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Some Liouville Theorems on Finsler Manifolds

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Abstract: We give some Liouville type theorems of L^p harmonic (resp. subharmonic, superharmonic) functions on a complete noncompact Finsler manifold. Using the geometric relationship between a Finsler metric and its reverse metric, we remove some restrictions on the reversibility. These improve the recent literature (Zhang and Xia, 2014).

Keywords: harmonic function; Finsler manifold; Liouville theorem; reversibility

MSC: Primary 53C60; Secondary 53B40

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1. Introduction

As is well known, Liouville theorems play an important role in analyzing the underlying manifolds. The classical Liouville theorem states that any nonnegative (or bounded) harmonic function on $\mathbb{R}^n$ must be constant. Up to now, there are many generalizations studied on complete Riemannian manifolds. Yau [1,2] proved that any positive (or bounded) harmonic function on a complete Riemannian manifold with nonnegative Ricci curvature must be constant and there are no nonnegative L^p subharmonic functions on such a Riemannian manifold for $p \in (1, \infty)$. Yau's results were then generalized by Sturm and Schoen, etc. See [3–5] and references therein. For general symmetric diffusion operators, Li [6] extended various Liouville theorems as above.

Recently, Zhang-Xia [7], Yin-He [8] and Yin-Zhang [9] extended the above Liouville theorems in the Finsler setting. Notice that, in [7,8], the Finsler manifolds discussed must have finite reversibility. In this paper, we show that this restriction can be removed. Specifically, we obtain the following results.

Theorem 1. *Let $(M, F, d\mu)$ be an n-dimensional forward complete noncompact Finsler manifold. If a positive function $u \in W^{2,2}_{loc}(M) \cap C^{1,\alpha}(M) \cap C^\infty(M_u)$ on M satisfies $\Delta \log u \geq 0$ on M_u and*

$$\limsup_{r\to\infty} \frac{r^2}{V_1(r)} = \infty,$$

where $M_u = \{x \in M | du(x) \neq 0\}$, then u is a constant. In particular, if $u \in L^1(M)$ and $\Delta \log u \geq 0$ on M_u, then u is a constant.

Theorem 2. *Let $(M, F, d\mu)$ be an n-dimensional complete noncompact Finsler manifold. Assume that*

$$\int_1^\infty \frac{r}{V_p(r)} dr = \infty.$$

1. *If $p \in (-\infty, 1)$ and $u \in W^{2,2}_{loc}(M) \cap C^{1,\alpha}(M) \cap C^{\infty}(M_u)$ is a nonnegative superharmonic function on M, then u is a constant.*
2. *If $p \in (1, \infty)$ and $u \in W^{2,2}_{loc}(M) \cap C^{1,\alpha}(M) \cap C^{\infty}(M_u)$ is a nonnegative subharmonic function on M, then u is a constant.*

Here, $V_p(r), p \in \mathbb{R}$ is defined in (2) below, and some important concepts such as Finsler metric, Finsler Laplacian and harmonic (resp. subharmonic, superharmonic) functions will be given in Section 2, respectively.

Remark 1. *If the Finsler manifold is compact, then, by the divergence theorem, we can prove all harmonic (resp. subharmonic, superharmonic) functions are constant. Theorem 1 can be regarded as a generalization of Theorem 1 in [2] when $p = 1$. If $(M, F, d\mu)$ is a Riemannian metric measure space, then Theorem 2 is exactly Theorem 1 in [5] or Theorem 13.1 in [10].*

Remark 2. *In comparison with [7], the condition on the reversibility is deleted in theorems above. There are many Finsler manifolds with infinity reversibility. Consider the Randers metric in $\mathbb{B}^3(1)$*

$$F(x, y) = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} + |x|y^1, \quad |x| < 1,$$

where $|\cdot|$ denotes the standard Euclid norm. Then the reversibility

$$\lambda_F(x) = \frac{1 + |x|}{1 - |x|} \to +\infty, \quad \text{if } x \to \partial\mathbb{B}^3(1).$$

