad + \cdots + s^{n-m-1} \lambda^{n-2} p_b(\varrho_0, \varrho_1) + s^{n-m-1} \lambda^{n-1} p_b(\varrho_0, \varrho_1) \\
&\leq s\lambda^m \left(1 + (s\lambda) + (s\lambda)^2 + \cdots + (s\lambda)^{n-m-2} + \frac{(s\lambda)^{n-m-1}}{s}\right) p_b(\varrho_0, \varrho_1) \\
&\leq s\lambda^m \left(\frac{1}{1 - s\lambda} + \frac{(s\lambda)^{n-m-1}}{s}\right) p_b(\varrho_0, \varrho_1) \\
&= \left(\frac{s\lambda^m}{1 - s\lambda} + (s\lambda)^{n-1}\right) p_b(\varrho_0, \varrho_1) \to 0 \ (n, m \to \infty),
\end{aligned}
\tag{8}
$$

which implies that $\{\varrho_n\}$ is $0 - p_b$-Cauchy, that is, $\{T^n \varrho_0\}_{n \in \mathbb{N}}$ is $0 - p_b$-Cauchy.
    Case 2. Let $\lambda \in [\frac{1}{s}, 1)$ $(s > 1)$. In this case, we have $\lambda^n \to 0$ as $n \to \infty$, then there is $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{s}$. Thus, by Case 1, we have that

$$\left\{T^{k+n} \varrho_0\right\}_{n=0}^{+\infty} := \{\varrho_k, \varrho_{k+1}, \varrho_{k+2}, ..., \varrho_{k+n}, ...\}, \tag{9}$$

is a $0 - p_b$-Cauchy sequence. Since

$$\{\varrho_n\}_{n=0}^{\infty} = \{\varrho_0, \varrho_1, ..., \varrho_{k-1}\} \cup \{\varrho_k, \varrho_{k+1}, \varrho_{k+2}, ..., \varrho_{k+n}, ...\} \tag{10}$$

we obtain that $\{\varrho_n\}_{n=1}^{\infty} = \{T^n \varrho_0\}_{n=1}^{+\infty}$ is a $0 - p_b$-Cauchy sequence in X.
    Case 3. Let $s = 1$, then (X, p_b) is a partial metric space. In this case, the result is valid and hence we omit the proof (see [21], Theorem 14.1). $\square$

Remark 2. *Lemma 5 generalizes Lemma 2.2 in [24] from b-metric spaces to partial b-metric spaces. However, the condition (6) implies that*

$$d_{p_b}(\varrho_n, \varrho_{n+1}) \leq \lambda d_{p_b}(\varrho_{n-1}, \varrho_n), \tag{11}$$

for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. If $\varrho_n = \varrho_{n-1}$, then $\varrho_k = \varrho_{n-1}$ for all $k \geq n$ and (11) holds. If $\varrho_n \neq \varrho_{n-1}$, then we get

$$d_{p_b}\left(\varrho_n, \varrho_{n+1}\right) \leq p_b\left(\varrho_n, \varrho_{n+1}\right) \leq \lambda p_b\left(\varrho_{n-1}, \varrho_n\right) = \lambda d_{p_b}\left(\varrho_{n-1}, \varrho_n\right),$$

that is, the result follows. This shows that according to [Lemma 2.2] [24], the sequence $\{\varrho_n\}$ is d_{p_b}-Cauchy in the d_{p_b}-metric space $\left(X, d_{p_b}\right)$.

Now, in the sequel we show that it is possible to simplify the proof of Theorem 3 if $s > 1$. This will be done without applying Lemma 4. Namely, we shall prove that the sequence $\{\varrho_n\}$ in X induced by $\varrho_{n+1} = T\varrho_n$ in [27] satisfies the condition (6), that is, $\{\varrho_n\}$ is $0 - p_b$-Cauchy. Indeed, in [27], in page 172 we get

$$\psi\left(sp_b\left(\varrho_{n+1}, \varrho_{n+2}\right)\right) \leq \psi\left(\max\left\{p_b\left(\varrho_n, \varrho_{n+1}\right), p_b\left(\varrho_{n+1}, \varrho_{n+2}\right)\right\}\right). \tag{12}$$

Assume that $\max\left\{p_b\left(\varrho_n, \varrho_{n+1}\right), p_b\left(\varrho_{n+1}, \varrho_{n+2}\right)\right\} = p_b\left(\varrho_{n+1}, \varrho_{n+2}\right)$ for some n. We get $s \leq 1$, which is a contradiction. Hence, from (12), it follows that $sp_b\left(\varrho_{n+1}, \varrho_{n+2}\right) \leq p_b\left(\varrho_n, \varrho_{n+1}\right)$, or equivalently

$$p_b\left(\varrho_{n+1}, \varrho_{n+2}\right) \leq \frac{1}{s}p_b\left(\varrho_n, \varrho_{n+1}\right), \text{ for all } n \in \mathbb{N}.$$

Now, according to Lemma 5, the sequence $\{\varrho_n\}$ is $0 - p_b$- Cauchy. The rest of the proof is the same as in the paper of Mukheimer.