The geometric quantities between F and its reverse metric $\overleftarrow{F}$ have many important relationships. For example,

1. $F(-df) = \overleftarrow{F}(df)$, $\overleftarrow{\nabla} f = -\nabla(-f)$, $\overleftarrow{\Delta} f = -\Delta(-f)$.
2. *A forward (backward) distance function w.r.t. F is a backward (forward) distance function w.r.t. $\overleftarrow{F}$, and vice versa.*
3. *A forward (backward) geodesic ball w.r.t. F is a backward (forward) geodesic ball w.r.t. $\overleftarrow{F}$, and vice versa.*
4. *If f is a superharmonic (subharmonic) function w.r.t. Δ, then $-f$ is a subharmonic (superharmonic) function w.r.t. $\overleftarrow{\Delta}$, and vice versa.*

To give a more elaborate improvement, we use these relations and thus avoid employing the reversibility. The remainder of the approaches adopted are similar to Zhang-Xia's paper [7]. See also in [2,5,10] for the Riemannian case.

The contents of the paper are arranged as follows. In Section 2, some fundamental concepts which are necessary for the present paper are given, and some lemmas are contained. In Section 3, we prove the main theorems and give some corollaries.

2. Preliminaries

Let M be an $n-$dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinates on TM with $y = y^i\partial/\partial x^i$. A *Finsler metric* on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) *Regularity*: $F(x, y)$ is smooth in $TM \setminus 0$;
(ii) *Positive homogeneity*: $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$;

(iii) *Strong convexity*: The fundamental quadratic form

$$g := g_{ij}(x,y)dx^i \otimes dx^j, \qquad g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$$

is positive definite.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field. Then, the *covariant derivative* of X by $v \in T_xM$ with reference vector $w \in T_xM \backslash 0$ is defined by

$$D_v^w X(x) := \left\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w) v^j X^k(x) \right\} \frac{\partial}{\partial x^i},$$

where Γ^i_{jk} denote the coefficients of the Chern connection.

For a smooth function u, the *gradient vector* of u is

$$\nabla u := \ell^{-1}(du),$$

where $\ell : TM \to T^*M$ is *Legendre transformation* defined as

$$\ell(y) := \begin{cases} g_y(y,\cdot), & y \in TM\backslash 0, \\ 0, & y = 0. \end{cases}$$

Let $V = V^i \frac{\partial}{\partial x^i}$ be a smooth vector field on M. The *divergence* of V with respect to an arbitrary smooth volume form $d\mu$ is defined by

$$\mathrm{div} V := \sum_{i=1}^n \left(\frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \Phi}{\partial x^i} \right),$$

where $d\mu = e^{\Phi} dx$. Then, the *Finsler Laplacian* of u can be defined by

$$\Delta u := \mathrm{div}(\nabla u).$$

Since Δu is undefined at x where $du(x) = 0$, the definition can be viewed in distributional sense. That is, for $u \in W^{1,2}(M)$,

$$\int_M \varphi \Delta u d\mu = -\int_M d\varphi(\nabla u) d\mu, \quad \forall \varphi \in C_0^\infty(M). \tag{1}$$

We note here that since the gradient operator ∇ is not linear operator in general, the Finsler Laplacian is quite a bit different from the Riemannian Laplacian. Given a vector field V such that $V \neq 0$ on M_u, where $M_u := \{x \in M | du(x) \neq 0\}$, the *weighted gradient vector* and the *weighted Laplacian* on the weighted Riemannian manifold (M, g_V) are defined by

$$\nabla^V u := \begin{cases} g^{ij}(V) \frac{\partial u}{\partial x^j} \frac{\partial}{\partial x^i}, & \text{on} \quad M_u, \\ 0, & \text{on} \quad M \backslash M_u, \end{cases} \qquad \Delta^V u = \mathrm{div}(\nabla^V u).$$

It follows that $\nabla^{\nabla u} u = \nabla u, \quad \Delta^{\nabla u} u = \Delta u$.

Let u be a positive harmonic function on M, $\Delta u = 0$. It was proved that $u \in W^{2,2}_{loc}(M) \cap C^{1,\alpha}(M) \cap C^\infty(M_u)$ (see [11]). We say that $u \in W^{2,2}_{loc}(M) \cap C^{1,\alpha}(M) \cap C^\infty(M_u)$ is a subharmonic (resp. superharmonic) function on M if $\Delta u \geq 0$ (resp. $\Delta u \leq 0$). In a weak sense, u is a subharmonic (resp. superharmonic) function in M if, for any positive function $\varphi \in C_0^\infty(M)$, it holds

$$\int_M \varphi \Delta u d\mu \geq (\text{resp.} \leq) 0.$$

Let (M, F) be a Finsler n-manifold. Fix a point $x_0 \in M$. We denote a forward (resp. backward) geodesic ball of radius r with center at x_0 by $B^+_{x_0}(r)$(resp.$B^-_{x_0}(r)$).

Lemma 1. *Let (M, F) be a Finsler n-manifold and $x_0 \in M$. Then, there exists a function defined by*

$$\varphi(x) = \begin{cases} 1, & x \in B^+_{x_0}(R); \\ \\ 0, & x \in M \backslash B^+_{x_0}(2R) \end{cases}$$

such that

$$F^*(-d\varphi) \leq \frac{C}{R},$$

where C is a positive constant.

Proof. Let $\omega(t)$ be a smooth function on the real line with $0 \leq \omega(t) \leq 1$ and $\omega'(t) \leq 0$ such that

$$\omega(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t \geq 2. \end{cases}$$

Clearly, $|\omega'(t)| \leq C$, where C is some positive constant. Define

$$\varphi(x) = \omega\left(\frac{r(x)}{R}\right),$$

where $r(x) = d_F(x_0, x)$ is the distance function form x_0. Then,

$$\begin{aligned} F^*(-d\varphi) &= F^*\left(-\frac{\omega'}{R}dr\right) = \frac{|\omega'|}{R}F^*(dr) \\ &= \frac{|\omega'|}{R} \leq \frac{C}{R} \quad \text{a.e. on} \quad M. \end{aligned}$$

□

Notice that $\varphi \in C^\infty_0(B^+_{x_0}(2R) \backslash (\{x_0\} \cup \text{cut}(x_0)))$ and it is differentiable almost everywhere on $B^+_{x_0}(2R)$ with bounded differential. Since a subharmonic (resp. superharmonic) function u belongs to $W^{2,2}_{loc}(M) \cap C^{1,\alpha}(M) \cap C^\infty(M_u)$, and $\Delta u = 0$ a.e. on $M \backslash M_u$ (Lemma 3.5 in [12]), we find the Formula (1) still holds for this φ.

3. Proof of the Main Theorems

For any nonnegative function u, set

$$V_p(r) = \begin{cases} \int_{B^+_{x_0}(r)} u^p d\mu, & p \geq 1; \\ \\ \int_{B^-_{x_0}(r)} u^p d\mu, & p < 1. \end{cases} \tag{2}$$

Note that $V_p(r) = \text{vol}^{d\mu}(B^-_{x_0}(r)) := V(r)$ if $p = 0$.

Proof of Theorem 1. Set $v = \sqrt{u}$. Then, in $M_u = M_v$, one obtains

$$0 \leq \frac{1}{2}\Delta \log u = \Delta \log v = \text{div}\left(\frac{\nabla v}{v}\right) = \frac{\Delta v}{v} - \frac{F(\nabla v)^2}{v^2},$$

which gives

$$v\Delta v \geq F(\nabla v)^2 \geq 0.$$

Let φ be the function defined in Lemma 1. Then, it is differentiable almost everywhere on $(B^+_{x_0}(2R))$ with a bounded differential. Note that $\Delta v = 0$ a.e. on $M\backslash M_v$ from Lemma 3.5 in [12]. Thus, by the divergence theorem, we have