On the other hand, a Sehgal-Guseman theorem for partial b-metric spaces is also true. Namely, we have the following.

Theorem 4. *Let $(X, p_b, s \geq 1)$ be a p_b-complete partial b-metric space and let $T : X \to X$ be such that: for every $x \in X$ there is $n(x) \in \mathbb{N}$ so that*

$$p_b\left(T^{n(x)}x, T^{n(x)}y\right) \leq \lambda p_b\left(x, y\right), \tag{13}$$

for all $y \in X$, where $\lambda \in (0, 1)$. Then T has a unique fixed point $u \in X$, and $\lim_{n \to \infty} p_b\left(T^n x, u\right) = p_b\left(u, u\right) = 0$ for each $x \in X$.

Proof of Theorem. We shall show that (13) implies

$$d_{p_b}\left(T^{n(x)}x, T^{n(x)}y\right) \leq \lambda d_{p_b}\left(x, y\right), \tag{14}$$

where d_{p_b} is a b-metric defined in Theorem 1.

Indeed, if $x = y$, then $d_{p_b}\left(T^{n(x)}x, T^{n(x)}y\right) = 0 \leq \lambda d_{p_b}\left(x, y\right)$. If $x \neq y$, then $d_{p_b}\left(x, y\right) = p_b\left(x, y\right)$ and we have $d_{p_b}\left(T^{n(x)}x, T^{n(x)}y\right) \leq p_b\left(T^{n(x)}x, T^{n(x)}y\right) \leq \lambda p_b\left(x, y\right) = \lambda d_{p_b}\left(x, y\right)$. Therefore, (14) holds for all $x, y \in X$. The result further follows according to (Theorem 2.2) [25]. □

It is known that a self-map T has the property P if $F(T) = F(T^n)$ for all $n \in \mathbb{N}$. For more details, see [19]. The first result for the property P in the context of partial b-metric spaces is the following.

Theorem 5. *Let T be a self-map on a partial b-metric space $(X, p_b, s \geq 1)$ satisfying*

$$p_b\left(T\varrho, T^2\varrho\right) \leq \lambda p_b\left(\varrho, T\varrho\right) \tag{15}$$

for some $\lambda \in (0, 1)$, either (i) for each $\varrho \in X$, or (ii) for each $\varrho \in X, \varrho \neq T\varrho$ and suppose that T has a fixed point. Then T has the property P.

Proof of Theorem. First, we shall prove that (15) implies that

$$d_{p_b}\left(T\varrho, T^2\varrho\right) \leq \lambda d_{p_b}\left(\varrho, T\varrho\right),$$ (16)

either for each $\varrho \in X$, or for each $\varrho \in X$ with $\varrho \neq T\varrho$. The result then follows by [Proposition 2] [11]. If $\varrho = T\varrho$, then (16) clearly holds. Let $\varrho \neq T\varrho$. In this case, we have

$$d_{p_b}\left(T\varrho, T^2\varrho\right) \leq p_b\left(T\varrho, T^2\varrho\right) \leq \lambda p_b\left(\varrho, T\varrho\right) = \lambda d_{p_b}\left(\varrho, T\varrho\right),$$

that is, (16) holds.　□

The following result generalizes a Boyd-Wong type theorem from both b-metric spaces and partial metric spaces to partial b-metric spaces.

Theorem 6. *Let $(X, p_b, s > 1)$ be a p_b-complete partial b-metric space, and suppose $T : X \to X$ satisfies*

$$p_b\left(T\varrho, T\delta\right) \leq \varphi\left(p_b\left(\varrho, \delta\right)\right)$$ (17)

for all $\varrho, \delta \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is increasing and satisfies

$$\lim_{n \to \infty} \varphi^n\left(t\right) = 0$$

for each $t > 0$. Then T has a unique fixed point $u \in X$, and $\lim_{n \to \infty} p_b\left(T^n\left(\varrho\right), u\right) = p_b\left(u, u\right)$ for each $\varrho \in X$.