$$\begin{aligned}\int_{B^+_{x_0}(2R)\cap M_v} \varphi^2 v\Delta v d\mu = \int_{B^+_{x_0}(2R)} \varphi^2 v\Delta v d\mu &= -\int_{B^+_{x_0}(2R)} d(\varphi^2 v)(\nabla v) d\mu \\ &= -\int_{B^+_{x_0}(2R)} \varphi^2 F(\nabla v)^2 d\mu - 2\int_{B^+_{x_0}(2R)} \varphi v d\varphi(\nabla v) d\mu \\ &\leq -\int_{B^+_{x_0}(2R)} \varphi^2 F(\nabla v)^2 d\mu + 2\int_{B^+_{x_0}(2R)} \varphi v F^*(-d\varphi) F(\nabla v) d\mu.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{B^+_{x_0}(2R)} \varphi^2 F(\nabla v)^2 d\mu &\leq 2\int_{B^+_{x_0}(2R)} \varphi v F^*(-d\varphi) F(\nabla v) d\mu \\ &\leq 2\left(\int_{B^+_{x_0}(2R)} \varphi^2 F(\nabla v)^2 d\mu\right)^{\frac{1}{2}} \left(\int_{B^+_{x_0}(2R)} v^2 F^*(-d\varphi)^2 d\mu\right)^{\frac{1}{2}},\end{aligned}$$

which implies that

$$\int_{B^+_{x_0}(2R)} \varphi^2 F(\nabla v)^2 d\mu \leq 4\int_{B^+_{x_0}(2R)} v^2 F^*(-d\varphi)^2 d\mu.$$

By using Lemma 1 and the definition of φ, we deduce

$$\int_{B^+_{x_0}(R)} \varphi^2 F(\nabla v)^2 d\mu \leq \frac{4C^2}{R^2}\int_{B^+_{x_0}(2R)} v^2 d\mu = \frac{16C^2}{(2R)^2}\int_{B^+_{x_0}(2R)} u d\mu \qquad = 16C^2\frac{V_1(2R)}{(2R)^2}.$$

Letting $R \to \infty$, it follows from $\limsup_{r\to\infty} \frac{r^2}{V_1(r)} = \infty$ that $F(\nabla v) = 0$ everywhere. Since M is connected, v is a constant on M and so is u. $\square$

Proof of Theorem 2. Without loss of generality, we might as well assume $u > 0$. Otherwise, we can replace it by $\tilde{u} = u + \varepsilon > 0$ for some positive number ε. We first prove (2) in Theorem 2. Let x_0 be a fixed point in M and r_0 be a number with $0 < r_0 < R$. Define

$$\psi(x) = \begin{cases} 1, & x \in \bar{B}^+_{x_0}(r_0); \\ 0, & x \in M\backslash B^+_{x_0}(R), \end{cases}$$

with $\psi(x) \in C^\infty_0(B^+_{x_0}(R)\backslash(\{x_0\}\cup \mathrm{cut}(x_0)))$ satisfying

$$F^*(-d\psi) \leq \frac{C}{R} \leq \frac{C}{r_0} := \widehat{C}. \tag{3}$$

Using the divergence theorem and similar arguments above, we have

$$\begin{aligned}
&(p-1)\int_{B^+_{x_0}(R)\cap M_u} u^{p-1}\psi^2\Delta u d\mu \\
=&(p-1)\int_{B^+_{x_0}(R)} u^{p-1}\psi^2\Delta u d\mu \\
=&-(p-1)\int_{B^+_{x_0}(R)} d(u^{p-1}\psi^2)(\nabla u)d\mu \\
=&-(p-1)^2\int_{B^+_{x_0}(R)} u^{p-2}\psi^2F(\nabla u)^2d\mu-2(p-1)\int_{B^+_{x_0}(R)} u^{p-1}\psi d\psi(\nabla u)d\mu.
\end{aligned} \tag{4}$$

Set $v=u^{\frac{p}{2}}$. Then, (4) becomes

$$\begin{aligned}
&(p-1)\int_{B^+_{x_0}(R)} u^{p-1}\psi^2\Delta u d\mu \\
=&-4(1-\frac{1}{p})^2\int_{B^+_{x_0}(R)}\psi^2F(\nabla v)^2d\mu-4(1-\frac{1}{p})\int_{B^+_{x_0}(R)} v\psi d\psi(\nabla v)d\mu.
\end{aligned}$$

From the conditions in Theorem 2 and (3), it follows that

$$\begin{aligned}
&(1-\frac{1}{p})^2\int_{B^+_{x_0}(R)}\psi^2F(\nabla v)^2d\mu \\
\leq&-(1-\frac{1}{p})\int_{B^+_{x_0}(R)\backslash\bar{B}^+_{x_0}(r_0)} v\psi d\psi(\nabla v)d\mu \\
\leq&\left(\int_{B^+_{x_0}(R)\backslash\bar{B}^+_{x_0}(r_0)} v^2F^*(-d\psi)^2d\mu\right)^{\frac{1}{2}}\left((1-\frac{1}{p})^2\int_{B^+_{x_0}(R)\backslash\bar{B}^+_{x_0}(r_0)}\psi^2F(\nabla v)^2d\mu\right)^{\frac{1}{2}} \\
\leq&\widehat{C}\left(\int_{B^+_{x_0}(R)\backslash\bar{B}^+_{x_0}(r_0)} u^pd\mu\right)^{\frac{1}{2}}\left((1-\frac{1}{p})^2\int_{B^+_{x_0}(R)\backslash\bar{B}^+_{x_0}(r_0)}\psi^2F(\nabla v)^2d\mu\right)^{\frac{1}{2}} \\
=&\widehat{C}(V_p(R)-V_p(r_0))^{\frac{1}{2}}\left((1-\frac{1}{p})^2\int_{B^+_{x_0}(R)\backslash\bar{B}^+_{x_0}(r_0)}\psi^2F(\nabla v)^2d\mu\right)^{\frac{1}{2}}.
\end{aligned} \tag{5}$$

Let

$$G(r)=(1-\frac{1}{p})^2\int_{B^+_{x_0}(r)}F(\nabla v)^2d\mu.$$

Then, by similar arguments as in [7], we can also reach

$$\frac{1}{G(r_0)}-\frac{1}{G(R)}\geq\frac{1}{\widehat{C}^2}\frac{(R-r_0)^2}{V_p(R)-V_p(r_0)}.$$

For fixed r_0, taking $R_k=2^kr_0, k\in N^+$, we have

$$\begin{aligned}
\frac{1}{G(r_0)}&\geq\frac{1}{G(R_n)}+\frac{1}{\widehat{C}^2}\sum_{k=1}^n\frac{(R_k-R_{k-1})^2}{V_p(R_k)-V_p(R_{k-1})}\geq\frac{1}{4\widehat{C}^2}\sum_{k=1}^n\frac{(R_k)^2}{V_p(R_k)} \\
&=\frac{1}{8\widehat{C}^2}\sum_{k=1}^n\frac{2R_k^2}{V_p(R_k)}=\frac{1}{8\widehat{C}^2}\sum_{k=1}^n\frac{2^{k+1}r_0}{V_p(2^kr_0)}\times 2^kr_0 \\
&\geq\frac{1}{8\widehat{C}^2}\sum_{k=1}^n\int_{2^kr_0}^{2^{k+1}r_0}\frac{r}{V_p(r)}dr.
\end{aligned}$$

Letting $n \to \infty$, we have
$$\frac{1}{G(r_0)} \geq \frac{1}{8\widehat{C}^2}\int_{2r_0}^{\infty}\frac{r}{V_p(r)}dr = \infty,$$
which means that
$$\int_{B^+_{x_0}(r_0)} F(\nabla v)^2 d\mu = 0.$$

Therefore, by arbitrariness of r_0, we conclude that v must be constant on M and so is u.

Now, we are to prove (1) according to the cases $0 < p < 1$, $p < 0$ and $p = 0$, respectively.

Case I: $0 < p < 1$.