Proof of Theorem. First, we observe that the assumption on φ implies that

$$\lim_{t \to 0^+} \varphi\left(t\right) = 0,$$

so we can take that $\varphi\left(0\right) = 0$, that is, $\varphi\left(t\right) = 0$ iff $t = 0$. Therefore, according to Theorem 1 the condition (17) implies

$$d_{p_b}\left(T\varrho, T\delta\right) \leq \varphi\left(d_{p_b}\left(\varrho, \delta\right)\right).$$ (18)

The result further follows by ([21], Theorem 12.2).　□

Now, for $s \geq 1$, we denote by $\mathcal{G}_s$ the set of functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ such that

$$\lim_{n \to \infty} \beta(t_n) = \frac{1}{s} \quad \text{implies} \quad \lim_{n \to \infty} t_n = 0.$$

A Geraghty type result in the context of partial b-metric spaces is as follows.

Theorem 7. *Let $(X, p_b, s > 1)$ be a p_b-complete partial b-metric space. Assume that $T : X \to X$ is such that*

$$p_b\left(T\varrho, T\delta\right) \leq g_s\left(p_b\left(\varrho, \delta\right)\right) p_b\left(\varrho, \delta\right)$$ (19)

for all $\varrho, \delta \in X$, where $g_s \in \mathcal{G}_s$. Then T has a unique fixed point $u \in X$ and for each $\varrho \in X$, $\{T^n\varrho\}$ converges to u in the partial b-metric space (X, p_b), that is, $\lim_{n \to \infty} p_b\left(T^n\varrho, u\right) = p_b\left(u, u\right) = 0$.

Proof of Theorem. Since $g_s\left(p_b\left(\varrho, \delta\right)\right) < \frac{1}{s}$ for all $\varrho, \delta \in X$, the condition (19) becomes

$$p_b\left(Tx, Ty\right) \leq \frac{1}{s} p_b\left(x, y\right)$$ (20)

for all $\varrho, \delta \in X$. Now, according to Theorem 1, the condition (20) implies

$$d_{p_b}(T\varrho, T\delta) \leq \frac{1}{s} d_{p_b}(\varrho, \delta) \tag{21}$$

for all $\varrho, \delta \in X$. Then the result comes from ([12], Corollary 2.7).　□

Now, we formulate and prove a Meir-Keeler type result in the context of partial *b*-metric spaces. It generalizes ones from metric spaces and partial metric spaces to partial *b*-metric spaces. For more details, see [2,23].

Theorem 8. *Let* $(X, p_b, s > 1)$ *be a* p_b*-complete partial b-metric space and let T be a self-mapping on X verifying:*
For $\varepsilon > 0$ *there is* $\tau > 0$ *so that*

$$\varepsilon \leq p_b(\varrho, \delta) < \varepsilon + \tau \text{ implies } s \cdot p_b(T\varrho, T\delta) < \varepsilon. \tag{22}$$

Then T has a unique fixed point $u \in X$, *and for each* $\varrho \in X$, $\lim_{n \to \infty} p_b(T^n\varrho, u) = p_b(u, u) = 0.$

Proof of Theorem. Note that (22) implies the following Banach contractive condition:

$$p_b(T\varrho, T\delta) \leq \frac{1}{s} \cdot p_b(\varrho, \delta), \tag{23}$$

for all $\varrho, \delta \in X$. Further, according to Theorem 1, the condition (23) becomes

$$d_{p_b}(T\varrho, T\delta) \leq \frac{1}{s} d_{p_b}(\varrho, \delta) = k \cdot d_{p_b}(\varrho, \delta), \tag{24}$$

where $k = \frac{1}{s} \in (0, 1)$ because $s > 1$. Now the result follows by known recent results (for example, see [11,12,21,35]).　□

Finally, we announce an open question:

Prove or disprove the following:

Let $(X, p_b, s > 1)$ *be a* p_b*-complete partial b-metric space and let T be a self-mapping on X satisfying:*
Given $\varepsilon > 0$, *there is* $\tau > 0$ *such that for all* $\varrho, \delta \in X$

$$\varepsilon \leq p_b(\varrho, \delta) < \varepsilon + \tau \text{ implies } p_b(T\varrho, T\delta) < \varepsilon. \tag{25}$$

Then T has a unique fixed point $u \in X$, *and for each* $\varrho \in X$, $\lim_{n \to \infty} p_b(T^n\varrho, u) = p_b(u, u) = 0.$

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