Let x_0 and r_0 be as above. Define
$$\psi(x) = \begin{cases} 1, & x \in \bar{B}^-_{x_0}(r_0); \\ \\ 0, & x \in M\backslash B^-_{x_0}(R), \end{cases}$$
with $\psi(x) \in C_0^\infty(B^-_{x_0}(R)\backslash(\{x_0\} \cup \mathrm{cut}(x_0)))$ satisfying
$$F^*(d\psi) \leq \frac{C}{R} \leq \frac{C}{r_0} := \widehat{C}.$$

By similar arguments, we also obtain (5) for the backward geodesic ball. The remainder of the proof is the same as above.

Case II: $p < 0$.

Set $v = -u^{\frac{p}{2}}$. Then,
$$F(\nabla v) = -\frac{p}{2}u^{\frac{p}{2}-1}F(\nabla u).$$

Let ψ be a function as in Case I. We can also obtain
$$\begin{aligned} 0 \leq & (p-1)\int_{B^-_{x_0}(R)} u^{p-1}\psi^2 \Delta u d\mu \\ = & -(p-1)^2\int_{B^-_{x_0}(R)} u^{p-2}\psi^2 F(\nabla u)^2 d\mu - 2(p-1)\int_{B^-_{x_0}(R)} u^{p-1}\psi d\psi(\nabla u) d\mu. \\ = & -4(1-\frac{1}{p})^2\int_{B^-_{x_0}(R)} \psi^2 F(\nabla v)^2 d\mu - 4(1-\frac{1}{p})\int_{B^-_{x_0}(R)} v\psi d\psi(\nabla v) d\mu. \end{aligned}$$

Therefore, we have
$$\begin{aligned} & (1-\frac{1}{p})^2\int_{B^-_{x_0}(R)} \psi^2 F(\nabla v)^2 d\mu \\ \leq & -(1-\frac{1}{p})\int_{B^-_{x_0}(R)\backslash \bar{B}^-_{x_0}(r_0)} v\psi d\psi(\nabla v) d\mu \\ \leq & \left(\int_{B^-_{x_0}(R)\backslash \bar{B}^-_{x_0}(r_0)} v^2 F^*(d\psi)^2 d\mu\right)^{\frac{1}{2}} \left((1-\frac{1}{p})^2\int_{B^-_{x_0}(R)\backslash \bar{B}^-_{x_0}(r_0)} \psi^2 F(\nabla v)^2 d\mu\right)^{\frac{1}{2}} \\ = & \widehat{C}(V_p(R) - V_p(r_0))^{\frac{1}{2}} \left((1-\frac{1}{p})^2\int_{B^-_{x_0}(R)\backslash \bar{B}^-_{x_0}(r_0)} \psi^2 F(\nabla v)^2 d\mu\right)^{\frac{1}{2}}. \end{aligned}$$

Then, by the same argument as above, one obtains that u is constant.

Case III: $p = 0$.

For every $k \in \mathbb{R}^+$, set

$$u_k = \begin{cases} k, & u \geq k; \\ u, & u < k. \end{cases}$$

Then, u_k is a nonnegative superharmonic function in a weak sense. We will prove this by following the arguments in ([10], p. 178). Let β be a symmetric, concave and bounded smooth function with $|\beta'| \leq 1$ and $\beta'' \geq 0$ (think of a smooth approximation of $x \longrightarrow |x|$). Define

$$\tilde{u}_k = \frac{u+k}{2} - \frac{\beta(u-k)}{2}$$

(think of a smooth approximation of u_k). Then, $d\tilde{u}_k = \frac{1}{2}(1-\beta')du$. Notice that $1-\beta' \geq 0$. By Legendre transformation, we have $\nabla\tilde{u}_k = \frac{1}{2}(1-\beta')\nabla u$. Since $\Delta\tilde{u}_k = 0$ a.e. on $M \backslash M_{\tilde{u}_k}$, for ψ defined in Case I, we have

$$\begin{aligned} 2\int_{B^-_{x_0}(R)} \psi\Delta\tilde{u}_k d\mu &= -2\int_{B^-_{x_0}(R)} d\psi(\nabla\tilde{u}_k)d\mu = -\int_{B^-_{x_0}(R)} (1-\beta')d\psi(\nabla u)d\mu \\ &= -\int_{B^-_{x_0}(R)} d[(1-\beta')\psi](\nabla u)d\mu - \int_{B^-_{x_0}(R)} \beta''\psi F(\nabla u)^2 d\mu \\ &\leq -\int_{B^-_{x_0}(R)} d[(1-\beta')\psi](\nabla u)d\mu = \int_{B^-_{x_0}(R)} (1-\beta')\psi\Delta u d\mu \\ &\leq 0. \end{aligned}$$

The last step holds because $(1-\beta')\psi$ is differentiable almost everywhere on $B^-_{x_0}(R)$ with bounded differential and u is superharmonic. Thus, the claim follows by approximation.

It is shown that $\nabla u_k = 0$ a.e. on $\{u \geq k\}$, $\nabla u_k = \nabla u$ on $\{u < k\}$ and $\mu(\{x \in M | u(x) = k, du(x) \neq 0\}) = 0$ w.r.t. to the measure $d\mu$ (see [7]). Thus, $\Delta u_k = 0$ a.e. on $\{u \geq k\}$ and $\Delta u_k = \Delta u$ on $\{u < k\}$. Notice that ψ is differentiable almost everywhere on $B^-_{x_0}(R)$ with a bounded differential. Therefore, by similar arguments, we can also obtain (4) for u_k on $B^-_{x_0}(R)$. Set $v_k = u_k^{\frac{q}{2}}$ for any $q \in (0,1)$. Then, we have (5) as follows:

$$\begin{aligned} &(1-\frac{1}{q})^2 \int_{B^-_{x_0}(R)} \psi^2 F(\nabla v_k)^2 d\mu \\ \leq &\widehat{C}\left(\int_{B^-_{x_0}(R)\backslash \bar{B}^-_{x_0}(r_0)} v_k^2\right)^{\frac{1}{2}} \left((1-\frac{1}{q})^2 \int_{B^-_{x_0}(R)\backslash \bar{B}^-_{x_0}(r_0)} \psi^2 F(\nabla v_k)^2 d\mu\right)^{\frac{1}{2}} \\ = &\widehat{C}(V_q(R) - V_q(r_0))^{\frac{1}{2}} \left((1-\frac{1}{q})^2 \int_{B^-_{x_0}(R)\backslash \bar{B}^-_{x_0}(r_0)} \psi^2 F(\nabla v_k)^2 d\mu\right)^{\frac{1}{2}}. \end{aligned}$$

On the other hand, note that

$$\int_{B^-_{x_0}(R)} u_k^q d\mu \leq \int_{B^-_{x_0}(R)} k^q d\mu = k^q V(R),$$

which implies that

$$\int_1^\infty \frac{r}{V_q(r)} dr = \infty.$$

Then, by the same discussion in the proof of (2) and Case I of (1), we show that this u_k is constant. Since k is arbitrary, the function u is also constant. □

Using Theorem 2, we can reach the following corollaries which extend Theorem 3 in [2] and Corollary 1 in [13], respectively.

Corollary 1. *Let* $(M, F, d\mu)$ *and* u *be as in Theorem 2.*

1. *If* $p \in (-\infty, 1)$, *then every nonnegative superharmonic function* $u \in L^p(M)$ *is a constant. In particular, if* $\mathrm{vol}^{d\mu}(M) < \infty$, *then every nonnegative superharmonic function on* M *is a constant.*
2. *If* $p \in (1, \infty)$, *then every nonnegative subharmonic function* $u \in L^p(M)$ *is a constant.*

Corollary 2. *Let* $(M, F, d\mu)$ *and* u *be as in Theorem 2 and* u *be a nonnegative superharmonic function. If, for a sequence* $r_k \to +\infty$,

$$V(r_k) \leq C r_k^2,$$

then u *is a constant, where* C *is a positive constant.*

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